SINGULAR ELLIPTIC KIRCHHOFF EQUATIONS WITH UNBALANCED GROWTH AND NONLINEAR BOUNDARY CONDITION

FRANZISKA BORER, IAN JESSE GLOTH, AND PATRICK WINKERT

Abstract. In this paper, we consider a singular Kirchhoff double phase problem with a right-hand side that consists of a singular and superlinear reaction term. Moreover, we allow critical growth on the nonlinear Neumann boundary condition. Using an equivalent norm in our function space and very general assumptions on the data, we prove the existence of two weak solutions whereby the first solution turns out to have negative energy sign while for the second one it is positive. Our proofs are based on variational methods and minimization of the associated energy functional over certain subsets of the Nehari manifold which are characterized by the corresponding fibering function.

1. Introduction

For a given bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we are interested in the multiplicity of weak solutions to singular Kirchhoff equations with nonlinear Neumann boundary condition of the form

$$\begin{aligned} -m(\Phi(u))\left(\nabla \cdot \mathcal{L}(u) - \alpha(x)u^{p-1}\right) &= \lambda u^{-\gamma} + u^{r-1} && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ m(\Phi(u))\mathcal{L}(u)\cdot \nu &= -\beta(x)u^{p_*-1} && \text{on } \partial\Omega, \end{aligned} \tag{K_λ}$$

where $\nabla \cdot \mathcal{L}$ is the double phase operator with

$$\mathcal{L}(u) = |\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u, \quad u \in W^{1,\mathcal{H}}(\Omega),$$

and

$$\Phi(u) = \Phi_{\mathcal{H}}(\nabla u) + \int_\Omega \alpha(x)\frac{1}{p}|u|^p \,\mathrm{d}x \quad \text{and} \quad \Phi_{\mathcal{H}}(u) = \int_\Omega \left(\frac{1}{p}|u|^p + \mu(x)\frac{1}{q}|u|^q\right)\mathrm{d}x.$$

Here, $\lambda > 0$ is a parameter to be specified and $\nu(x)$ is the outer unit normal of Ω at $x \in \partial\Omega$. Moreover, we suppose the following hypotheses:

(H) (i) $1 < p < N$, $p < q < p^* = \frac{Np}{N-p}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;
(ii) $0 < \gamma < 1$;
(iii) $\vartheta \geq 1$, $p_* < r < p^*$ and $\max\{2,\vartheta\}q < r$, where $p_* = \frac{(N-1)p}{N-p}$;
(iv) $a_0 \geq 0$, $b_0 > 0$ and m is a function given by
$$m\colon [0,\infty) \to [0,\infty), \quad t \mapsto a_0 + b_0 t^{\vartheta-1};$$
(v) $\alpha \in L^\infty(\Omega) \setminus \{0\}$ with $\alpha \geq 0$ a.e. in Ω;
(vi) $\beta \in L^\infty(\partial\Omega)$ with $\beta \geq 0$ a.e. on $\partial\Omega$.

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A function $u \in W^{1,\mathcal{H}}(\Omega)$ is said to be a weak solution of problem (K_λ) if $u > 0$ a.e. in Ω, $u^{-\gamma}\varphi \in L^1(\Omega)$ and if

$$\begin{aligned} &m(\Phi(u)) \left(\int_\Omega \mathcal{L}(u) \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega \alpha(x) u^{p-1} \varphi \,\mathrm{d}x \right) + \int_{\partial\Omega} \beta(x) u^{p_*-1} \varphi \,\mathrm{d}\sigma \\ &= \lambda \int_\Omega u^{-\gamma} \varphi \,\mathrm{d}x + \int_\Omega u^{r-1} \varphi \,\mathrm{d}x, \end{aligned} \tag{1.1}$$

is satisfied for all test functions $\varphi \in W^{1,\mathcal{H}}(\Omega)$, where $W^{1,\mathcal{H}}(\Omega)$ denotes an appropriate Musielak–Orlicz Sobolev space, see Section 2 for more details.

Our main result is the following one.

Theorem 1.1. *Let hypotheses* (H) *be satisfied. Then there exists* $\hat{\Lambda} > 0$ *such that for all* $\lambda \in (0, \hat{\Lambda})$ *problem* (K_λ) *has at least two weak solutions* u_λ, $v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ *with opposite energy sign.*

The proof of Theorem 1.1 relies on an appropriate usage of the fibering function together with the associated Nehari manifold corresponding to problem (K_λ). To be more precise, the idea is the splitting of the Nehari manifold into three disjoint subsets and then we minimize the associated energy functional over two of them to get the claimed solutions whereby the first solution turns out to have negative energy sign while for the second one it is positive. We also make use of an equivalent norm in $W^{1,\mathcal{H}}(\Omega)$ given by

$$\inf \left\{ \tau > 0 \colon \int_\Omega \left(\left(\frac{|\nabla u|}{\tau} \right)^p + \mu(x) \left(\frac{|\nabla u|}{\tau} \right)^q + \alpha(x) \left(\frac{|u|}{\tau} \right)^p \right) \mathrm{d}x \le 1 \right\}, \tag{1.2}$$

which has been recently proved by Amoroso–Crespo-Blanco–Pucci–Winkert [3]. It should be noted that the fibering method is a very powerful tool, not only applicable for singular problems but also for superlinear right-hand sides with subcritical and critical growth. As a starting point, the works of Drábek–Pohozaev [20] and Sun–Wu–Long [48] should be mentioned and later, this technique has been applied to different problems of singular and nonsingular type. We refer to the papers by Alves–Santos–Silva [2], Arora–Fiscella–Mukherjee–Winkert [5], Chen–Kuo–Wu [11], Crespo-Blanco–Papageorgiou–Winkert [17], Fiscella–Mishra [24], Kumar–Rădulescu–Sreenadh [34], Liu–Dai–Papageorgiou–Winkert [36], Papageorgiou–Repovš–Vetro [42], Tang–Chen [49], Wang–Zhao–Zhao [50], see also the references therein.

We point out that problem (K_λ) combines several interesting phenomena. First the appearing differential operator in (K_λ) is the so-called double phase operator defined by

$$\nabla \cdot \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right),$$

whose energy functional is given by

$$\omega \mapsto \int_\Omega \left(\frac{1}{p} |\nabla \omega|^p + \frac{\mu(x)}{q} |\nabla \omega|^q \right) \mathrm{d}x. \tag{1.3}$$

We emphasize that functionals of the shape (1.3) first appeared in the works of Marcellini [38, 39] concerning general (p,q)-growth. It is used to characterize models for strongly anisotropic materials in the context of homogenization and elasticity, and it also appears in duality theory and in the study of the Lavrentiev gap phenomenon, see the works of Zhikov [55, 56, 57]. First mathematical treatments of functionals given in (1.3) have been done in a remarkable series of papers about the regularity of local minimizers of such functionals, see the works by Baroni–Colombo–Mingione [7, 8] and Colombo–Mingione [13, 14].

A second fascinating phenomenon arises due to the appearance of the nonlocal Kirchhoff term in (K_λ) which is given by the function $m(t) = a_0 + b_0 t^{\vartheta-1}$ for $a_0 \geq 0$, $b_0 > 0$ and $\vartheta \geq 1$. The issues associated with this type of problem can be traced back to a model initially proposed by Kirchhoff [33] in 1883 which is a generalization of the D'Alembert equation and has the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \,\mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0.$$

It is noteworthy that problem (K_λ) is a generalization of several models that describe intriguing phenomena studied in the field of mathematical physics. Note that in our setting, the constant a_0 in the definition of the Kirchhoff function m may be zero which makes problem (K_λ) degenerate and which leads to the most intriguing models in practical applications. There is also a long list of references dealing with different types of Kirchhoff problems, we refer, for example, to the papers by Alves–Corrêa–Ma [1], Autuori–Pucci–Salvatori [6], D'Ancona–Spagnolo [19], Figueiredo [22], Fiscella–Valdinoci [26], He–Zou [31], Lions [35], Mao–Zhang [37], Mingqi–Rădulescu–Zhang [40], Perera–Zhang [46], Pucci–Xiang–Zhang [47], and Xiang–Zhang–Rădulescu [51].

In contrast, there are only a few works dealing with Kirchhoff problems of double phase type. As far as we know the first work in this direction has been published by Fiscella–Pinamonti [25] who obtained a mountain-pass type solution of Kirchhoff problems given by

$$-m\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x)\frac{|\nabla u|^q}{q} \right) \mathrm{d}x \right] \nabla \cdot \mathcal{L}(u) = f(x,u) \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = 0,$$

supposing the Ambrosetti-Rabinowitz condition and a subcritical growth on the nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$. We also refer to the work by Arora–Fiscella–Mukherjee–Winkert [5] (see also [4] by the same authors in the critical case) who studied a singular double phase problem with Kirchhoff function given by

$$-m\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x)\frac{|\nabla u|^q}{q} \right) \mathrm{d}x \right] \nabla \cdot \mathcal{L}(u) = \lambda u^{-\gamma} + u^{r-1} \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = 0,$$

whereby the existence of two solutions has been shown. Further results for Kirchhoff double phase problems have been obtained in the papers by Cen–Vetro-Zeng [10], Cheng–Bai [12], Crespo-Blanco–Gasiński–Winkert [16], Ho–Winkert [32], and Yang–Liu–Meng [52].

A further noteworthy aspect of problem (K_λ) is the presence of a nonlinear Neumann boundary condition which renders the treatment of problem (K_λ) more complex. The first work for a Kirchhoff double phase problem with a nonlinear Neumann boundary condition has been published by Fiscella–Marino–Pinamonti–Verzellesi [23] who proved various existence results based on variational tools and a version of the fountain theorem of the problem given by

$$-M\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x)\frac{|\nabla u|^q}{q} \right) \mathrm{d}x \right] \nabla \cdot \mathcal{L}(u) = h_1(x,u) \quad \text{in } \Omega,$$

$$-M\left[\int_\Omega \left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{L}(u)\cdot\nu = h_2(x,u) \quad \text{in } \partial\Omega,$$

where h_1 and h_2 fulfill different structure conditions. Recently, Borer–Pimenta–Winkert [9] proved the existence of a least energy sign-changing solution based on variational tools in combination with the quantitative deformation lemma and the Poincaré-Miranda existence theorem to the degenerate Kirchhoff problem

$$\begin{aligned} -\phi(\Xi(u))\left(\nabla\cdot\mathcal{L}(u)-|u|^{p-2}u\right) &= f(x,u) && \text{in } \Omega,\\ \phi(\Xi(u))\mathcal{L}(u)\cdot\nu &= g(x,u) && \text{on } \partial\Omega, \end{aligned}$$

where $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $g\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions that grow superlinearly and subcritically. Existence results for double phase problems with nonlinear Neumann boundary condition but without a nonlocal Kirchhoff term can be found in the papers by Cui–Sun [18], El Manouni–Marino–Winkert [21], Gasiński–Winkert [28], Guarnotta–Livrea–Winkert [29], Papageorgiou–Rădulescu–Repovš [41], Papageorgiou–Vetro–Vetro [43], Papageorgiou–Zhang [45], and Zeng–Rădulescu–Winkert [53, 54].

The paper is organized as follows. In Section 2 we introduce Musielak–Orlicz Sobolev spaces and its properties as well as a new equivalent norm as given in (1.2). Section 3 gives a detailed analysis of the fibering map and presents several results about suitable subsets of the Nehari manifold. Finally, Section 4 is devoted to the proof of Theorem 1.1 which is made by several lemmas and propositions.

2. Preliminaries

In this section we will present all the necessary tools that will be needed for the proof of our main result stated in Theorem 1.1. To this end, for $s\in[1,\infty]$ we denote by $L^s(\Omega)$ and $L^s(\Omega;\mathbb{R}^N)$ the standard Lebesgue spaces and by $L^s(\partial\Omega)$ the boundary Lebesgue spaces equipped with their usual norms $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. Denoting by $M(\Omega)$ and $M(\partial\Omega)$ the sets of all measurable functions $\Omega\to\mathbb{R}$ and $\partial\Omega\to\mathbb{R}$, respectively, and suppose from now on hypotheses (H), we introduce the following seminormed Lebesgue spaces

$$\begin{aligned} L^p_\alpha(\Omega) &= \left\{u\in M(\Omega)\colon \int_\Omega \alpha(x)|u|^p\,\mathrm{d}x<\infty\right\}, \quad \|u\|_{p,\alpha}=\left(\int_\Omega \alpha(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}},\\ L^{p_*}_\beta(\partial\Omega) &= \left\{u\in M(\partial\Omega)\colon \int_{\partial\Omega}\beta(x)|u|^{p_*}\,\mathrm{d}\sigma<\infty\right\},\\ \|u\|_{p_*,\beta,\partial\Omega} &= \left(\int_{\partial\Omega}\beta(x)|u|^{p_*}\,\mathrm{d}\sigma\right)^{\frac{1}{p_*}},\\ L^q_\mu(\Omega) &= \left\{u\in M(\Omega)\colon \int_\Omega \mu(x)|u|^q\,\mathrm{d}x<\infty\right\}, \quad \|u\|_{q,\mu}=\left(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}, \end{aligned}$$

where σ is the $(N-1)$-dimensional Hausdorff surface measure. Further, for $1<p<\infty$ we denote by $W^{1,p}(\Omega)$ the usual Sobolev space and equip it with the equivalent norm

$$\|u\|_{1,p}=\left(\|\nabla u\|_p^p+\|u\|_{p,\alpha}^p\right)^{\frac{1}{p}}.$$

In addition, due to the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we denote the best embedding constant by S, i.e.,

$$S := \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{1,p}^p}{\|u\|_{p^*}^p}.$$

This implies

$$\|u\|_{p^*} \leq S^{-\frac{1}{p}} \|u\|_{1,p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Next, we introduce the needed Musielak–Orlicz Sobolev space based on the monographs by Harjulehto–Hästö [30] and Papageorgiou–Winkert [44] as well as the paper by Crespo-Blanco–Gasiński–Harjulehto–Winkert [15]. Under the assumption (H), we introduce the nonlinear map

$$\mathcal{H} \colon \Omega \times [0,\infty) \to [0,\infty), \quad (x,t) \mapsto t^p + \mu(x) t^q$$

and $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x = \|u\|_p^p + \|u\|_{q,\mu}^q.$$

The Musielak-Orlicz Lebesgue space is defined as

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) \colon \rho_{\mathcal{H}}(u) < \infty\}$$

endowed with the Luxemburg norm given by

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 \colon \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

The corresponding Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is then given by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \colon |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

and equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. We know that both spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces.

Using Proposition 3.1 by Amoroso–Crespo-Blanco–Pucci–Winkert [3], we can equip the space $W^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\| = \inf\left\{\tau > 0 \colon \int_\Omega \left(\left(\frac{|\nabla u|}{\tau}\right)^p + \mu(x)\left(\frac{|\nabla u|}{\tau}\right)^q + \alpha(x)\left(\frac{|u|}{\tau}\right)^p\right)\mathrm{d}x \leq 1\right\}$$

and the related modular ρ to $\|\cdot\|$ is given by

$$\begin{aligned}\rho(u) &= \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q\right)\mathrm{d}x + \int_\Omega \alpha(x)|u|^p\,\mathrm{d}x \\ &= \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\alpha}^p = \rho_{\mathcal{H}}(\nabla u) + \|u\|_{p,\alpha}^p\end{aligned}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$.

The next proposition is taken from Amoroso–Crespo-Blanco–Pucci–Winkert [3, Proposition 3.2].

Proposition 2.1. *Let hypotheses (H)(i), (v) be satisfied, $u \in W^{1,\mathcal{H}}(\Omega)$ and $\lambda \in \mathbb{R}$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\| = \lambda$ if and only if $\rho\left(\frac{u}{\lambda}\right) = 1$;*
(ii) *$\|u\| < 1$ (resp. > 1, $= 1$) if and only if $\rho(u) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|u\| < 1$, then $\|u\|^q \leq \rho(u) \leq \|u\|^p$;*

(iv) *If* $\|u\|>1$, *then* $\|u\|^p \leq \rho(u) \leq \|u\|^q$;
(v) $\|u\| \to 0$ *if and only if* $\rho(u) \to 0$;
(vi) $\|u\| \to \infty$ *if and only if* $\rho(u) \to \infty$.

The next proposition summarizes the main embeddings related to the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco–Gasiński–Harjulehto–Winkert [17, Proposition 2.16].

Proposition 2.2. *Let hypotheses* (H)(i), (v) *be satisfied. Then the following hold:*
(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^s(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,s}(\Omega)$ *are continuous for all* $s \in [1,p]$,
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^s(\Omega)$ *is continuous for all* $s \in [1,p^*]$ *and compact for all* $s \in [1,p^*)$,
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^s(\partial\Omega)$ *is continuous for all* $s \in [1,p_*]$ *and compact for all* $s \in [1,p_*)$,
(iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_\mu(\Omega)$ *is continuous,*
(v) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for a function $u \in W^{1,\mathcal{H}}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It holds $|u| = u^+ + u^-$ and $u = u^+ - u^-$. Moreover, we know that $u^\pm \in W^{1,\mathcal{H}}(\Omega)$ whenever $u \in W^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [15, Proposition 2.17]. The Lebesgue measure of a set $V \subseteq \mathbb{R}^N$ will be denoted by $|V|$.

3. Analysis Of The Fibering Function

In this section, we give a detailed study of the fibering map. To this end, we first introduce the energy functional $J_\lambda\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ related to problem (K_λ) given by

$$J_\lambda(u) = M\left(\Phi(u)\right) + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u\|_r^r$$

where $M\colon [0,\infty) \to [0,\infty)$ is defined by

$$M(t) = \int_0^t m(\tau)\,\mathrm{d}\tau = a_0 t + \frac{b_0}{\vartheta} t^\vartheta.$$

Recall that

$$\Phi(u) = \Phi_{\mathcal{H}}(\nabla u) + \frac{1}{p}\|u\|_{p,\alpha}^p \quad \text{and} \quad \Phi_{\mathcal{H}}(u) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_{q,\mu}^q.$$

It is clear that J_λ is not C^1 because of the appearance of the singular term. Next, for $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$, we introduce the fibering function $\psi_u\colon [0,\infty) \to \mathbb{R}$ by $\psi_u(t) = J_\lambda(tu)$ for all $t \geq 0$, that is

$$\psi_u(t) = a_0\Phi(tu) + \frac{b_0}{\vartheta}\Phi^\vartheta(tu) + \frac{t^{p_*}}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{t^r}{r}\|u\|_r^r.$$

We observe that $\psi_u \in \mathcal{C}^\infty(0,\infty)$. For all $t>0$ the first and second derivatives are given by

$$\begin{aligned}\psi_u'(t) &= a_0 t^{-1}\rho(tu) + b_0\Phi^{\vartheta-1}(tu)t^{-1}\rho(tu) + t^{p_*-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\ &\quad - \lambda t^{-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - t^{r-1}\|u\|_r^r\\ &= \left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right) + t^{p_*-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\ &\quad - \lambda t^{-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - t^{r-1}\|u\|_r^r\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\psi_u''(t) &= \left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left((p-1)t^{p-2}\|u\|_{1,p}^p + (q-1)t^{q-2}\|\nabla u\|_{q,\mu}^q\right)\\ &\quad + b_0(\vartheta-1)\Phi^{\vartheta-2}(tu)\left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right)^2\\ &\quad + (p_*-1)t^{p_*-2}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma t^{-\gamma-1}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\\ &\quad - (r-1)t^{r-2}\|u\|_r^r.\end{aligned} \tag{3.2}$$

Next, we define the Nehari manifold related to problem (K_λ) which is given by

$$\mathcal{N}_\lambda := \left\{u \in W^{1,\mathcal{H}}(\Omega)\setminus\{0\} \colon \psi_u'(1) = 0\right\},$$

where ψ_u' is stated in (3.1). By the definition of $\mathcal{N}_\lambda$ it is clear that $\mathcal{N}_\lambda$ contains all weak solutions of problem (K_λ). On the contrary, if we suppose $u \in \mathcal{N}_\lambda$, then equation (1.1) holds true for u, if $\varphi = u$. Moreover, we split $\mathcal{N}_\lambda$ into three disjoint sets given by

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda \colon \psi_u''(1) > 0\},\\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda \colon \psi_u''(1) < 0\},\\ \mathcal{N}_\lambda^\circ &= \{u \in \mathcal{N}_\lambda \colon \psi_u''(1) = 0\}\end{aligned}$$

with ψ_u'' as in (3.2).

Remark 3.1. *Let $u \in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$ and $t>0$. Then the following hold:*

(i) *It holds $tu \in \mathcal{N}_\lambda$ if and only if $\psi_u'(t) = 0$ since*

$$\begin{aligned}\psi_u'(t) &= \left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right)\\ &\quad + t^{p_*-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda t^{-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - t^{r-1}\|u\|_r^r\\ &= \frac{1}{t}\bigg(\left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left(\|tu\|_{1,p}^p + \|t\nabla u\|_{q,\mu}^q\right) + \|tu\|_{p_*,\beta,\partial\Omega}^{p_*}\\ &\quad - \lambda\int_\Omega |tu|^{1-\gamma}\,\mathrm{d}x - \|tu\|_r^r\bigg) = \frac{1}{t}\psi_{tu}'(1).\end{aligned}$$

(ii) *It holds $tu \in \mathcal{N}_\lambda^\pm$ if and only if $\psi_u'(t) = 0$ and $\pm\psi_u''(t) > 0$. Also, $tu \in \mathcal{N}_\lambda^\circ$ if and only if $\psi_u'(t) = 0$ and $\psi_u''(t) = 0$. These facts follow from the simple calculation*

$$\begin{aligned}&\psi_u''(t)\\ &= \left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left((p-1)t^{p-2}\|u\|_{1,p}^p + (q-1)t^{q-2}\|\nabla u\|_{q,\mu}^q\right)\\ &\quad + b_0(\vartheta-1)\Phi^{\vartheta-2}(tu)\left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right)^2\end{aligned}$$

$$
\begin{aligned}
&\quad + (p_*-1)t^{p_*-2}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma t^{-\gamma-1}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&\quad - (r-1)t^{r-2}\|u\|_r^r \\
&= \frac{1}{t^2}\bigg(\left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left((p-1)\|tu\|_{1,p}^p + (q-1)\|t\nabla u\|_{q,\mu}^q\right) \\
&\quad + b_0(\vartheta-1)\Phi^{\vartheta-2}(tu)\left(\|tu\|_{1,p}^p + \|t\nabla u\|_{q,\mu}^q\right)^2 \\
&\quad + (p_*-1)\|tu\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |tu|^{1-\gamma}\,\mathrm{d}x - (r-1)\|tu\|_r^r\bigg) \\
&= \frac{1}{t^2}\psi_{tu}''(1).
\end{aligned}
$$

The following lemma shows that J_λ restricted to $\mathcal{N}_\lambda$ is coercive.

Lemma 3.2. *Suppose hypothesis (H) is satisfied and let $\lambda > 0$. Then, J_λ restricted on $\mathcal{N}_\lambda$ is coercive and bounded from below.*

Proof. Taking $u \in \mathcal{N}_\lambda$ with $\|u\| > 1$, the definition of $\mathcal{N}_\lambda$ yields

$$
-\frac{1}{r}\|u\|_r^r = -\frac{1}{r}\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) + \frac{\lambda}{r}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}. \tag{3.3}
$$

By Proposition 2.2 the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^1(\Omega)$ is continuous. This implies

$$
\|u\|_1 \le \tilde{C}_2\|u\| \tag{3.4}
$$

for some constant $\tilde{C}_2 > 0$. Using $a_0 \ge 0$, $b_0 > 0$, $\vartheta - 1 \ge 0$, $\frac{1}{\vartheta q} - \frac{1}{r} > 0$, $\frac{1}{p_*} - \frac{1}{r} > 0$, $\frac{1}{1-\gamma} - \frac{1}{r} > 0$, along with $\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \le |\Omega|^\gamma \|u\|_1^{1-\gamma}$, $\Phi(u) > \frac{1}{q}\rho(u)$, (3.3), (3.4), and Proposition 2.1 we obtain

$$
\begin{aligned}
J_\lambda(u) &= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u\|_r^r \\
&= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&\quad - \frac{1}{r}\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) + \frac{\lambda}{r}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&= a_0\left(\Phi(u) - \frac{1}{r}\rho(u)\right) + b_0\Phi^{\vartheta-1}(u)\left(\frac{1}{\vartheta}\Phi(u) - \frac{1}{r}\rho(u)\right) \\
&\quad + \left(\frac{1}{p_*} - \frac{1}{r}\right)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&\ge a_0\left(\frac{1}{q}\rho(u) - \frac{1}{r}\rho(u)\right) + b_0\Phi^{\vartheta-1}(u)\left(\frac{1}{\vartheta q}\rho(u) - \frac{1}{r}\rho(u)\right) \\
&\quad + \left(\frac{1}{p_*} - \frac{1}{r}\right)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&\ge a_0\left(\frac{1}{q} - \frac{1}{r}\right)\rho(u) + b_0\Phi^{\vartheta-1}(u)\left(\frac{1}{\vartheta q} - \frac{1}{r}\right)\rho(u) \\
&\quad - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)|\Omega|^\gamma\|u\|_1^{1-\gamma} \\
&\ge b_0\frac{1}{q^{\vartheta-1}}\left(\frac{1}{\vartheta q} - \frac{1}{r}\right)\rho^\vartheta(u) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)|\Omega|^\gamma\tilde{C}_2^{1-\gamma}\|u\|^{1-\gamma}
\end{aligned}
$$

$$\geq b_0 \frac{1}{q^{\vartheta-1}}\left(\frac{1}{\vartheta q}-\frac{1}{r}\right)\|u\|^{\vartheta p}-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)|\Omega|^{\gamma}\tilde{C}_2^{1-\gamma}\|u\|^{1-\gamma}$$
$$=\|u\|\omega\left(\|u\|\right),$$

where

$$\omega(t):=t^{\vartheta p-1}\left(C_1-C_2t^{1-\gamma-\vartheta p}\right) \quad \text{for all } t>0,$$

and C_1 and C_2 are positive constants given by

$$C_1=b_0\frac{1}{q^{\vartheta-1}}\left(\frac{1}{\vartheta q}-\frac{1}{r}\right) \quad \text{and} \quad C_2=\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)|\Omega|^{\gamma}\tilde{C}_2^{1-\gamma}.$$

Since $\vartheta p-1>0$ and $1-\gamma-\vartheta p<0$ we have $\lim_{t\to\infty}\omega(t)=\infty$. Hence, $J_\lambda\big|_{\mathcal{N}_\lambda}$ is coercive.

Next, we define

$$h(t)=t\omega(t)=C_1t^{\vartheta p}-C_2t^{1-\gamma} \quad \text{for all } t>0.$$

By computing the first and second derivatives of h, we see that h is strictly convex and attains a unique global minimum in

$$t_0=\left(\frac{C_2(1-\gamma)}{C_1\vartheta p}\right)^{\frac{1}{\vartheta p-1+\gamma}}.$$

Therefore, $J_\lambda\big|_{\mathcal{N}_\lambda}$ is bounded from below. □

In the next lemma we are going to show the emptiness of $\mathcal{N}_\lambda^\circ$ for $\lambda>0$ sufficiently small.

Lemma 3.3. *Suppose hypothesis* (H) *is satisfied. Then there exists $\Lambda_1>0$ such that for all $\lambda\in(0,\Lambda_1)$ we have $\mathcal{N}_\lambda^\circ=\emptyset$.*

Proof. Let $\lambda>0$ and suppose $u\in\mathcal{N}_\lambda^\circ$. Then we have $u\in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$ with $\psi_u'(1)=0$ and $\psi_u''(1)=0$, i.e.

$$(a_0+b_0\Phi^{\vartheta-1}(u))\rho(u)=\lambda\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x+\|u\|_r^r-\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \tag{3.5}$$

and

$$\begin{aligned}(a_0&+b_0\Phi^{\vartheta-1}(u))((p-1)\|u\|_{1,p}^p+(q-1)\|\nabla u\|_{q,\mu}^q)\\&+b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)\\=&-\lambda\gamma\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x+(r-1)\|u\|_r^r-(p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.\end{aligned} \tag{3.6}$$

Now, we multiply (3.5) by γ, add the result to (3.6) and obtain

$$\begin{aligned}(a_0&+b_0\Phi^{\vartheta-1}(u))((p-1+\gamma)\|u\|_{1,p}^p+(q-1+\gamma)\|\nabla u\|_{q,\mu}^q)\\&+b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)\\=&(r-1+\gamma)\|u\|_r^r-(p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.\end{aligned} \tag{3.7}$$

Next, multiplying (3.5) by $r-1$ and subtracting (3.6) gives

$$\begin{aligned}(a_0&+b_0\Phi^{\vartheta-1}(u))((r-p)\|u\|_{1,p}^p+(r-q)\|\nabla u\|_{q,\mu}^q)\\&-b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)\\=&(r-1+\gamma)\lambda\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x+(p_*-r)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.\end{aligned} \tag{3.8}$$

We consider the functional $T_\lambda\colon \mathcal{N}_\lambda \to \mathbb{R}$ defined by

$$T_\lambda(u) = \frac{(a_0 + b_0\Phi^{\vartheta-1}(u))((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q)}{r-1+\gamma} + \frac{b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)}{r-1+\gamma} + \frac{(p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}}{r-1+\gamma} - \|u\|_r^r.$$

Equation (3.7) implies

$$T_\lambda(u) = 0 \quad \text{for all } u \in \mathcal{N}_\lambda^\circ. \tag{3.9}$$

By using the estimates $a_0 \geq 0$, $b_0 > 0$, $p-1+\gamma$, $q-1+\gamma > 0$, $p_*-1+\gamma > 0$, $r-1+\gamma > 0$ and $\vartheta - 1 \geq 0$ together with $\Phi(u) \geq \frac{1}{p}\|u\|_{1,p}^p$, $\|u\|_r^r \leq |\Omega|^{1-\frac{r}{p^*}}\|u\|_{p^*}^r$ and $\|u\|_{p^*} \leq S^{-\frac{1}{p}}\|u\|_{1,p}$ we get

$$\begin{aligned}
T_\lambda(u) &= \frac{(a_0 + b_0\Phi^{\vartheta-1}(u))((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q)}{r-1+\gamma} \\
&\quad + \frac{b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)}{r-1+\gamma} + \frac{(p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}}{r-1+\gamma} - \|u\|_r^r \\
&\geq \frac{p-1+\gamma}{r-1+\gamma} b_0 \Phi^{\vartheta-1}(u)\|u\|_{1,p}^p - \|u\|_r^r \\
&\geq \frac{p-1+\gamma}{r-1+\gamma}\frac{b_0}{p^{\vartheta-1}}\|u\|_{1,p}^{p(\vartheta-1)}\|u\|_{1,p}^p - |\Omega|^{1-\frac{r}{p^*}}\|u\|_{p^*}^r \\
&\geq \frac{p-1+\gamma}{r-1+\gamma}\frac{b_0}{p^{\vartheta-1}}\|u\|_{1,p}^{\vartheta p} - S^{-\frac{r}{p}}|\Omega|^{1-\frac{r}{p^*}}\|u\|_{1,p}^r \\
&= \|u\|_{1,p}^r\left(\frac{p-1+\gamma}{r-1+\gamma}\frac{b_0}{p^{\vartheta-1}}\|u\|_{1,p}^{\vartheta p - r} - S^{-\frac{r}{p}}|\Omega|^{1-\frac{r}{p^*}}\right) \\
&= \|u\|_{1,p}^r\left(A\|u\|_{1,p}^{\vartheta p - r} - B\right),
\end{aligned} \tag{3.10}$$

where A and B are positive constants given by

$$A := \frac{b_0(p-1+\gamma)}{p^{\vartheta-1}(r-1+\gamma)} \quad \text{and} \quad B := S^{-\frac{r}{p}}|\Omega|^{1-\frac{r}{p^*}}.$$

Furthermore, from $a_0 \geq 0$, $b_0 > 0$, $\vartheta - 1 \geq 0$, $r - \vartheta q > 0$, $r-p > 0$, $r-q > 0$, $r-1+\gamma > 0$, and $q-p > 0$ along with $\Phi(u) \geq \frac{1}{p}\|u\|_{1,p}^p$, $q\Phi(u) > \rho(u)$, $\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \leq |\Omega|^{1-\frac{1-\gamma}{p^*}}\|u\|_{p^*}^{1-\gamma}$, $\|u\|_{p^*} \leq S^{-\frac{1}{p}}\|u\|_{1,p}$ and (3.8) we infer that

$$\begin{aligned}
&\frac{b_0(r-\vartheta q + q - p)}{p^{\vartheta-1}}\|u\|_{1,p}^{\vartheta p} \\
&= b_0(r-\vartheta q+q-p)\|u\|_{1,p}^p\left(\frac{\|u\|_{1,p}^p}{p}\right)^{\vartheta-1} \\
&\leq b_0\Phi^{\vartheta-1}(u)(r-\vartheta q+q-p)\|u\|_{1,p}^p \\
&\leq a_0\left((r-p)\|u\|_{1,p}^p + (r-q)\|\nabla u\|_{q,\mu}^q\right) \\
&\qquad + b_0\Phi^{\vartheta-1}(u)\left((r-\vartheta q+q-p)\|u\|_{1,p}^p + (r-\vartheta q)\|\nabla u\|_{q,\mu}^q\right) \\
&= \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((r-p)\|u\|_{1,p}^p + (r-q)\|\nabla u\|_{q,\mu}^q\right) \\
&\qquad - b_0(\vartheta-1)\Phi^{\vartheta-2}(u)q\Phi(u)\rho(u) \\
&\leq \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((r-p)\|u\|_{1,p}^p + (r-q)\|\nabla u\|_{q,\mu}^q\right)
\end{aligned}$$

$$
\begin{aligned}
&- b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u)\\
&= (r-1+\gamma)\lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + (p_*-r)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\le (r-1+\gamma)\lambda|\Omega|^{1-\frac{1-\gamma}{p^*}}\|u\|_{p^*}^{1-\gamma}\\
&\le (r-1+\gamma)\lambda|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}.
\end{aligned}
$$

Solving this inequality for $\|u\|_{1,p}$ and taking $r>\vartheta q$ and $q>p$ into consideration yields

$$\|u\|_{1,p} \le C\lambda^{\frac{1}{\vartheta p-1+\gamma}},$$

where C is a positive constant given by

$$C := \left(\frac{(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}p^{\vartheta-1}}{b_0(r-\vartheta q+q-p)}\right)^{\frac{1}{\vartheta p-1+\gamma}}.$$

Since $r>\vartheta p$, this implies

$$\|u\|_{1,p}^{-(r-\vartheta p)} \ge \left(C\lambda^{\frac{1}{\vartheta p-1+\gamma}}\right)^{-(r-\vartheta p)} = C^{-(r-\vartheta p)}\lambda^{\frac{-(r-\vartheta p)}{\vartheta p-1+\gamma}}$$

and together with (3.10) we get

$$T_\lambda(u) \ge \|u\|_{1,p}^r\left(A\|u\|_{1,p}^{-(r-\vartheta p)} - B\right) \ge \|u\|_{1,p}^r\left(AC^{-(r-\vartheta p)}\lambda^{\frac{-(r-\vartheta p)}{\vartheta p-1+\gamma}} - B\right).$$

Finally, we define

$$\Lambda_1 := \left(\frac{B}{AC^{-(r-\vartheta p)}}\right)^{\frac{\vartheta p-1+\gamma}{-(r-\vartheta p)}} > 0.$$

But then we have $T_\lambda(u)>0$ for $\lambda\in(0,\Lambda_1)$ which contradicts (3.9). This shows the assertion. ☐

For the next result we introduce the function $\sigma_u\colon(0,\infty)\to\mathbb{R}$ given by

$$
\begin{aligned}
\sigma_u(t) &= t^\gamma\psi_u'(t) + \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\\
&= \left(a_0 + b_0\Phi^{\vartheta-1}(tu)\right)\left(t^{p-1+\gamma}\|u\|_{1,p}^p + t^{q-1+\gamma}\|\nabla u\|_{q,\mu}^q\right)\\
&\quad + t^{p_*-1+\gamma}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - t^{r-1+\gamma}\|u\|_r^r.
\end{aligned}
\tag{3.11}
$$

From (3.11) we observe that σ_u does not depend on the parameter λ. Also, by (3.11) and the following Remark 3.1, for all $u\in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$, $\lambda>0$ and $t>0$, we have

$$tu\in\mathcal{N}_\lambda \quad\text{if and only if}\quad \sigma_u(t) = \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x. \tag{3.12}$$

Lemma 3.4. *Suppose hypothesis (H) is satisfied. Then, for $u\in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$, there exists a unique $t_{\max}^u>0$ such that*

$$\sigma_u(t_{\max}^u) = \max_{t>0}\sigma_u(t).$$

Furthermore, we can find $\Lambda_2>0$ such that for all $\lambda\in(0,\Lambda_2)$ and $u\in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$ there exist unique $0<t_1^u<t_{\max}^u<t_2^u$ with

$$t_1^u u\in\mathcal{N}_\lambda^+ \quad\text{and}\quad t_2^u u\in\mathcal{N}_\lambda^-.$$

Proof. Let $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$. First, we will show that there exists a unique $t^u_{\max} > 0$ such that $\sigma_u(t^u_{\max}) = \max_{t>0} \sigma_u(t)$. For this purpose we define a function $T_u \colon (0,\infty) \to \mathbb{R}$ via

$$T_u(t) = \frac{1}{t^{r-2+\gamma}} \sigma_u'(t) + (r-1+\gamma)\|u\|_r^r.$$

For all $t > 0$ we have

$$\begin{aligned}
\sigma_u'(t) &= (a_0 + b_0\Phi^{\vartheta-1}(tu)) \left((p-1+\gamma)t^{p-2+\gamma}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-2+\gamma}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)\left(t^{p-1+\gamma}\|u\|_{1,p}^p + t^{q-1+\gamma}\|\nabla u\|_{q,\mu}^q\right) \\
&\qquad \times \left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (p_*-1+\gamma)t^{p_*-2+\gamma}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - (r-1+\gamma)t^{r-2+\gamma}\|u\|_r^r
\end{aligned}$$

and so

$$\begin{aligned}
T_u(t) &= (a_0 + b_0\Phi^{\vartheta-1}(tu)) \left((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)\left(t^{p-r+1}\|u\|_{1,p}^p + t^{q-r+1}\|\nabla u\|_{q,\mu}^q\right) \\
&\qquad \times \left(t^{p-1}\|u\|_{1,p}^p + t^{q-1}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (p_*-1+\gamma)t^{p_*-r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&= (a_0 + b_0\Phi^{\vartheta-1}(tu)) \left((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-r}\rho^2(tu) + (p_*-1+\gamma)t^{p_*-r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.
\end{aligned}$$

Using $\Phi(tu) \geq \frac{1}{p}t^p\|u\|_{1,p}^p$ along with $p-1+\gamma>0$, $q-1+\gamma>0$, $p_*-1+\gamma>0$ and $\vartheta \geq 1$, we find for all $t>0$ that

$$\begin{aligned}
T_u(t) &= (a_0 + b_0\Phi^{\vartheta-1}(tu))\big((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p \\
&\quad + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\big) \\
&\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-r}\rho^2(tu) + (p_*-1+\gamma)t^{p_*-r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\geq b_0\left(\frac{1}{p}t^p\|u\|_{1,p}^p\right)^{\vartheta-1}(p-1+\gamma)t^{p-r}\|u\|_{1,p}^p \\
&= \frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)\|u\|_{1,p}^{\vartheta p}t^{\vartheta p-r}.
\end{aligned} \tag{3.13}$$

Because of $u \neq 0$, $\frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)>0$ and $\vartheta p - r < 0$, the estimate (3.13) implies

$$\lim_{t \searrow 0} T_u(t) = \infty. \tag{3.14}$$

Since $a_0 \geq 0$, $b_0 > 0$, $p-1+\gamma>0$, $q-1+\gamma>0$, $p_*-1+\gamma>0$ and $\vartheta \geq 1$ together with $\|u\|_{1,p}^p \leq \rho(u)$ and $\|\nabla u\|_{q,\mu}^q \leq \rho(u)$ as well as

$$t^p \leq t^q \quad \text{and} \quad \Phi(tu) \leq \frac{1}{p}\rho(tu) \leq \frac{t^q}{p}\rho(u) \quad \text{if } t \geq 1,$$

we obtain for all $t \geq 1$

$$\begin{aligned}
|T_u(t)| &= T_u(t) \\
&= (a_0 + b_0\Phi^{\vartheta-1}(tu)) \left((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\right) \\
&\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-r}\rho^2(tu) + (p_*-1+\gamma)t^{p_*-r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(a_0 + b_0 \left(\frac{t^q}{p}\rho(u)\right)^{\vartheta-1}\right) \big((q-1+\gamma)t^{q-r}\rho(u) + (q-1+\gamma)t^{q-r}\rho(u)\big) \\
&\qquad + (\vartheta-1)b_0 \left(\frac{t^q}{p}\rho(u)\right)^{\vartheta} t^{-r}\frac{\rho^2(tu)}{\Phi^2(tu)} + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} t^{p_*-r},
\end{aligned}
$$

that is,

$$
\begin{aligned}
|T_u(t)| &\leq 2a_0(q-1+\gamma)\rho(u)t^{q-r} + 2\frac{b_0}{p^{\vartheta-1}}\rho^{\vartheta}(u)(q-1+\gamma)t^{\vartheta q-r} \\
&\qquad + (\vartheta-1)\frac{b_0}{p^{\vartheta}}\rho^{\vartheta}(u)\frac{\rho^2(tu)}{\Phi^2(tu)}t^{\vartheta q-r} + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}t^{p_*-r}.
\end{aligned}
\tag{3.15}
$$

Because of $q-r<0$, $p_*-r<0$, $\vartheta q - r < 0$ and

$$
\lim_{t\to\infty}\frac{\rho^2(tu)}{\Phi^2(tu)} = \lim_{t\to\infty}\left(\frac{t^{p-q}\|u\|_{1,p}^p + \|\nabla u\|_{q,\mu}^q}{t^{p-q}\left(\frac{1}{p}\|u\|_{1,p}^p\right) + \frac{1}{q}\|\nabla u\|_{q,\mu}^q}\right)^2 = \begin{cases} q^2 & \text{if } \|\nabla u\|_{q,\mu}^q \neq 0, \\ p^2 & \text{if } \|\nabla u\|_{q,\mu}^q = 0, \end{cases}
$$

the estimate (3.15) implies

$$
\lim_{t\to\infty} T_u(t) = 0. \tag{3.16}
$$

Thus, by the intermediate value theorem, there exists $t_{\max}^u > 0$ such that

$$
T_u(t_{\max}^u) = (r-1+\gamma)\|u\|_r^r.
$$

In order to verify $\sigma_u(t_{\max}^u) = \max_{t>0}\sigma_u(t)$ and obtain the uniqueness of $t_{\max}^u$, we show via calculating the first derivative that T_u is strictly decreasing. For $t>0$ we have

$$
\begin{aligned}
T_u'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big((a_0 + b_0\Phi^{\vartheta-1}(tu))\big((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\big) \\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-r}\rho^2(tu) + (p_*-1+\gamma)t^{p_*-r}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\Big) \\
&= (a_0 + b_0\Phi^{\vartheta-1}(tu))(p-r)(p-1+\gamma)t^{p-r-1}\|u\|_{1,p}^p \\
&\qquad + (a_0 + b_0\Phi^{\vartheta-1}(tu))(q-r)(q-1+\gamma)t^{q-r-1}\|\nabla u\|_{q,\mu}^q \\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-1}\rho(tu) \\
&\qquad\qquad \times \big((p-1+\gamma)t^{p-r}\|u\|_{1,p}^p + (q-1+\gamma)t^{q-r}\|\nabla u\|_{q,\mu}^q\big) \\
&\qquad + (\vartheta-1)(\vartheta-2)b_0\Phi^{\vartheta-3}(tu)t^{-1}\rho(tu)t^{-r}\rho^2(tu) \\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)(-r)t^{-r-1}\rho^2(tu) \\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(tu)t^{-r}2\rho(tu)\big(pt^{p-1}\|u\|_{1,p}^p + qt^{q-1}\|\nabla u\|_{q,\mu}^q\big) \\
&\qquad + (p_*-r)(p_*-1+\gamma)t^{p_*-r-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.
\end{aligned}
$$

After using $a_0 \geq 0$, $p-r<0$, $q-r<0$, $p_*-r<0$, $p-1+\gamma>0$, $q-1+\gamma>0$, and $p_*-1+\gamma>0$ to find a first estimate from above and then splitting off common positive factors we get

$$
\begin{aligned}
&T_u'(t)\\
&\leq b_0\Phi^{\vartheta-1}(tu)t^{-r-1}\Big((p-r)(p-1+\gamma)t^p\|u\|_{1,p}^p\\
&\quad+(q-r)(q-1+\gamma)t^q\|\nabla u\|_{q,\mu}^q\\
&\quad+(\vartheta-1)\Phi^{-1}(tu)\rho(tu)\left((p-1+\gamma)t^p\|u\|_{1,p}^p+(q-1+\gamma)t^q\|\nabla u\|_{q,\mu}^q\right)\\
&\quad+(\vartheta-1)(\vartheta-2)\Phi^{-2}(tu)\rho^2(tu)\rho(tu)-(\vartheta-1)\Phi^{-1}(tu)\rho(tu)\rho(tu)r\\
&\quad+(\vartheta-1)\Phi^{-1}(tu)\rho(tu)2\left(pt^p\|u\|_{1,p}^p+qt^q\|\nabla u\|_{q,\mu}^q\right)\Big).
\end{aligned}
\tag{3.17}
$$

In the case $\vartheta=1$, by $u\neq 0$, $b_0>0$, $p-r<0$, $q-r<0$, $p-1+\gamma>0$ and $q-1+\gamma>0$ we immediately observe from (3.17) that

$$
T_u'(t)\leq b_0t^{-r-1}\left((p-r)(p-1+\gamma)t^p\|u\|_{1,p}^p+(q-r)(q-1+\gamma)t^q\|\nabla u\|_{q,\mu}^q\right)<0.
$$

So, let us consider the case $\vartheta>1$. We continue with splitting off common positive factors in (3.17) and derive that

$$
\begin{aligned}
T_u'(t)\leq b_0(\vartheta-1)\Phi^{\vartheta-1}(tu)\frac{\rho(tu)}{\Phi(tu)}t^{-r-1}\Bigg(&\frac{\Phi(tu)}{\rho(tu)}\frac{p-r}{\vartheta-1}(p-1+\gamma)t^p\|u\|_{1,p}^p\\
&+\frac{\Phi(tu)}{\rho(tu)}\frac{q-r}{\vartheta-1}(q-1+\gamma)t^q\|\nabla u\|_{q,\mu}^q\\
&+(p-1+\gamma)t^p\|u\|_{1,p}^p+(q-1+\gamma)t^q\|\nabla u\|_{q,\mu}^q\\
&+\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)\rho(tu)-r\rho(tu)+2\left(pt^p\|u\|_{1,p}^p+qt^q\|\nabla u\|_{q,\mu}^q\right)\Bigg).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\left(b_0(\vartheta-1)\Phi^{\vartheta-1}(tu)\frac{\rho(tu)}{\Phi(tu)}t^{-r-1}\right)^{-1}T_u'(t)\\
&\leq(p-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{p-r}{\vartheta-1}\right)t^p\|u\|_{1,p}^p\\
&\quad+(q-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{q-r}{\vartheta-1}\right)t^q\|\nabla u\|_{q,\mu}^q\\
&\quad+\left(\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2p\right)t^p\|u\|_{1,p}^p\\
&\quad+\left(\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2q\right)t^q\|\nabla u\|_{q,\mu}^q\\
&=A_1t^p\|u\|_{1,p}^p+A_2t^q\|\nabla u\|_{q,\mu}^q+B_1t^p\|u\|_{1,p}^p+B_2t^q\|\nabla u\|_{q,\mu}^q,
\end{aligned}
\tag{3.18}
$$

where A_1, A_2, B_1 and B_2 are given by

$$
\begin{aligned}
A_1&:=(p-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{p-r}{\vartheta-1}\right), &\quad B_1&:=\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2p,\\
A_2&:=(q-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{q-r}{\vartheta-1}\right), &\quad B_2&:=\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2q.
\end{aligned}
$$

By considering $p-r<0$, $p-q<0$, $\vartheta-1>0$, $\vartheta q-r<0$ and $q-r<0$ along with $\Phi(tu)>\frac{1}{q}\rho(tu)$ we find the estimates

$$\begin{aligned}
A_1 &= (p-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{p-r}{\vartheta-1}\right) < (p-1+\gamma)\left(1+\frac{p-r}{q(\vartheta-1)}\right)\\
&= (p-1+\gamma)\,\frac{\vartheta q-r+p-q}{q(\vartheta-1)} < 0,\\
A_2 &= (q-1+\gamma)\left(1+\frac{\Phi(tu)}{\rho(tu)}\frac{q-r}{\vartheta-1}\right) < (q-1+\gamma)\left(1+\frac{q-r}{q(\vartheta-1)}\right)\\
&= (q-1+\gamma)\,\frac{\vartheta q-r}{q(\vartheta-1)} < 0.
\end{aligned}$$

Then, by using $\vartheta-1>0$, $\vartheta q-r<0$, and $p-q<0$ accompanied by $p\Phi(tu)<\rho(tu)<q\Phi(tu)$, we furthermore get

$$\begin{aligned}
B_1 &= \frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2p = \frac{\rho(tu)}{\Phi(tu)}(\vartheta-1)-\frac{\rho(tu)}{\Phi(tu)}-r+2p\\
&< q(\vartheta-1)-p-r+2p = \vartheta q-r+p-q<0.
\end{aligned}$$

In order to show $B_2<0$ there are two distinct cases. First, let $\vartheta-2\geq 0$. Then, by using similar arguments as before we obtain

$$B_2=\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2q<q(\vartheta-2)-r+2q=\vartheta q-r<0.$$

On the other hand, if $\vartheta-2<0$, we recall that $r>\max\{2,\vartheta\}q=2q$ and conclude that

$$B_2=\frac{\rho(tu)}{\Phi(tu)}(\vartheta-2)-r+2q<-r+2q<0.$$

Now, applying $A_1<0$, $A_2<0$, $B_1<0$, and $B_2<0$ in (3.18) results in $T_u'(t)<0$. Hence, T_u is strictly decreasing. Then, because T_u is injective, we have that $t^u_{\max}$ is unique. We recall from the definition of T_u that

$$\sigma_u'(t)=t^{r-2+\gamma}\left(T_u(t)-(r-1+\gamma)\|u\|_r^r\right). \tag{3.19}$$

Then, from $T_u(t^u_{\max})=(r-1+\gamma)\|u\|_r^r$ and equation (3.19) we obtain $\sigma_u'(t^u_{\max})=0$. Moreover, since T_u is strictly decreasing, equation (3.19) yields

$$\sigma_u'(t)>0 \quad \text{for all } t\in(0,t^u_{\max}) \quad \text{and} \quad \sigma_u'(t)<0 \quad \text{for all } t\in(t^u_{\max},\infty). \tag{3.20}$$

Therefore, σ_u is strictly increasing on $(0,t^u_{\max})$ and strictly decreasing on $(t^u_{\max},\infty)$. We conclude that $\sigma_u(t^u_{\max})=\max_{t>0}\sigma_u(t)$.

Next we will show, for small $\lambda>0$, the existence of unique $0<t_1^u<t^u_{\max}<t_2^u$ with $t_1^u u\in\mathcal{N}_\lambda^+$ and $t_2^u u\in\mathcal{N}_\lambda^-$. Taking into account $a_0\geq 0$, $b_0>0$, $(p-1+\gamma)>0$, $(q-1+\gamma)>0$, $(p_*-1+\gamma)>0$, $(r-1+\gamma)>0$ and $\vartheta-1\geq 0$ as well as $\|u\|_r^r\leq|\Omega|^{1-\frac{r}{p^*}}\|u\|_{p^*}^r$ and $\|u\|_{p^*}\leq S^{-\frac{1}{p}}\|u\|_{1,p}$ we derive for all $t>0$ that

$$\begin{aligned}
\sigma_u'(t) &= \left(a_0+b_0\Phi^{\vartheta-1}(tu)\right)\left((p-1+\gamma)t^{p-2+\gamma}\|u\|_{1,p}^p+(q-1+\gamma)t^{q-2+\gamma}\|\nabla u\|_{q,\mu}^q\right)\\
&\quad+(\vartheta-1)b_0\Phi^{\vartheta-2}(tu)\left(t^{p-1+\gamma}\|u\|_{1,p}^p+t^{q-1+\gamma}\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad\times\left(t^{p-1}\|u\|_{1,p}^p+t^{q-1}\|\nabla u\|_{q,\mu}^q\right)\\
&\quad+(p_*-1+\gamma)t^{p_*-2+\gamma}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}-(r-1+\gamma)t^{r-2+\gamma}\|u\|_r^r\\
&\geq b_0\Phi^{\vartheta-1}(tu)(p-1+\gamma)t^{p-2+\gamma}\|u\|_{1,p}^p-(r-1+\gamma)t^{r-2+\gamma}\|u\|_r^r
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{b_0}{p^{\vartheta-1}} t^{p(\vartheta-1)}(p-1+\gamma)t^{p-2+\gamma}\|u\|_{1,p}^p\|u\|_{1,p}^{p(\vartheta-1)}\\
&\qquad -(r-1+\gamma)t^{r-2+\gamma}|\Omega|^{1-\frac{r}{p^*}}\|u\|_{p^*}^r\\
&\geq \frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)t^{\vartheta p-2+\gamma}\|u\|_{1,p}^{\vartheta p}-(r-1+\gamma)t^{r-2+\gamma}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^r.
\end{aligned}
$$

Thus, by requiring

$$
\frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)t^{\vartheta p-2+\gamma}\|u\|_{1,p}^{\vartheta p}-(r-1+\gamma)t^{r-2+\gamma}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^r\geq 0
$$

and solving this inequality for $t_0^u=t>0$, we observe that for

$$
t_0^u:=\frac{1}{\|u\|_{1,p}}\left(\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}}{p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}}\right)^{\frac{1}{r-\vartheta p}}>0,
$$

we have

$$
\begin{aligned}
\sigma_u'(t_0^u)\geq{}& \frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)\,(t_0^u)^{\vartheta p-2+\gamma}\,\|u\|_{1,p}^{\vartheta p}\\
&-(r-1+\gamma)\,(t_0^u)^{r-2+\gamma}\,|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^r\geq 0.
\end{aligned}
$$

Applying (3.20) yields $t_{\max}^u\geq t_0^u$, and since σ_u is increasing on $(0,t_{\max}^u)$ together with the same arguments as before we infer that

$$
\begin{aligned}
&\sigma_u\left(t_{\max}^u\right)\\
&\geq \sigma_u\left(t_0^u\right)\\
&=\left(a_0+b_0\Phi^{\vartheta-1}\left(t_0^u u\right)\right)\left(\left(t_0^u\right)^{p-1+\gamma}\|u\|_{1,p}^p+\left(t_0^u\right)^{q-1+\gamma}\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad+\left(t_0^u\right)^{p_*-1+\gamma}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}-\left(t_0^u\right)^{r-1+\gamma}\|u\|_r^r\\
&\geq b_0\Phi^{\vartheta-1}\left(t_0^u u\right)\left(t_0^u\right)^{p-1+\gamma}\|u\|_{1,p}^p-\left(t_0^u\right)^{r-1+\gamma}\|u\|_r^r\\
&\geq b_0\frac{1}{p^{\vartheta-1}}\left(t_0^u\right)^{p(\vartheta-1)}\|u\|_{1,p}^{p(\vartheta-1)}\left(t_0^u\right)^{p-1+\gamma}\|u\|_{1,p}^p\\
&\qquad-\left(t_0^u\right)^{r-1+\gamma}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^r\\
&=\frac{b_0}{p^{\vartheta-1}}\left(t_0^u\right)^{\vartheta p-1+\gamma}\|u\|_{1,p}^{\vartheta p}-\left(t_0^u\right)^{r-1+\gamma}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^r\\
&=\left(t_0^u\right)^{\vartheta p-1+\gamma}\|u\|_{1,p}^{\vartheta p}\left(\frac{b_0}{p^{\vartheta-1}}-\left(t_0^u\right)^{r-\vartheta p}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^{r-\vartheta p}\right)\\
&=\left(t_0^u\right)^{\vartheta p-1+\gamma}\|u\|_{1,p}^{\vartheta p}\left(\frac{b_0}{p^{\vartheta-1}}-\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}|\Omega|^{1-\frac{r}{p^*}}S^{-\frac{r}{p}}\|u\|_{1,p}^{r-\vartheta p}}{\|u\|_{1,p}^{r-\vartheta p}p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}}\right)\\
&=\left(t_0^u\right)^{\vartheta p-1+\gamma}\|u\|_{1,p}^{\vartheta p}\left(\frac{b_0}{p^{\vartheta-1}}-\frac{b_0\,(p-1+\gamma)}{p^{\vartheta-1}\,(r-1+\gamma)}\right)\\
&=\left(t_0^u\right)^{\vartheta p-1+\gamma}\|u\|_{1,p}^{\vartheta p}\frac{b_0(r-p)}{p^{\vartheta-1}(r-1+\gamma)}\\
&=\frac{1}{\|u\|_{1,p}^{\vartheta p-1+\gamma}}\left(\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}}{p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}}\right)^{\frac{\vartheta p-1+\gamma}{r-\vartheta p}}\|u\|_{1,p}^{\vartheta p}\frac{b_0(r-p)}{p^{\vartheta-1}(r-1+\gamma)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b_0(r-p)}{p^{\vartheta-1}(r-1+\gamma)} \left(\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}}{p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}} \right)^{\frac{\vartheta p-1+\gamma}{r-\vartheta p}} \|u\|_{1,p}^{1-\gamma} \\
&\geq \frac{S^{\frac{1-\gamma}{p}}}{|\Omega|^{1-\frac{1-\gamma}{p^*}}} \frac{b_0(r-p)}{p^{\vartheta-1}(r-1+\gamma)} \left(\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}}{p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}} \right)^{\frac{\vartheta p-1+\gamma}{r-\vartheta p}} \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&= \Lambda_2 \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x,
\end{aligned}
$$

where Λ_2 is a positive constant given by

$$
\Lambda_2 := \frac{S^{\frac{1-\gamma}{p}}}{|\Omega|^{1-\frac{1-\gamma}{p^*}}} \frac{b_0(r-p)}{p^{\vartheta-1}(r-1+\gamma)} \left(\frac{b_0\,(p-1+\gamma)\,S^{\frac{r}{p}}}{p^{\vartheta-1}\,(r-1+\gamma)\,|\Omega|^{1-\frac{r}{p^*}}} \right)^{\frac{\vartheta p-1+\gamma}{r-\vartheta p}}.
$$

Let $\lambda \in (0,\Lambda_2)$. From the above estimate we observe that

$$
\sigma_u(t^u_{\max}) \geq \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x.
$$

Note that a similar proof can be done as in (3.14) and (3.16) in order to show that

$$
\lim_{t\to\infty} \sigma_u(t) = -\infty \quad \text{and} \quad \lim_{t\searrow 0} \sigma_u(t) = 0.
$$

Thus, because of $\sigma_u(t^u_{\max}) = \max_{t>0}\sigma_u(t)$, the continuity of σ_u and the injectivity of σ_u in $(0,t^u_{\max})$ and $(t^u_{\max},\infty)$ we find unique

$$
0 < t^u_1 < t^u_{\max} < t^u_2 \tag{3.21}
$$

such that

$$
\sigma_u(t^u_1) = \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x = \sigma_u(t^u_2).
$$

Recalling (3.12) this implies $t^u_1 u,\, t^u_2 u \in \mathcal{N}_\lambda$. From (3.21) and (3.20) we infer that

$$
\sigma_u'(t^u_2) < 0 < \sigma_u'(t^u_1). \tag{3.22}
$$

Using the formula $\sigma_u(t) = t^\gamma \psi_u'(t) + \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x$ given in (3.11), we compute

$$
\sigma_u'(t) = \gamma t^{\gamma-1}\psi_u'(t) + t^\gamma \psi_u''(t) \quad \text{for all } t>0.
$$

This calculation along with (3.22) and $t^u_1 u,\, t^u_2 u \in \mathcal{N}_\lambda$ leads to

$$
\psi_u''(t^u_1) = (t^u_1)^{-\gamma}\sigma_u'(t^u_1) > 0, \quad \psi_u''(t^u_2) = (t^u_2)^{-\gamma}\sigma_u'(t^u_2) < 0.
$$

We conclude that $t^u_1 u \in \mathcal{N}_\lambda^+$ and $t^u_2 u \in \mathcal{N}_\lambda^-$. □

Remark 3.5. *Let $\lambda \in (0,\min\{\Lambda_1,\Lambda_2\})$ and $u \in W^{1,\mathcal{H}}(\Omega)\setminus\{0\}$. From $\lambda < \Lambda_2$ and Lemma 3.4 we obtain the existence of unique $0 < t^u_1 < t^u_2$ such that $t^u_1 u \in \mathcal{N}_\lambda^+$ and $t^u_2 u \in \mathcal{N}_\lambda^-$, which by Remark 3.1 means we have $\psi_u'(t^u_1) = 0 = \psi_u'(t^u_2)$ and $\psi_u''(t^u_1) > 0 > \psi_u''(t^u_2)$. Since $\lambda < \Lambda_1$, from Lemma 3.3 and Remark 3.1 we know that there is no $t>0$ such that $\psi_u'(t) = 0 = \psi_u''(t)$. Due to the uniqueness of t^u_1 and t^u_2, this implies that the map ψ_u' has besides those two numbers no other zeros. From elementary calculus, we derive the following properties:*

(i) *We have $\psi_u' < 0$ in $(0,t^u_1)$, i.e. the map ψ_u is strictly decreasing in $(0,t^u_1)$.*
(ii) *We have $\psi_u' > 0$ in (t^u_1,t^u_2), i.e. the map ψ_u is strictly increasing in (t^u_1,t^u_2).*
(iii) *We have $\psi_u' < 0$ in (t^u_2,∞), i.e. the map ψ_u is strictly decreasing in (t^u_2,∞).*

Furthermore, from the above properties and the fact that $\psi_u(0) = 0$*, we see that* $\psi_u(t) \leq 0$ *for all* $t \in [0, t_1^u]$ *as well as* $\max_{t\in[t_1^u,t_2^u]} \psi_u(t) = \psi_u(t_2^u)$*. Hence, we obtain the following property:*

(iv) *If* $\psi_u(t_2^u) > 0$*, then we have* $\max_{t\geq 0} \psi_u(t) = \psi_u(t_2^u)$*.*

In the next lemma we will prove that on $\mathcal{N}_\lambda^+$ the modulus ρ is bounded from above while on $\mathcal{N}_\lambda^-$ it is bounded from below. We will see that the lower bound on $\mathcal{N}_\lambda^-$ does not depend on the parameter λ, which will be useful in the proof of Lemma 4.6, where we will show that on $\mathcal{N}_\lambda^-$ the energy functional J_λ is strictly positive for small λ.

Lemma 3.6. *Suppose hypothesis* (H) *is satisfied. Then for all* $\lambda > 0$ *there are constants* $D_1 = D_1(\lambda) > 0$ *and* $D_2 > 0$ *such that*

$$\rho(u) < D_1 \quad \text{for all } u \in \mathcal{N}_\lambda^+ \quad \text{and} \quad \|u\|_{1,p}^p > D_2 \quad \text{for all } u \in \mathcal{N}_\lambda^-,$$

where D_2 *is independent of* λ*.*

Proof. Let $\lambda > 0$ and $u \in \mathcal{N}_\lambda^+$. Then we have $\psi_u'(1) = 0$ and $\psi_u''(1) > 0$, i.e.

$$\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x = \|u\|_r^r \tag{3.23}$$

and

$$\begin{aligned}(r-1)\|u\|_r^r &< \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1)\|u\|_{1,p}^p + (q-1)\|\nabla u\|_{q,\mu}^q\right)\\ &\quad + (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x.\end{aligned}$$

This implies

$$\begin{aligned}&(r-1)\left(\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\right)\\ &= (r-1)\|u\|_r^r\\ &< \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1)\|u\|_{1,p}^p + (q-1)\|\nabla u\|_{q,\mu}^q\right) + (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u)\\ &\qquad + (p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x.\end{aligned}$$

We bring $\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1)\|u\|_{1,p}^p + (q-1)\|\nabla u\|_{q,\mu}^q\right)$ and $(p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}$ to the left-hand side and put $-\lambda(r-1)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x$ to the right-hand side, apply $\Phi(u) > \frac{1}{q}\rho(u)$ together with $\vartheta - 1 \geq 0$ and obtain

$$\begin{aligned}&\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((r-p)\|u\|_{1,p}^p + (r-q)\|\nabla u\|_{q,\mu}^q\right) + (r-p_*)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\ &< \lambda(r-1+\gamma)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u)\\ &= \lambda(r-1+\gamma)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + (\vartheta-1)b_0\Phi^{\vartheta-1}(u)\Phi^{-1}(u)\rho^2(u)\\ &< \lambda(r-1+\gamma)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + (\vartheta-1)b_0\Phi^{\vartheta-1}(u)q\rho(u).\end{aligned}$$

Subtracting $(\vartheta-1)b_0\Phi^{\vartheta-1}(u)q\rho(u)$ from this inequality and using $\lambda(r-1+\gamma)>0$ together with $\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \leq |\Omega|^{1-\frac{1-\gamma}{p^*}}\|u\|_{p^*}^{1-\gamma}$ and $\|u\|_{p^*}\leq S^{-\frac{1}{p}}\|u\|_{1,p}$ yield

$$
\begin{aligned}
& a_0\Big((r-p)\|u\|_{1,p}^p+(r-q)\|\nabla u\|_{q,\mu}^q\Big)\\
&+b_0\Phi^{\vartheta-1}(u)\left((r-\vartheta q+q-p)\|u\|_{1,p}^p+(r-\vartheta q)\|\nabla u\|_{q,\mu}^q\right)\\
&+(r-p_*)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&<\lambda(r-1+\gamma)\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\\
&\leq \lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}.
\end{aligned}
\tag{3.24}
$$

First, we are concerned with finding an upper bound for $\|u\|_{1,p}^p$. Since $a_0\geq 0$, $b_0>0$, $r-p>0$, $r-q>0$, $r-p_*>0$, $r-\vartheta q>0$, $q-p>0$, $\vartheta\geq 1$ and $\lambda(r-1+\gamma)>0$ as well as $\Phi(u)>\frac{1}{p}\|u\|_{1,p}^p$, we conclude from (3.24) that

$$
\begin{aligned}
&\frac{b_0}{p^{\vartheta-1}}(r-\vartheta q+q-p)\|u\|_{1,p}^{\vartheta p}\\
&=b_0\left(\frac{\|u\|_{1,p}^p}{p}\right)^{\vartheta-1}(r-\vartheta q+q-p)\|u\|_{1,p}^p\\
&<b_0\Phi^{\vartheta-1}(u)(r-\vartheta q+q-p)\|u\|_{1,p}^p\\
&<a_0\left((r-p)\|u\|_{1,p}^p+(r-q)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad+b_0\Phi^{\vartheta-1}(u)(r-\vartheta q+q-p)\|u\|_{1,p}^p\\
&\qquad+b_0\Phi^{\vartheta-1}(u)(r-\vartheta q)\|\nabla u\|_{q,\mu}^q+(r-p_*)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&<\lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}.
\end{aligned}
$$

Solving this inequality for $\|u\|_{1,p}^p$ results in

$$\|u\|_{1,p}^p<A_1, \tag{3.25}$$

where A_1 is a positive constant given by

$$A_1:=\left(\frac{p^{\vartheta-1}\lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}}{b_0(r-\vartheta q+q-p)}\right)^{\frac{p}{\vartheta p-1+\gamma}}.$$

In order to find an upper bound for $\|\nabla u\|_{q,\mu}^q$, we use once more the estimate (3.24) along with (3.25), and apply similar arguments as before to obtain

$$
\begin{aligned}
&\frac{b_0}{q^{\vartheta-1}}(r-\vartheta q)\|\nabla u\|_{q,\mu}^{\vartheta q}\\
&=b_0\left(\frac{\|\nabla u\|_{q,\mu}^q}{q}\right)^{\vartheta-1}(r-\vartheta q)\|\nabla u\|_{q,\mu}^q\\
&<b_0\Phi(u)^{\vartheta-1}(r-\vartheta q)\|\nabla u\|_{q,\mu}^q\\
&<a_0\left((r-p)\|u\|_{1,p}^p+(r-q)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad+b_0\Phi^{\vartheta-1}(u)\left((r-\vartheta q+q-p)\|u\|_{1,p}^p+(r-\vartheta q)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad+(r-p_*)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&<\lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}}S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}
\end{aligned}
$$

$$< \lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}} S^{-\frac{1-\gamma}{p}} A_1^{\frac{1-\gamma}{p}}.$$

This yields

$$\|\nabla u\|_{q,\mu}^q < A_2, \tag{3.26}$$

where A_2 is a positive constant given by

$$A_2 := \left(\frac{q^{\vartheta-1}\lambda(r-1+\gamma)|\Omega|^{1-\frac{1-\gamma}{p^*}} S^{-\frac{1-\gamma}{p}} A_1^{\frac{1-\gamma}{p}}}{b_0(r-\vartheta q)} \right)^{\frac{1}{\vartheta}}.$$

So, with (3.25) and (3.26) we get that

$$\rho(u) = \|u\|_{1,p}^p + \|\nabla u\|_{q,\mu}^q < A_1 + A_2 := D_1.$$

In order to prove the second assertion let $u \in \mathcal{N}_\lambda^-$. Then we have $\psi_u'(1) = 0$ and $\psi_u''(1) < 0$, i.e. (3.23) still holds, but this time we have

$$\begin{aligned}
&\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1)\|u\|_{1,p}^p + (q-1)\|\nabla u\|_{q,\mu}^q\right)\\
&+ (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x \\
&< (r-1)\|u\|_r^r.
\end{aligned} \tag{3.27}$$

Multiplying (3.23) by γ, adding the result to (3.27) and applying $r-1+\gamma > 0$ together with $\|u\|_r^r \leq |\Omega|^{1-\frac{r}{p^*}}\|u\|_{p^*}^r$ and $\|u\|_{p^*} \leq S^{-\frac{1}{p}}\|u\|_{1,p}$ gives

$$\begin{aligned}
&\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q\right)\\
&+ (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&< (r-1+\gamma)\|u\|_r^r \leq (r-1+\gamma)|\Omega|^{1-\frac{r}{p^*}} S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}.
\end{aligned}$$

Therefore, with the same arguments as we used in the proof of (3.25), we get

$$\begin{aligned}
&\frac{b_0}{p^{\vartheta-1}}(p-1+\gamma)\|u\|_{1,p}^{\vartheta p}\\
&= b_0\left(\frac{\|u\|_{1,p}^p}{p}\right)^{\vartheta-1}(p-1+\gamma)\|u\|_{1,p}^p\\
&< b_0\Phi^{\vartheta-1}(u)(p-1+\gamma)\|u\|_{1,p}^p\\
&< \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&< (r-1+\gamma)|\Omega|^{1-\frac{r}{p^*}} S^{-\frac{1-\gamma}{p}}\|u\|_{1,p}^{1-\gamma}.
\end{aligned}$$

We solve this inequality for $\|u\|_{1,p}^p$ and end up with

$$\|u\|_{1,p}^p > D_2,$$

where D_2 is a positive constant, independent of λ, given by

$$D_2 := \left(\frac{b_0(p-1+\gamma)S^{\frac{1-\gamma}{p}}}{p^{\vartheta-1}(r-1+\gamma)|\Omega|^{1-\frac{r}{p^*}}}\right)^{\frac{p}{1-\gamma-\vartheta p}}.$$

This finishes the proof. $\square$

4. Existence of Weak Solutions

In this section, we are going to apply the results of Section 3 in order to prove Theorem 1.1, i.e. we will show that there are at least two weak solutions u_λ and v_λ of problem (K_λ) with opposite energy sign. In addition, we will see that

$$J_\lambda(u_\lambda) = \min_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \quad \text{and} \quad J_\lambda(v_\lambda) = \min_{v\in\mathcal{N}_\lambda^-} J_\lambda(v).$$

For simplification of the notation, we define

$$\Theta_\lambda^+ := \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \quad \text{and} \quad \Theta_\lambda^- := \inf_{v\in\mathcal{N}_\lambda^-} J_\lambda(v).$$

In the first lemma of this section, we will show that for all $\lambda > 0$ the energy functional J_λ is strictly negative on $\mathcal{N}_\lambda^+$.

Lemma 4.1. *Suppose hypothesis (H) is satisfied. Then for all $\lambda > 0$ and all $u \in \mathcal{N}_\lambda^+$ we have $J_\lambda(u) < 0$. In particular, if $\mathcal{N}_\lambda^+$ is nonempty for some $\lambda > 0$, then $\Theta_\lambda^+ < 0$.*

Proof. Let $\lambda > 0$ and $u \in \mathcal{N}_\lambda^+$. Then we have $\psi_u'(1) = 0$ and $\psi_u''(1) > 0$, i.e.

$$\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x = \|u\|_r^r \tag{4.1}$$

and

$$\begin{aligned}
&(r-1)\|u\|_r^r\\
&< \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1)\|u\|_{1,p}^p + (q-1)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad + (\vartheta-1)b_0\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x.
\end{aligned} \tag{4.2}$$

Multiplying (4.1) by γ and adding the result to (4.2) implies

$$\begin{aligned}
(r-1+\gamma)\|u\|_r^r &< \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q\right)\\
&\qquad + b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}.
\end{aligned}$$

Using this estimate and (4.1) together with $b_0 > 0$, $\vartheta - 1 \geq 0$, $p - 1 + \gamma > 0$, $p_* - 1 + \gamma > 0$, $1 - \gamma > 0$, $\frac{1}{r} - \frac{1}{p} < 0$, $\frac{1}{r} - \frac{1}{p_*} < 0$ and $\frac{1}{q}\rho(u) < \Phi(u)$ we infer that

$$\begin{aligned}
J_\lambda(u) &= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u\|_r^r\\
&= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\qquad + \frac{1}{1-\gamma}\left(\|u\|_r^r - \left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u) - \|u\|_{p_*,\beta,\partial\Omega}^{p_*}\right) - \frac{1}{r}\|u\|_r^r\\
&= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1-\gamma-p_*}{p_*(1-\gamma)}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{1-\gamma}\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u)\\
&\qquad + \frac{r-1+\gamma}{r(1-\gamma)}\|u\|_r^r\\
&< a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^\vartheta(u) + \frac{1-\gamma-p_*}{p_*(1-\gamma)}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{1-\gamma}\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\rho(u)\\
&\qquad + \frac{1}{r(1-\gamma)}\left(a_0 + b_0\Phi^{\vartheta-1}(u)\right)\left((p-1+\gamma)\|u\|_{1,p}^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q\right)
\end{aligned}$$

$$\begin{aligned}
&\quad + \frac{1}{r(1-\gamma)}\Big(b_0(\vartheta-1)\Phi^{\vartheta-2}(u)\rho^2(u) + (p_*-1+\gamma)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\Big)\\
&= a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^{\vartheta}(u) + b_0\frac{\vartheta-1}{r(1-\gamma)}\Phi^{\vartheta-1}(u)\Phi^{-1}(u)\rho^2(u)\\
&\quad + \big(a_0 + b_0\Phi^{\vartheta-1}(u)\big)\left(\frac{p-r+\gamma-1}{r(1-\gamma)}\|u\|_{1,p}^p + \frac{q-r+\gamma-1}{r(1-\gamma)}\|\nabla u\|_{q,\mu}^q\right)\\
&\quad + \frac{p_*-1+\gamma}{1-\gamma}\left(\frac{1}{r}-\frac{1}{p_*}\right)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\le a_0\Phi(u) + \frac{b_0}{\vartheta}\Phi^{\vartheta}(u) + b_0\frac{q(\vartheta-1)}{r(1-\gamma)}\Phi^{\vartheta-1}(u)\rho(u)\\
&\quad + \big(a_0 + b_0\Phi^{\vartheta-1}(u)\big)\left(\frac{p-r+\gamma-1}{r(1-\gamma)}\|u\|_{1,p}^p + \frac{q-r+\gamma-1}{r(1-\gamma)}\|\nabla u\|_{q,\mu}^q\right)\\
&= \Bigg[a_0\left(\frac{1}{p} + \frac{p-r+\gamma-1}{r(1-\gamma)}\right)\\
&\quad + b_0\Phi^{\vartheta-1}(u)\left(\frac{1}{\vartheta p} + \frac{p-r+\gamma-1}{r(1-\gamma)} + \frac{q(\vartheta-1)}{r(1-\gamma)}\right)\Bigg]\|u\|_{1,p}^p\\
&\quad + \Bigg[a_0\left(\frac{1}{q} + \frac{q-r+\gamma-1}{r(1-\gamma)}\right)\\
&\quad + b_0\Phi^{\vartheta-1}(u)\left(\frac{1}{\vartheta q} + \frac{q-r+\gamma-1}{r(1-\gamma)} + \frac{q(\vartheta-1)}{r(1-\gamma)}\right)\Bigg]\|\nabla u\|_{q,\mu}^q\\
&= \Big(a_0\tilde{A}_1 + b_0\Phi^{\vartheta-1}(u)\tilde{B}_1\Big)\|u\|_{1,p}^p + \Big(a_0\tilde{A}_2 + b_0\Phi^{\vartheta-1}(u)\tilde{B}_2\Big)\|\nabla u\|_{q,\mu}^q,
\end{aligned}$$

where $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$ and $\tilde{B}_2$ are defined as

$$\tilde{A}_1 := \frac{1}{p} + \frac{p-r+\gamma-1}{r(1-\gamma)}, \qquad \tilde{B}_1 := \frac{1}{\vartheta p} + \frac{p-r+\gamma-1}{r(1-\gamma)} + \frac{q(\vartheta-1)}{r(1-\gamma)},$$

$$\tilde{A}_2 := \frac{1}{q} + \frac{q-r+\gamma-1}{r(1-\gamma)}, \qquad \tilde{B}_2 := \frac{1}{\vartheta q} + \frac{q-r+\gamma-1}{r(1-\gamma)} + \frac{q(\vartheta-1)}{r(1-\gamma)}.$$

We take $r-p>0$, $1-\gamma-p<0$ and $1-\gamma>0$ into account and compute

$$\tilde{A}_1 = \frac{(r-p)(1-\gamma-p)}{pr(1-\gamma)} < 0 \quad \text{and} \quad \tilde{A}_2 = \frac{(r-q)(1-\gamma-q)}{qr(1-\gamma)} < 0.$$

In order to take care of $\tilde{B}_1$ and $\tilde{B}_2$, we recall that $r-\vartheta p>0$, $r-\vartheta q>0$, $1-\gamma>0$, $1-\gamma-\vartheta q<0$ and $1-\vartheta\le 0$ and calculate

$$\begin{aligned}
\tilde{B}_1 &= \frac{(r-\vartheta p)(1-\gamma-p) + p(r-\vartheta q)(1-\vartheta)}{\vartheta pr(1-\gamma)} < 0,\\
\tilde{B}_2 &= \frac{(r-\vartheta q)(1-\gamma-\vartheta q)}{\vartheta qr(1-\gamma)} < 0.
\end{aligned}$$

Since $u\neq 0$ we conclude that for all $u\in\mathcal{N}_\lambda^+$ we have

$$J_\lambda(u) < \Big(a_0\tilde{A}_1 + b_0\Phi^{\vartheta-1}(u)\tilde{B}_1\Big)\|u\|_{1,p}^p + \Big(a_0\tilde{A}_2 + b_0\Phi^{\vartheta-1}(u)\tilde{B}_2\Big)\|\nabla u\|_{q,\mu}^q < 0,$$

which concludes the proof. □

Next we will prove that for $\lambda>0$ small enough there exists a function $u_\lambda\in\mathcal{N}_\lambda^+$ such that $\Theta_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) = J_\lambda(u_\lambda)$.

Proposition 4.2. *Suppose hypothesis* (H) *is satisfied and* $\lambda \in (0, \min\{\Lambda_1, \Lambda_2\})$*, where* $\Lambda_1 > 0$ *is from Lemma* 3.3 *and* $\Lambda_2 > 0$ *is from Lemma* 3.4*. Then there exists* $u_\lambda \in \mathcal{N}_\lambda^+$ *such that* $J_\lambda(u_\lambda) = \Theta_\lambda^+$ *with* $u_\lambda \geq 0$ *a.e. in* Ω*. In particular, by Lemma* 4.1 *we have* $J_\lambda(u_\lambda) = \Theta_\lambda^+ < 0$*.*

Proof. Since $\lambda < \Lambda_2$ we know from Lemma 3.4 that $\mathcal{N}_\lambda^+$ is nonempty. Applying Lemma 4.1 yields $\Theta_\lambda^+ < 0$. Because $\mathcal{N}_\lambda^+$ is nonempty, we are allowed to choose a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}_\lambda^+$ which minimizes the energy functional J_λ, i.e. $\{J_\lambda(u_n)\}_{n\in\mathbb{N}}$ is decreasing and we have

$$\lim_{n\to\infty} J_\lambda(u_n) = \Theta_\lambda^+ < 0. \tag{4.3}$$

If $\{u_n\}_{n\in\mathbb{N}}$ was unbounded in $W^{1,\mathcal{H}}(\Omega)$, Lemma 3.2 would imply $\lim_{n\to\infty} J_\lambda(u_n) = \infty$ for a subsequence in contradiction to (4.3). Thus, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\Omega)$. By the reflexivity of $W^{1,\mathcal{H}}(\Omega)$ we obtain a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$\begin{aligned} u_n &\to u_\lambda \quad \text{in } L^r(\Omega), & u_n &\rightharpoonup u_\lambda \quad \text{in } L^{p_*}_\beta(\partial\Omega),\\ u_n &\rightharpoonup u_\lambda \quad \text{in } W^{1,p}(\Omega), & \nabla u_n &\rightharpoonup \nabla u_\lambda \quad \text{in } L^q_\mu(\Omega) \end{aligned}$$

as $n \to \infty$. We can also assume that

$$u_n \to u_\lambda \quad \text{a.e. in } \Omega \quad \text{and} \quad |u_n| \leq f \quad \text{for all } n \in \mathbb{N} \text{ and a.e. in } \Omega$$

as $n \to \infty$ and for some $f \in L^1(\Omega)$. Applying Lebesgue's dominated convergence theorem to the latter property gives

$$\lim_{n\to\infty} \int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x = \int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x. \tag{4.4}$$

Next, we will show that $u_\lambda \neq 0$. From the weak lower semicontinuity of the corresponding norms and seminorms, the fact that $u_n \to u_\lambda$ in $L^r(\Omega)$ as $n \to \infty$, and (4.4) we derive that

$$\begin{aligned}
&\liminf_{n\to\infty} J_\lambda(u_n)\\
&= \liminf_{n\to\infty} \Bigg(a_0\Phi(u_n) + \frac{b_0}{\vartheta}\Phi^\vartheta(u_n) + \frac{1}{p_*}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\qquad\qquad - \frac{\lambda}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|u_n\|_r^r \Bigg)\\
&\geq \liminf_{n\to\infty} \left(a_0\Phi(u_n)\right) + \liminf_{n\to\infty} \left(\frac{b_0}{\vartheta}\Phi^\vartheta(u_n)\right) + \liminf_{n\to\infty} \left(\frac{1}{p_*}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\
&\quad + \liminf_{n\to\infty} \left(-\frac{\lambda}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\right) + \liminf_{n\to\infty} \left(-\frac{1}{r}\|u_n\|_r^r\right)\\
&\geq a_0 \left(\frac{1}{p}\liminf_{n\to\infty} \|u_n\|_{1,p}^p + \frac{1}{q}\liminf_{n\to\infty} \|\nabla u_n\|_{q,\mu}^q\right)\\
&\quad + \frac{b_0}{\vartheta}\left(\frac{1}{p}\liminf_{n\to\infty} \|u_n\|_{1,p}^p + \frac{1}{q}\liminf_{n\to\infty} \|\nabla u_n\|_{q,\mu}^q\right)^\vartheta\\
&\quad + \frac{1}{p_*}\liminf_{n\to\infty} \|u_n\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda}{1-\gamma}\lim_{n\to\infty}\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\lim_{n\to\infty}\|u_n\|_r^r
\end{aligned}$$

$$\begin{aligned}
&\geq a_0\left(\frac{1}{p}\|u_\lambda\|_{1,p}^p+\frac{1}{q}\|\nabla u_\lambda\|_{q,\mu}^q\right)+\frac{b_0}{\vartheta}\left(\frac{1}{p}\|u_\lambda\|_{1,p}^p+\frac{1}{q}\|\nabla u_\lambda\|_{q,\mu}^q\right)^\vartheta\\
&\qquad+\frac{1}{p_*}\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}-\frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x-\frac{1}{r}\|u_\lambda\|_r^r\\
&=J_\lambda(u_\lambda).
\end{aligned}$$

Therefore

$$J_\lambda(u_\lambda)\leq \liminf_{n\to\infty} J_\lambda(u_n)=\lim_{n\to\infty} J_\lambda(u_n)=\Theta_\lambda^+<0=J_\lambda(0).$$

Thus, we have shown that $u_\lambda\neq 0$. By taking further into account that $\lambda<\Lambda_2$, we obtain by Lemma 3.4 the existence of $t_1^{u_\lambda}>0$ such that $t_1^{u_\lambda}u_\lambda\in\mathcal{N}_\lambda^+$.

The next step is to show that we have $\lim_{n\to\infty} J_\lambda(u_n)=J_\lambda(u_\lambda)$ for a subsequence. For this purpose we will prove that

$$\begin{aligned}
\liminf_{n\to\infty}\|u_n\|_{1,p}^p&=\|u_\lambda\|_{1,p}^p,\\
\liminf_{n\to\infty}\|\nabla u_n\|_{q,\mu}^q&=\|\nabla u_\lambda\|_{q,\mu}^q,\\
\liminf_{n\to\infty}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}&=\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}.
\end{aligned}\tag{4.5}$$

Using the weak lower semicontinuity of the corresponding norms and seminorms, we assume by contradiction that one of the following statements is true:

$$\begin{aligned}
&\liminf_{n\to\infty}\|u_n\|_{1,p}^p>\|u_\lambda\|_{1,p}^p\\
\text{or}\quad&\liminf_{n\to\infty}\|\nabla u_n\|_{q,\mu}^q>\|\nabla u_\lambda\|_{q,\mu}^q\\
\text{or}\quad&\liminf_{n\to\infty}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}>\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}.
\end{aligned}\tag{4.6}$$

It is clear that

$$\liminf_{n\to\infty}\Phi^{\vartheta-1}\left(t_1^{u_\lambda}u_n\right)\geq\left(\frac{1}{p}\left(t_1^{u_\lambda}\right)^p\liminf_{n\to\infty}\|u_n\|_{1,p}^p+\frac{1}{q}\left(t_1^{u_\lambda}\right)^q\liminf_{n\to\infty}\|\nabla u_n\|_{q,\mu}^q\right)^{\vartheta-1}$$

and

$$\liminf_{n\to\infty}\rho\left(t_1^{u_\lambda}u_n\right)\geq\left(t_1^{u_\lambda}\right)^p\liminf_{n\to\infty}\|u_n\|_{1,p}^p+\left(t_1^{u_\lambda}\right)^q\liminf_{n\to\infty}\|\nabla u_n\|_{q,\mu}^q.$$

These two estimates together with (4.4), (4.6), the weak lower semicontinuity of the corresponding norms and seminorms, and the fact that $u_n\to u_\lambda$ in $L^r(\Omega)$ as $n\to\infty$ yield

$$\begin{aligned}
&\liminf_{n\to\infty}\psi_{u_n}'\left(t_1^{u_\lambda}\right)\\
&=\liminf_{n\to\infty}\Big(\left(a_0+b_0\Phi^{\vartheta-1}\left(t_1^{u_\lambda}u_n\right)\right)\left(t_1^{u_\lambda}\right)^{-1}\rho\left(t_1^{u_\lambda}u_n\right)+\left(t_1^{u_\lambda}\right)^{p_*-1}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\qquad-\lambda\left(t_1^{u_\lambda}\right)^{-\gamma}\int_\Omega|u_n|^{1-\gamma}\,\mathrm{d}x-\left(t_1^{u_\lambda}\right)^{r-1}\|u_n\|_r^r\Big)\\
&\geq\liminf_{n\to\infty}\Big(\left(a_0+b_0\Phi^{\vartheta-1}\left(t_1^{u_\lambda}u_n\right)\right)\left(t_1^{u_\lambda}\right)^{-1}\rho\left(t_1^{u_\lambda}u_n\right)\Big)\\
&\qquad+\liminf_{n\to\infty}\left(t_1^{u_\lambda}\right)^{p_*-1}\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*}-\lambda\left(t_1^{u_\lambda}\right)^{-\gamma}\lim_{n\to\infty}\int_\Omega|u_n|^{1-\gamma}\,\mathrm{d}x\\
&\qquad-\left(t_1^{u_\lambda}\right)^{r-1}\lim_{n\to\infty}\|u_n\|_r^r\\
&>\left(a_0+b_0\Phi^{\vartheta-1}\left(t_1^{u_\lambda}u_\lambda\right)\right)\left(t_1^{u_\lambda}\right)^{-1}\rho\left(t_1^{u_\lambda}u_\lambda\right)+\left(t_1^{u_\lambda}\right)^{p_*-1}\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \left(t_1^{u_\lambda}\right)^{-\gamma} \int_\Omega |u_\lambda|^{1-\gamma} \,\mathrm{d}x - \left(t_1^{u_\lambda}\right)^{r-1} \|u_\lambda\|_r^r \\
&= \psi'_{u_\lambda}\left(t_1^{u_\lambda}\right) = \left(t_1^{u_\lambda}\right)^{-1} \psi'_{t_1^{u_\lambda} u_\lambda}(1) = 0.
\end{aligned}
$$

Thus, there exists $n \in \mathbb{N}$ with the property that $\psi'_{u_n}(t_1^{u_\lambda}) > 0$. Since $\lambda < \Lambda_2$ and $u_n \neq 0$, there exists a unique $t_1^{u_n} > 0$ with the property that $t_1^{u_n} u_n \in \mathcal{N}_\lambda^+$. Note that $u_n \in \mathcal{N}_\lambda^+$ implies $t_1^{u_n} = 1$. Then, from $\lambda < \min\{\Lambda_1, \Lambda_2\}$ and Remark 3.5 we know $\psi'_{u_n}(t) \leq 0$ for all $t \in (0, t_1^{u_n}] = (0,1]$. This implies $t_1^{u_\lambda} > 1$. By Remark 3.5 and $u_\lambda \neq 0$ we infer that ψ_{u_λ} is decreasing on $[0, t_1^{u_\lambda}] \supseteq [1, t_1^{u_\lambda}]$. Hence, we obtain the contradiction

$$
\Theta_\lambda^+ \leq J_\lambda\left(t_1^{u_\lambda} u_\lambda\right) = \psi_{u_\lambda}\left(t_1^{u_\lambda}\right) \leq \psi_{u_\lambda}(1) = J_\lambda(u_\lambda) < \liminf_{n\to\infty} J_\lambda(u_n) = \Theta_\lambda^+.
$$

In consequence, (4.5) holds for a subsequence, which we still denote by $\{u_n\}_{n\in\mathbb{N}}$. Taking further into consideration that $u_n \to u_\lambda$ in $L^r(\Omega)$ as $n \to \infty$ along with (4.3) and (4.4), we conclude that

$$
\begin{aligned}
\Theta_\lambda^+ &= \lim_{n\to\infty} J_\lambda(u_n) \\
&= \lim_{n\to\infty} \Bigg(a_0 \Phi(u_n) + \frac{b_0}{\vartheta} \Phi^\vartheta(u_n) + \frac{1}{p_*} \|u_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\qquad\qquad - \frac{\lambda}{1-\gamma} \int_\Omega |u_n|^{1-\gamma} \,\mathrm{d}x - \frac{1}{r} \|u_n\|_r^r \Bigg) \\
&= J_\lambda(u_\lambda).
\end{aligned}
$$

In the next step, we will prove $u_\lambda \in \mathcal{N}_\lambda^+$. We recall that for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\psi'_{u_n}(1) &= \left(a_0 + b_0 \Phi^{\vartheta-1}(u_n)\right) \left(\|u_n\|_{1,p}^p + \|\nabla u_n\|_{q,\mu}^q\right) + \|u_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\quad - \lambda \int_\Omega |u_n|^{1-\gamma} \,\mathrm{d}x - \|u_n\|_r^r
\end{aligned}
$$

and

$$
\begin{aligned}
\psi''_{u_n}(1) &= \left(a_0 + b_0 \Phi^{\vartheta-1}(u_n)\right) \left((p-1)\|u_n\|_{1,p}^p + (q-1)\|\nabla u_n\|_{q,\mu}^q\right) \\
&\quad + b_0(\vartheta-1)\Phi^{\vartheta-2}(u_n) \left(\|u_n\|_{1,p}^p + \|\nabla u_n\|_{q,\mu}^q\right)^2 + (p_*-1)\|u_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\quad + \lambda\gamma \int_\Omega |u_n|^{1-\gamma} \,\mathrm{d}x - (r-1)\|u_n\|_r^r.
\end{aligned}
$$

Since $u_n \in \mathcal{N}_\lambda^+$ we have $\psi'_{u_n}(1) = 0$ and $\psi''_{u_n}(1) > 0$ for all $n \in \mathbb{N}$. Thus, by taking the limit as $n \to \infty$ in the above representations of ψ'_{u_n} and ψ''_{u_n}, we infer from the same arguments as we used in the proof of $\lim_{n\to\infty} J_\lambda(u_n) = J_\lambda(u_\lambda)$ that $\psi'_{u_\lambda}(1) = 0$ and $\psi''_{u_\lambda}(1) \geq 0$. Taking into account $\lambda < \Lambda_1$ and Lemma 3.3 we get $\mathcal{N}_\lambda^\circ = \emptyset$ and therefore $\psi''_{u_\lambda}(1) > 0$. Hence, we obtain $u_\lambda \in \mathcal{N}_\lambda^+$.

It remains to show that $u_\lambda \geq 0$ a.e. in Ω. We know that $|u_\lambda| \in W^{1,\mathcal{H}}(\Omega)$ and since

$$
J_\lambda(|u_\lambda|) = J_\lambda(u_\lambda) = \Theta_\lambda^+, \quad \psi'_{|u_\lambda|}(1) = \psi'_{u_\lambda}(1) = 0, \quad \text{and} \quad \psi''_{|u_\lambda|}(1) = \psi''_{u_\lambda}(1) > 0,
$$

we get $|u_\lambda| \in \mathcal{N}_\lambda^+$. Hence, we can assume $u_\lambda \geq 0$ a.e. in Ω. ☐

The next two results are needed later in order to prove Theorem 1.1.

Lemma 4.3. *Suppose hypothesis* (H) *is satisfied. Let $\lambda > 0$ and $u \in \mathcal{N}_\lambda^+$. Then there are $\varepsilon > 0$ and a continuous function $\zeta\colon B_\varepsilon(0) \to (0,\infty)$ with the property that*

$$\zeta(0) = 1 \quad \text{and} \quad \zeta(v)(u+v) \in \mathcal{N}_\lambda^+ \quad \text{for all } v \in B_\varepsilon(0),$$

where $B_\varepsilon(0) \subset W^{1,\mathcal{H}}(\Omega)$ denotes the open ball at 0 with radius ε. The same assertion holds true if we replace $\mathcal{N}_\lambda^+$ by $\mathcal{N}_\lambda^-$.

Proof. We only show the proof for the case $u \in \mathcal{N}_\lambda^+$. The other case works in a similar way. We consider the map

$$F\colon W^{1,\mathcal{H}}(\Omega) \times (0,\infty) \to \mathbb{R}, \quad (v,t) \mapsto t^\gamma \psi_{u+v}'(t).$$

Then F is continuous and has a continuous partial derivative with respect to the second variable which is given by

$$\frac{\partial F}{\partial t}(v,t) = \gamma t^{\gamma-1}\psi_{u+v}'(t) + t^\gamma \psi_{u+v}''(t) \quad \text{for all } (v,t) \in W^{1,\mathcal{H}}(\Omega) \times (0,\infty). \tag{4.7}$$

Because of $u \in \mathcal{N}_\lambda^+$ it holds

$$F(0,1) = \psi_u'(1) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}(0,1) = \psi_u''(1) > 0.$$

By the aforementioned considerations and the implicit function theorem (see, for example, Fusco–Marcellini–Sbordone [27, p. 569]) we get the existence of $\varepsilon > 0$ and a continuous function $\zeta\colon B_\varepsilon(0) \to (0,\infty)$ such that for all $v \in B_\varepsilon(0)$ and $t \in \zeta(B_\varepsilon(0))$ we have

$$F(v,t) = 0 \quad \text{if and only if} \quad t = \zeta(v).$$

This implies

$$\zeta(0) = 1 \quad \text{and} \quad \psi_{u+v}'(\zeta(v)) = F(v,\zeta(v)) = 0 \quad \text{for all } v \in B_\varepsilon(0). \tag{4.8}$$

Recalling the definition of F, we obtain $\zeta(v)(u+v) \in \mathcal{N}_\lambda$ for all $v \in B_\varepsilon(0)$. Moreover, combining (4.7) with (4.8) gives

$$\frac{\partial F}{\partial t}(v,\zeta(v)) = \zeta^\gamma(v)\psi_{u+v}''(\zeta(v)) \quad \text{for all } v \in B_\varepsilon(0).$$

Hence, by taking into account $\partial_t F(0,1) > 0$ along with $\zeta(0) = 1$ and applying the continuity of $\partial_t F$, we can choose ε small enough to establish that $\zeta^\gamma(v)\psi_{u+v}''(\zeta(v)) > 0$ and therefore

$$\zeta(v)(u+v) \in \mathcal{N}_\lambda^+ \quad \text{for all } v \in B_\varepsilon(0),$$

which concludes the proof. □

Proposition 4.4. *Suppose hypothesis* (H) *is satisfied and let $\lambda \in (0, \min\{\Lambda_1, \Lambda_2\})$, where $\Lambda_1 > 0$ is from Lemma 3.3 and $\Lambda_2 > 0$ from Lemma 3.4. Then, for all $h \in W^{1,\mathcal{H}}(\Omega)$, there exists $\delta_h > 0$ such that*

$$J_\lambda(u_\lambda) \le J_\lambda(u_\lambda + th) \quad \text{for all } t \in [0,\delta_h),$$

where $u_\lambda \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ is from Proposition 4.2.

Proof. Let $h \in W^{1,\mathcal{H}}(\Omega)$. Because of $u_\lambda \in \mathcal{N}_\lambda^+$ we have $\psi_{u_\lambda}''(1) > 0$. First we show the existence of $\delta_h > 0$ and $\eta_h \in (0,1)$ such that for all $t \in [0,\delta_h)$, the map $\psi_{u_\lambda+th}$ is strictly convex in $(1-\eta_h, 1+\eta_h)$, i.e., $\psi_{u_\lambda+th}''(s) > 0$ for all $t \in [0,\delta_h)$ and $s \in (1-\eta_h, 1+\eta_h)$. Since the map $(s,t) \mapsto \psi_{u_\lambda+th}''(s)$ is continuous in $(1,0)$, for $\frac{1}{2}\psi''_{u_\lambda}(1) > 0$, there exists $\delta_0 > 0$ with the property that for all $t \geq 0$ and $s > 0$ we have

$$|\psi''_{u_\lambda+th}(s) - \psi''_{u_\lambda}(1)| < \frac{1}{2}\psi''_{u_\lambda}(1) \tag{4.9}$$

whenever $|(s,t)-(1,0)| < \delta_0$. We define

$$\delta_h := \frac{\delta_0}{2} > 0 \quad \text{and} \quad \eta_h := \min\left\{\sqrt{\delta_0^2 - \delta_h^2}, \frac{1}{2}\right\} \in (0,1). \tag{4.10}$$

Then, for all $t \in [0, \delta_h)$ and $s \in (1-\eta_h, 1+\eta_h)$ we find

$$|(s,t)-(1,0)| = \sqrt{(s-1)^2+t^2} < \sqrt{\eta_h^2 + \delta_h^2} \leq \delta_0.$$

Applying (4.9) yields

$$\psi''_{u_\lambda+th}(s) > \psi''_{u_\lambda}(1) - \frac{1}{2}\psi''_{u_\lambda}(1) = \frac{1}{2}\psi''_{u_\lambda}(1) > 0.$$

By Lemma 4.3 we obtain the existence of $\varepsilon > 0$, independent of h, and a continuous map $\zeta \colon B_\varepsilon(0) \to (0,\infty)$ such that

$$\zeta(0\cdot h) = 1 \quad \text{and} \quad \zeta(th)(u_\lambda + th) \in \mathcal{N}_\lambda^+ \quad \text{for all } t \in [0,\delta_h),$$

where we choose δ_h small enough in order to ensure $th \in B_\varepsilon(0)$ for all $t \in [0,\delta_h)$. The property $\zeta(th)(u_\lambda+th) \in \mathcal{N}_\lambda^+$ gives $\psi'_{u_\lambda+th}(\zeta(th)) = 0$ and $\psi''_{u_\lambda+th}(\zeta(th)) > 0$, i.e. $\psi_{u_\lambda+th}$ attains in $\zeta(th)$ a local minimum for all $t \in [0,\delta_h)$. Note that the definition of η_h given in (4.10) implies that η_h increases if we choose δ_h to be smaller. Thus, as a consequence of the continuity of $t \mapsto \zeta(th)$ along with $\zeta(0\cdot h) = 1$, we can choose δ_h even smaller to ensure that $\zeta(th) \in (1-\eta_h, 1+\eta_h)$ for all $t \in [0,\delta_h)$. Putting all the arguments mentioned above together, we know that for all $t \in [0,\delta_h)$ the map $\psi_{u_\lambda+th}$ is strictly convex and attains in $(1-\eta_h, 1+\eta_h)$ a global minimum in $\zeta(th) \in (1-\eta_h, 1+\eta_h)$. Hence, by remembering $J(u_\lambda) = \Theta_\lambda^+$ from Proposition 4.2, we conclude for all $t \in [0,\delta_h)$ that

$$J_\lambda(u_\lambda) = \Theta_\lambda^+ \leq J_\lambda(\zeta(th)(u_\lambda+th)) = \psi_{u_\lambda+th}(\zeta(th)) \leq \psi_{u_\lambda+th}(1) = J_\lambda(u_\lambda+th)$$

which completes the proof. ☐

Now, we are able to prove that u_λ is a weak solution of problem (K_λ).

Proposition 4.5. *Suppose hypothesis* (H) *is satisfied and let* $\lambda \in (0, \min\{\Lambda_1, \Lambda_2\})$, *where* $\Lambda_1 > 0$ *is from Lemma 3.3 and* $\Lambda_2 > 0$ *from Lemma 3.4. Then,* u_λ *from Proposition 4.2 is a weak solution of problem* (K_λ).

Proof. Since we already know from Proposition 4.2 that $u_\lambda \geq 0$ a.e. in Ω, it is sufficient to prove $u_\lambda \neq 0$ a.e. in Ω. We argue by contradiction and assume there is a measurable set $K \subseteq \Omega$ with $|K| > 0$ and $u_\lambda = 0$ in K. Let $h \in W^{1,\mathcal{H}}(\Omega)$ with $h > 0$. Because of $\gamma < 1$, for all $t > 0$, we infer that $(u_\lambda + th)^{1-\gamma} > u_\lambda^{1-\gamma}$ a.e. in $\Omega \setminus K$ and therefore

$$\int_{\Omega\setminus K} \left((u_\lambda+th)^{1-\gamma} - u_\lambda^{1-\gamma}\right) \,\mathrm{d}x \geq 0. \tag{4.11}$$

By Proposition 4.4, there exists $\delta_h > 0$ such that for all $t \in [0,\delta_h)$, we have the estimate $J_\lambda(u_\lambda) \leq J_\lambda(u_\lambda+th)$. Using this estimate at the same time as $u_\lambda = 0$ in K and (4.11), for all $t \in [0,\delta_h)$, we obtain

$$0 \leq J_\lambda(u_\lambda+th) - J_\lambda(u_\lambda)$$

$$
\begin{aligned}
&= M\left(\Phi(u_\lambda+th)\right)-M\left(\Phi(u_\lambda)\right)-\frac{\lambda}{1-\gamma}\int_\Omega\left((u_\lambda+th)^{1-\gamma}-u_\lambda^{1-\gamma}\right)\mathrm{d}x\\
&\quad+\frac{1}{p_*}\left(\|u_\lambda+th\|_{p_*,\beta,\partial\Omega}^{p_*}-\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}\right)-\frac{1}{r}\left(\|u_\lambda+th\|_r^r-\|u_\lambda\|_r^r\right)\\
&= M\left(\Phi(u_\lambda+th)\right)-M\left(\Phi(u_\lambda)\right)\\
&\quad-\frac{\lambda}{1-\gamma}\int_{\Omega\setminus K}\left((u_\lambda+th)^{1-\gamma}-u_\lambda^{1-\gamma}\right)\mathrm{d}x\\
&\quad-\frac{\lambda}{1-\gamma}\int_K\left((u_\lambda+th)^{1-\gamma}-u_\lambda^{1-\gamma}\right)\mathrm{d}x\\
&\quad+\frac{1}{p_*}\left(\|u_\lambda+th\|_{p_*,\beta,\partial\Omega}^{p_*}-\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}\right)-\frac{1}{r}\left(\|u_\lambda+th\|_r^r-\|u_\lambda\|_r^r\right)\\
&\le M\left(\Phi(u_\lambda+th)\right)-M\left(\Phi(u_\lambda)\right)-\frac{\lambda t^{1-\gamma}}{1-\gamma}\int_K h^{1-\gamma}\,\mathrm{d}x\\
&\quad+\frac{1}{p_*}\left(\|u_\lambda+th\|_{p_*,\beta,\partial\Omega}^{p_*}-\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}\right)-\frac{1}{r}\left(\|u_\lambda+th\|_r^r-\|u_\lambda\|_r^r\right).
\end{aligned}
$$

Dividing this estimate by $t>0$, taking the limit as $t\searrow 0$ and applying Lebesgue's dominated convergence theorem along with $1-\gamma>0$, $\lim_{t\searrow 0}t^{-\gamma}=\infty$ and $\int_K h^{1-\gamma}\,\mathrm{d}x>0$, we derive the contradiction

$$
0\le\lim_{t\searrow 0}\frac{J_\lambda(u_\lambda+th)-J_\lambda(u_\lambda)}{t}=-\infty.
$$

Hence, $u_\lambda>0$ a.e. in Ω.

Next, for all $h\in W^{1,\mathcal{H}}(\Omega)$ with $h\ge 0$ in Ω, we will show $u_\lambda^{-\gamma}h\in L^1(\Omega)$ and

$$
\begin{aligned}
&m(\Phi(u_\lambda))\left(\int_\Omega\mathcal{L}(u_\lambda)\cdot\nabla h\,\mathrm{d}x+\int_\Omega\alpha(x)u_\lambda^{p-1}h\,\mathrm{d}x\right)+\int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}h\,\mathrm{d}\sigma\\
&\ge\lambda\int_\Omega u_\lambda^{-\gamma}h\,\mathrm{d}x+\int_\Omega u_\lambda^{r-1}h\,\mathrm{d}x.
\end{aligned}\tag{4.12}
$$

Let $h\in W^{1,\mathcal{H}}(\Omega)$ with $h\ge 0$. We consider a decreasing sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}t_n=0$ and define a sequence of measurable functions $\{\zeta_n\}_{n\in\mathbb{N}}$ by

$$
\zeta_n\colon\Omega\to\mathbb{R},\quad\zeta_n(x):=\frac{(u_\lambda(x)+t_nh(x))^{1-\gamma}-u_\lambda(x)^{1-\gamma}}{t_n}.
$$

Since $h\ge 0$, it is obvious that ζ_n is nonnegative for all $n\in\mathbb{N}$. Furthermore, for a.a. $x\in\Omega$ we observe that

$$
\lim_{n\to\infty}\zeta_n(x)=(1-\gamma)u_\lambda(x)^{-\gamma}h(x).\tag{4.13}
$$

Using Proposition 4.4 once again, for $n\in\mathbb{N}$ large enough , we get

$$
\begin{aligned}
0&\le\frac{J_\lambda(u_\lambda+t_nh)-J_\lambda(u_\lambda)}{t_n}\\
&=\frac{M\left(\Phi(u_\lambda+t_nh)\right)-M\left(\Phi(u_\lambda)\right)}{t_n}-\frac{\lambda}{1-\gamma}\int_\Omega\zeta_n\,\mathrm{d}x\\
&\quad+\frac{1}{p_*}\frac{\|u_\lambda+t_nh\|_{p_*,\beta,\partial\Omega}^{p_*}-\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t_n}-\frac{1}{r}\frac{\|u_\lambda+t_nh\|_r^r-\|u_\lambda\|_r^r}{t_n}.
\end{aligned}
$$

Taking the limit superior of this estimate as $n \to \infty$ and recalling Lebesgue's dominated convergence theorem results in

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \Bigg(\frac{M\left(\Phi(u_\lambda + t_n h)\right) - M\left(\Phi(u_\lambda)\right)}{t_n} - \frac{\lambda}{1-\gamma}\int_\Omega \zeta_n \,\mathrm{d}x \\
&\quad + \frac{1}{p_*}\frac{\|u_\lambda + t_n h\|^{p_*}_{p_*,\beta,\partial\Omega} - \|u_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega}}{t_n} - \frac{1}{r}\frac{\|u_\lambda + t_n h\|_r^r - \|u_\lambda\|_r^r}{t_n}\Bigg) \\
&\leq \lim_{n\to\infty} \frac{M\left(\Phi(u_\lambda + t_n h)\right) - M\left(\Phi(u_\lambda)\right)}{t_n} + \limsup_{n\to\infty}\left(-\frac{\lambda}{1-\gamma}\int_\Omega \zeta_n\,\mathrm{d}x\right) \\
&\quad + \lim_{n\to\infty}\frac{1}{p_*}\frac{\|u_\lambda + t_n h\|^{p_*}_{p_*,\beta,\partial\Omega} - \|u_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega}}{t_n} - \lim_{n\to\infty}\frac{1}{r}\frac{\|u_\lambda + t_n h\|_r^r - \|u_\lambda\|_r^r}{t_n} \\
&= m(\Phi(u_\lambda))\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla h\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}h\,\mathrm{d}x\right) - \frac{\lambda}{1-\gamma}\liminf_{n\to\infty}\int_\Omega \zeta_n\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}h\,\mathrm{d}\sigma - \int_\Omega u_\lambda^{r-1}h\,\mathrm{d}x.
\end{aligned}$$

By rearranging this inequality and applying Fatou's Lemma coupled with (4.13) we infer that

$$\begin{aligned}
\lambda\int_\Omega u_\lambda^{-\gamma}h\,\mathrm{d}x &\leq \frac{\lambda}{1-\gamma}\liminf_{n\to\infty}\int_\Omega \zeta_n\,\mathrm{d}x \\
&\leq m(\Phi(u_\lambda))\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla h\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}h\,\mathrm{d}x\right) \\
&\quad + \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}h\,\mathrm{d}\sigma - \int_\Omega u_\lambda^{r-1}h\,\mathrm{d}x.
\end{aligned}$$

Hence, we conclude that $u_\lambda^{-\gamma}h \in L^1(\Omega)$ and (4.12) holds. For arbitrary $\varphi \in W^{1,\mathcal{H}}(\Omega)$ we consider $\varphi = \varphi^+ - \varphi^-$ and obtain $u_\lambda^{-\gamma}\varphi \in L^1(\Omega)$.

Finally, for all $\varphi \in W^{1,\mathcal{H}}(\Omega)$ we will verify

$$\begin{aligned}
m(\Phi(u_\lambda))\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}\varphi\,\mathrm{d}x\right) + \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}\varphi\,\mathrm{d}\sigma \\
= \lambda\int_\Omega u_\lambda^{-\gamma}\varphi\,\mathrm{d}x + \int_\Omega u_\lambda^{r-1}\varphi\,\mathrm{d}x.
\end{aligned} \tag{4.14}$$

Let $\varphi \in W^{1,\mathcal{H}}(\Omega)$ and let $\varepsilon > 0$. We define $w_\varepsilon := u_\lambda + \varepsilon\varphi$. We apply (4.12) to $h = w_\varepsilon^+ \geq 0$ and along with $u_\lambda \in \mathcal{N}_\lambda$, $w_\varepsilon^+ = w_\varepsilon + w_\varepsilon^-$, $w_\varepsilon^- = 0$ in $\{w_\varepsilon > 0\}$, $w_\varepsilon^- = -w_\varepsilon$ in $\{w_\varepsilon \leq 0\}$, $u_\lambda > 0$ a.e. in Ω, and $\Omega = \{w_\varepsilon > 0\} \cup \{w_\varepsilon \leq 0\}$ we obtain

$$\begin{aligned}
0 &\leq m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla w_\varepsilon^+\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}w_\varepsilon^+\,\mathrm{d}x\right) \\
&\quad + \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}w_\varepsilon^+\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}w_\varepsilon^+\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}w_\varepsilon^+\,\mathrm{d}x \\
&= m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla w_\varepsilon\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}w_\varepsilon\,\mathrm{d}x\right) \\
&\quad + \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}w_\varepsilon\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}w_\varepsilon\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}w_\varepsilon\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla w_\varepsilon^-\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}w_\varepsilon^-\,\mathrm{d}x\right)\\
&+ \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}w_\varepsilon^-\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}w_\varepsilon^-\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}w_\varepsilon^-\,\mathrm{d}x\\
={}& m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x\right)\\
&+ \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x\\
&- \int_\Omega u_\lambda^{r-1}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x\\
&- m\left(\Phi(u_\lambda)\right)\left(\int_{\{w_\varepsilon\le 0\}} \mathcal{L}(u_\lambda)\cdot\nabla w_\varepsilon\,\mathrm{d}x + \int_{\{w_\varepsilon\le 0\}} \alpha(x)u_\lambda^{p-1}w_\varepsilon\,\mathrm{d}x\right)\\
&- \int_{\{w_\varepsilon\le 0\}}\beta(x)u_\lambda^{p_*-1}w_\varepsilon\,\mathrm{d}\sigma + \lambda\int_{\{w_\varepsilon\le 0\}} u_\lambda^{-\gamma}w_\varepsilon\,\mathrm{d}x + \int_{\{w_\varepsilon\le 0\}} u_\lambda^{r-1}w_\varepsilon\,\mathrm{d}x\\
={}& m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla(\varepsilon\varphi)\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}(\varepsilon\varphi)\,\mathrm{d}x\right)\\
&+ \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}(\varepsilon\varphi)\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}(\varepsilon\varphi)\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}(\varepsilon\varphi)\,\mathrm{d}x\\
&- m\left(\Phi(u_\lambda)\right)\left(\int_{\{w_\varepsilon\le 0\}} \mathcal{L}(u_\lambda)\cdot\nabla(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x\right.\\
&\qquad\qquad\qquad\left. + \int_{\{w_\varepsilon\le 0\}} \alpha(x)u_\lambda^{p-1}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}x\right)\\
&- \int_{\{w_\varepsilon\le 0\}}\beta(x)u_\lambda^{p_*-1}(u_\lambda+\varepsilon\varphi)\,\mathrm{d}\sigma + \lambda\int_{\{w_\varepsilon\le 0\}} u_\lambda^{-\gamma}w_\varepsilon\,\mathrm{d}x\\
&+ \int_{\{w_\varepsilon\le 0\}} u_\lambda^{r-1}w_\varepsilon\,\mathrm{d}x\\
\le{}& m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla(\varepsilon\varphi)\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}(\varepsilon\varphi)\,\mathrm{d}x\right)\\
&+ \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}(\varepsilon\varphi)\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}(\varepsilon\varphi)\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}(\varepsilon\varphi)\,\mathrm{d}x\\
&- m\left(\Phi(u_\lambda)\right)\left(\int_{\{w_\varepsilon\le 0\}} \mathcal{L}(u_\lambda)\cdot\nabla(\varepsilon\varphi)\,\mathrm{d}x + \int_{\{w_\varepsilon\le 0\}} \alpha(x)u_\lambda^{p-1}(\varepsilon\varphi)\,\mathrm{d}x\right)\\
&- \int_{\{w_\varepsilon\le 0\}}\beta(x)u_\lambda^{p_*-1}(\varepsilon\varphi)\,\mathrm{d}\sigma\\
={}& \varepsilon\Bigg[m\left(\Phi(u_\lambda)\right)\left(\int_\Omega \mathcal{L}(u_\lambda)\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega \alpha(x)u_\lambda^{p-1}\varphi\,\mathrm{d}x\right)\\
&+ \int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}\varphi\,\mathrm{d}\sigma - \lambda\int_\Omega u_\lambda^{-\gamma}\varphi\,\mathrm{d}x - \int_\Omega u_\lambda^{r-1}\varphi\,\mathrm{d}x\\
&- m\left(\Phi(u_\lambda)\right)\left(\int_{\{w_\varepsilon\le 0\}} \mathcal{L}(u_\lambda)\cdot\nabla\varphi\,\mathrm{d}x + \int_{\{w_\varepsilon\le 0\}} \alpha(x)u_\lambda^{p-1}\varphi\,\mathrm{d}x\right)
\end{aligned}
$$

$$-\int_{\{w_\varepsilon\leq 0\}}\beta(x)u_\lambda^{p_*-1}\varphi\,\mathrm{d}\sigma\Bigg].$$

Thanks to $u_\lambda>0$ a.e. in Ω we observe that

$$\lim_{\varepsilon\searrow 0}|\{w_\varepsilon\leq 0\}|=\lim_{\varepsilon\searrow 0}|\{u_\lambda+\varepsilon\varphi\leq 0\}|=0$$

and therefore

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_{\{w_\varepsilon\leq 0\}}\mathcal{L}(u_\lambda)\cdot\nabla\varphi\,\mathrm{d}x&=0,\\ \lim_{\varepsilon\searrow 0}\int_{\{w_\varepsilon\leq 0\}}\alpha(x)u_\lambda^{p-1}\varphi\,\mathrm{d}x&=0,\\ \lim_{\varepsilon\searrow 0}\int_{\{w_\varepsilon\leq 0\}}\beta(x)u_\lambda^{p_*-1}\varphi\,\mathrm{d}\sigma&=0.\end{aligned}$$

Thus, dividing the above estimate by ε and taking the limit as $\varepsilon\searrow 0$ results in

$$\begin{aligned}m\left(\Phi(u_\lambda)\right)\left(\int_\Omega\mathcal{L}(u_\lambda)\cdot\nabla\varphi\,\mathrm{d}x+\int_\Omega\alpha(x)u_\lambda^{p-1}\varphi\,\mathrm{d}x\right)+\int_{\partial\Omega}\beta(x)u_\lambda^{p_*-1}\varphi\,\mathrm{d}\sigma\\ \geq\lambda\int_\Omega u_\lambda^{-\gamma}\varphi\,\mathrm{d}x+\int_\Omega u_\lambda^{r-1}\varphi\,\mathrm{d}x.\end{aligned}$$

Since φ is arbitrary, we are allowed to consider the same estimate with $-\varphi$ instead of φ. Therefore, we conclude that (4.14) is true. □

In the remaining part of this section, we will prove that there exists a second solution $v_\lambda\in\mathcal{N}_\lambda^-$ of problem (K_λ) such that $J_\lambda(v_\lambda)=\Theta_\lambda^-$. We start by stating a lemma, which establishes that the energy functional J_λ is strictly positive on $\mathcal{N}_\lambda^-$, but in contrast to Lemma 4.1 this time we have to choose λ to be small enough.

Lemma 4.6. *Suppose hypothesis* (H) *is satisfied. Then there exists* $\Lambda_3>0$ *such that for all* $\lambda\in(0,\Lambda_3)$ *and* $v\in\mathcal{N}_\lambda^-$ *we have* $J_\lambda(v)>0$*. In particular, if* $\mathcal{N}_\lambda^-$ *is nonempty for some* $\lambda\in(0,\Lambda_3)$*, then* $\Theta_\lambda^->0$*.*

Proof. In order to prove the assertion we will argue by contradiction. Let $\Lambda>0$. We assume there are $\lambda\in(0,\Lambda)$ and $v\in\mathcal{N}_\lambda^-$ with the property that $J_\lambda(v)\leq 0$. Then the definition of J_λ implies

$$a_0\Phi(v)+\frac{b_0}{\vartheta}\Phi^\vartheta(v)+\frac{1}{p_*}\|v\|_{p_*,\beta,\partial\Omega}^{p_*}\leq\frac{\lambda}{1-\gamma}\int_\Omega|v|^{1-\gamma}\,\mathrm{d}x+\frac{1}{r}\|v\|_r^r \tag{4.15}$$

and from $\psi_v'(1)=0$ we get

$$\left(a_0+b_0\Phi^{\vartheta-1}(v)\right)\rho(v)+\|v\|_{p_*,\beta,\partial\Omega}^{p_*}=\lambda\int_\Omega|v|^{1-\gamma}\,\mathrm{d}x+\|v\|_r^r. \tag{4.16}$$

Dividing (4.16) by $-r$ and adding the result to (4.15) as well as applying the estimates $\lambda\frac{r-1+\gamma}{r(1-\gamma)}>0$, $\int_\Omega|v|^{1-\gamma}\,\mathrm{d}x\leq|\Omega|^{1-\frac{1-\gamma}{p^*}}\|v\|_{p^*}^{1-\gamma}$ and $\|v\|_{p^*}\leq S^{-\frac{1}{p}}\|v\|_{1,p}$ yields

$$\begin{aligned}&a_0\Phi(v)+\frac{b_0}{\vartheta}\Phi^\vartheta(v)+\frac{1}{p_*}\|v\|_{p_*,\beta,\partial\Omega}^{p_*}-\frac{1}{r}\left(\left(a_0+b_0\Phi^{\vartheta-1}(v)\right)\rho(v)+\|v\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\ &\leq\frac{\lambda}{1-\gamma}\int_\Omega|v|^{1-\gamma}\,\mathrm{d}x+\frac{1}{r}\|v\|_r^r-\frac{1}{r}\left(\lambda\int_\Omega|v|^{1-\gamma}\,\mathrm{d}x+\|v\|_r^r\right)\end{aligned}$$

$$
\begin{aligned}
&= \lambda \frac{r-1+\gamma}{r(1-\gamma)} \int_\Omega |v|^{1-\gamma}\,\mathrm{d}x \leq \lambda \frac{r-1+\gamma}{r(1-\gamma)} |\Omega|^{1-\frac{1-\gamma}{p^*}} \|v\|_{p^*}^{1-\gamma}\\
&\leq \lambda \frac{r-1+\gamma}{r(1-\gamma)} |\Omega|^{1-\frac{1-\gamma}{p^*}} S^{-\frac{1-\gamma}{p}} \|v\|_{1,p}^{1-\gamma}.
\end{aligned}
$$

Furthermore, we remember that $a_0 \geq 0$, $b_0 > 0$, $\frac{1}{p} - \frac{1}{r} > 0$, $\frac{1}{q} - \frac{1}{r} > 0$, $\frac{1}{p_*} - \frac{1}{r} > 0$, $\frac{1}{\vartheta q} - \frac{1}{r} > 0$ and $\vartheta - 1 \geq 0$ along with $\Phi(v) \geq \frac{1}{p}\|v\|_{1,p}^p$, and obtain

$$
\begin{aligned}
&a_0\Phi(v) + \frac{b_0}{\vartheta}\Phi^\vartheta(v) + \frac{1}{p_*}\|v\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{r}\left(\left(a_0 + b_0\Phi^{\vartheta-1}(v)\right)\rho(v) + \|v\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\
&\quad = a_0\left(\frac{1}{p} - \frac{1}{r}\right)\|v\|_{1,p}^p + a_0\left(\frac{1}{q} - \frac{1}{r}\right)\|\nabla v\|_{q,\mu}^q + b_0\Phi^{\vartheta-1}(v)\left(\frac{1}{\vartheta p} - \frac{1}{r}\right)\|v\|_{1,p}^p\\
&\qquad + b_0\Phi^{\vartheta-1}(v)\left(\frac{1}{\vartheta q} - \frac{1}{r}\right)\|\nabla v\|_{q,\mu}^q + \left(\frac{1}{p_*} - \frac{1}{r}\right)\|v\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&\quad \geq b_0\Phi^{\vartheta-1}(v)\left(\frac{1}{\vartheta p} - \frac{1}{r}\right)\|v\|_{1,p}^p \geq \frac{b_0}{p^{\vartheta-1}}\|v\|_{1,p}^{p(\vartheta-1)}\left(\frac{1}{\vartheta p} - \frac{1}{r}\right)\|v\|_{1,p}^p\\
&\quad = \frac{b_0(r-\vartheta p)}{\vartheta p^\vartheta r}\|v\|_{1,p}^{\vartheta p}.
\end{aligned}
$$

We combine the aforementioned estimates and derive that

$$
\frac{b_0(r-\vartheta p)}{\vartheta p^\vartheta r}\|v\|_{1,p}^{\vartheta p} \leq \lambda \frac{r-1+\gamma}{r(1-\gamma)} |\Omega|^{1-\frac{1-\gamma}{p^*}} S^{-\frac{1-\gamma}{p}} \|v\|_{1,p}^{1-\gamma}.
$$

Rearranging this inequality results in

$$
\|v\|_{1,p}^{\vartheta p - 1 + \gamma} \leq D_3\lambda, \tag{4.17}
$$

where D_3 is a positive constant given by

$$
D_3 := \frac{(r-1+\gamma)\vartheta p^\vartheta r}{r(1-\gamma)b_0(r-\vartheta p)} |\Omega|^{1-\frac{1-\gamma}{p^*}} S^{-\frac{1-\gamma}{p}}.
$$

From Lemma 3.6 we know there exists a positive constant D_2, independent of λ, such that

$$
\|v\|_{1,p}^p > D_2. \tag{4.18}
$$

With the aim of deriving a contradiction we define

$$
\Lambda := \frac{D_2^{\vartheta p - 1 + \gamma}}{D_3} > 0.
$$

Then, by considering $\lambda < \Lambda$ and putting (4.17) and (4.18) together, we end up with

$$
D_2^{\vartheta p - 1 + \gamma} < \|v\|_{1,p}^{\vartheta p - 1 + \gamma} \leq D_3\lambda < D_3\Lambda = D_2^{\vartheta p - 1 + \gamma},
$$

which is a contradiction. □

Next we will prove that for $\lambda > 0$ small enough there exists a function $v_\lambda \in \mathcal{N}_\lambda^-$ such that $\Theta_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) = J_\lambda(v_\lambda)$. To this end, let

$$
\hat{\Lambda} := \min\{\Lambda_1, \Lambda_2, \Lambda_3\}, \tag{4.19}
$$

where Λ_1 is from Lemma 3.3, Λ_2 from Lemma 3.4, and Λ_3 from Lemma 4.6.

Proposition 4.7. *Suppose hypothesis (H) is satisfied and let $\lambda \in (0, \hat{\Lambda})$. Then there exists $v_\lambda \in \mathcal{N}_\lambda^-$ such that $J_\lambda(v_\lambda) = \Theta_\lambda^-$ with $v_\lambda \geq 0$ a.e. in Ω. In particular, by Lemma 4.6 we have $\Theta_\lambda^- = J_\lambda(v_\lambda) > 0$.*

Proof. Because of $\lambda < \Lambda_2$, by Lemma 3.4, the set $\mathcal{N}_\lambda^-$ is nonempty. Together with $\lambda < \Lambda_3$ and Lemma 4.6 this implies $\Theta_\lambda^- > 0$. Since $\mathcal{N}_\lambda^-$ is nonempty, we are allowed to choose a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}_\lambda^-$ which minimizes the energy functional J_λ, i.e. $\{J_\lambda(v_n)\}_{n\in\mathbb{N}}$ is decreasing and we have

$$\lim_{n\to\infty} J_\lambda(v_n) = \Theta_\lambda^- > 0. \tag{4.20}$$

If $\{v_n\}_{n\in\mathbb{N}}$ was unbounded in $W^{1,\mathcal{H}}(\Omega)$, Lemma 3.2 would imply $\lim_{n\to\infty} J_\lambda(v_n) = \infty$ for a subsequence in contradiction to (4.20). Thus, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\Omega)$. By the reflexivity of $W^{1,\mathcal{H}}(\Omega)$ we obtain a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, such that

$$\begin{aligned} &v_n \to v_\lambda \quad \text{in } L^r(\Omega), && v_n \rightharpoonup v_\lambda \quad \text{in } L^{p_*}_\beta(\partial\Omega),\\ &v_n \rightharpoonup v_\lambda \quad \text{in } W^{1,p}(\Omega), && \nabla v_n \rightharpoonup \nabla v_\lambda \text{ in } L^q_\mu(\Omega), \end{aligned}$$

as $n \to \infty$. We can also assume

$$v_n \to v_\lambda \text{ a.e. in } \Omega \quad \text{and} \quad |v_n| \le f \text{ for all } n \in \mathbb{N} \text{ a.e. in } \Omega$$

as $n \to \infty$ and for some $f \in L^1(\Omega)$. Applying Lebesgue's dominated convergence theorem to the latter property gives

$$\lim_{n\to\infty} \int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x = \int_\Omega |v_\lambda|^{1-\gamma}\,\mathrm{d}x. \tag{4.21}$$

Next, we will argue that $v_\lambda \neq 0$. Let $n \in \mathbb{N}$. By using $a_0 \ge 0$, $b_0 > 0$, $p-1 > 0$, $q-1 > 0$, $p_* - 1 > 0$ as well as $\Phi(v_n) \ge \frac{1}{p}\|v_n\|_{1,p}^p$, $\|v_n\|_r^r \le |\Omega|^{1-\frac{r}{p^*}}\|v_n\|_{p^*}^r$, $\psi''_{v_n}(1) < 0$ and $\|v_n\|_{p^*} \le S^{-1}\|v_n\|_{1,p}$ we get

$$\begin{aligned} &\frac{b_0(p-1)S^{\vartheta p}}{p^{\vartheta-1}|\Omega|^{\left(1-\frac{r}{p^*}\right)\frac{\vartheta p}{r}}}\|v_n\|_r^{\vartheta p} \le \frac{b_0(p-1)}{p^{\vartheta-1}}\|v_n\|_{1,p}^{\vartheta p}\\ &\le b_0(p-1)\frac{\Phi^{\vartheta-1}(v_n)}{\|v_n\|_{1,p}^{p(\vartheta-1)}}\|v_n\|_{1,p}^{\vartheta p} = b_0\Phi^{\vartheta-1}(v_n)(p-1)\|v_n\|_{1,p}^p\\ &\le (\vartheta-1)b_0\Phi^{\vartheta-2}(v_n)\rho^2(v_n)\\ &\qquad + \left(a_0 + b_0\Phi^{\vartheta-1}(v_n)\right)\left((p-1)\|v_n\|_{1,p}^p + (q-1)\|\nabla v_n\|_{q,\mu}^q\right)\\ &\qquad + (p_*-1)\|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\gamma\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x\\ &= \psi''_{v_n}(1) + (r-1)\|v_n\|_r^r < (r-1)\|v_n\|_r^r. \end{aligned}$$

Solving this inequality for $\|v_n\|_r$ leads to

$$\|v_n\|_r > \left(\frac{b_0(p-1)S^{\vartheta p}}{(r-1)p^{\vartheta-1}|\Omega|^{\left(1-\frac{r}{p^*}\right)\frac{\vartheta p}{r}}}\right)^{\frac{1}{r-\vartheta p}} > 0.$$

Since we have the strong convergence $v_n \to v_\lambda$ in $L^r(\Omega)$ as $n \to \infty$, we conclude from this estimate that $v_\lambda \neq 0$. Thus, by taking further into account that $\lambda < \Lambda_2$, we obtain by Lemma 3.4 the existence of a unique $t_2^{v_\lambda} > 0$ such that $t_2^{v_\lambda} v_\lambda \in \mathcal{N}_\lambda^-$.

The next step is to show that we have $\lim_{n\to\infty} J_\lambda(v_n) = J_\lambda(v_\lambda)$ for a subsequence. To this end we are going to prove that

$$\begin{aligned} \liminf_{n\to\infty} \|v_n\|_{1,p}^p &= \|v_\lambda\|_{1,p}^p, \\ \liminf_{n\to\infty} \|\nabla v_n\|_{q,\mu}^q &= \|\nabla v_\lambda\|_{q,\mu}^q, \\ \liminf_{n\to\infty} \|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} &= \|v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}. \end{aligned} \tag{4.22}$$

We argue by contradiction. Thanks to the weak lower semicontinuity of the corresponding norms and seminorms, it is sufficient to assume that one of the following statements is true:

$$\begin{aligned} &\liminf_{n\to\infty} \|v_n\|_{1,p}^p > \|v_\lambda\|_{1,p}^p, \\ \text{or} \quad &\liminf_{n\to\infty} \|\nabla v_n\|_{q,\mu}^q > \|\nabla v_\lambda\|_{q,\mu}^q, \\ \text{or} \quad &\liminf_{n\to\infty} \|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} > \|v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}. \end{aligned}$$

This assumption in addition to $v_n \to v_\lambda$ in $L^r(\Omega)$ as $n \to \infty$, (4.21), and the weak lower semicontinuity of the corresponding norms and seminorms implies

$$\begin{aligned} &J_\lambda\left(t_2^{v_\lambda} v_\lambda\right) \\ &= a_0 \left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \|v_\lambda\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \|\nabla v_\lambda\|_{q,\mu}^q\right) \\ &\quad + \frac{b_0}{\vartheta}\left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \|v_\lambda\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \|\nabla v_\lambda\|_{q,\mu}^q\right)^\vartheta \\ &\quad + \frac{1}{p_*}\left(t_2^{v_\lambda}\right)^{p_*} \|v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda\left(t_2^{v_\lambda}\right)^{1-\gamma}}{1-\gamma}\int_\Omega |v_\lambda|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\left(t_2^{v_\lambda}\right)^r \|v_\lambda\|_r^r \\ &< a_0 \left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \liminf_{n\to\infty} \|v_n\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \liminf_{n\to\infty} \|\nabla v_n\|_{q,\mu}^q\right) \\ &\quad + \frac{b_0}{\vartheta}\left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \liminf_{n\to\infty} \|v_n\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \liminf_{n\to\infty} \|\nabla v_n\|_{q,\mu}^q\right)^\vartheta \\ &\quad + \frac{1}{p_*}\left(t_2^{v_\lambda}\right)^{p_*} \liminf_{n\to\infty} \|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\ &\quad - \frac{\lambda\left(t_2^{v_\lambda}\right)^{1-\gamma}}{1-\gamma} \lim_{n\to\infty}\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\left(t_2^{v_\lambda}\right)^r \lim_{n\to\infty} \|v_n\|_r^r \\ &\le \liminf_{n\to\infty} \Bigg(a_0 \left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \|v_n\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \|\nabla v_n\|_{q,\mu}^q\right) \\ &\qquad + \frac{b_0}{\vartheta}\left(\frac{1}{p}\left(t_2^{v_\lambda}\right)^p \|v_n\|_{1,p}^p + \frac{1}{q}\left(t_2^{v_\lambda}\right)^q \|\nabla v_n\|_{q,\mu}^q\right)^\vartheta \\ &\quad + \frac{1}{p_*}\left(t_2^{v_\lambda}\right)^{p_*} \|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{\lambda\left(t_2^{v_\lambda}\right)^{1-\gamma}}{1-\gamma}\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\left(t_2^{v_\lambda}\right)^r \|v_n\|_r^r \Bigg) \\ &= \liminf_{n\to\infty} J_\lambda\left(t_2^{v_\lambda} v_\lambda\right). \end{aligned}$$

Let $n \in \mathbb{N}$. Since $\lambda < \Lambda_3$ and $v_n \in \mathcal{N}_\lambda^-$, by Lemma 4.6 we know $\psi_{v_n}(1) = J_\lambda(v_n) > 0$. Furthermore, from $\lambda < \Lambda_2$ and $v_n \in \mathcal{N}_\lambda^-$ we know $t_2^{v_n} = 1$, where

$t_2^{v_n} > 0$ is the unique number from Lemma 4.6 such that $t_2^{v_n} v_n \in \mathcal{N}_\lambda^-$. Thus, Remark 3.5 implies

$$\max_{t\geq 0} \psi_{v_n}(t) = \psi_{v_n}(1). \tag{4.23}$$

Putting the aforementioned considerations together we obtain the contradiction

$$\begin{aligned}\Theta_\lambda^- &\leq J_\lambda\left(t_2^{v_\lambda} v_\lambda\right) < \liminf_{n\to\infty} J_\lambda\left(t_2^{v_\lambda} v_n\right) = \liminf_{n\to\infty} \psi_{v_n}\left(t_2^{v_\lambda}\right) \\ &\leq \liminf_{n\to\infty} \psi_{v_n}(1) = \liminf_{n\to\infty} J_\lambda(v_n) = \Theta_\lambda^-.\end{aligned}$$

In consequence, (4.22) holds for a subsequence, which we again denote by $\{v_n\}_{n\in\mathbb{N}}$. Taking further into consideration that $v_n \to v_\lambda$ in $L^r(\Omega)$ as $n \to \infty$ along with (4.20) and (4.21), we conclude that

$$\begin{aligned}\Theta_\lambda^- &= \lim_{n\to\infty} J_\lambda(v_n) \\ &= \lim_{n\to\infty}\Bigg(a_0\Phi(v_n) + \frac{b_0}{\vartheta}\Phi^\vartheta(v_n) + \frac{1}{p_*}\|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\ &\qquad\qquad - \frac{\lambda}{1-\gamma}\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x - \frac{1}{r}\|v_n\|_r^r\Bigg) \\ &= J_\lambda(v_\lambda).\end{aligned}$$

Next, we are going to prove that $v_\lambda \in \mathcal{N}_\lambda^-$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\psi'_{v_n}(1) = &\left(a_0 + b_0\Phi^{\vartheta-1}(v_n)\right)\left(\|v_n\|_{1,p}^p + \|\nabla v_n\|_{q,\mu}^q\right) + \|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\ &- \lambda\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x - \|v_n\|_r^r\end{aligned}$$

and

$$\begin{aligned}\psi''_{v_n}(1) = &\left(a_0 + b_0\Phi^{\vartheta-1}(v_n)\right)\left((p-1)\|v_n\|_{1,p}^p + (q-1)\|\nabla v_n\|_{q,\mu}^q\right) \\ &+ b_0(\vartheta-1)\Phi^{\vartheta-2}(v_n)\left(\|v_n\|_{1,p}^p + \|\nabla v_n\|_{q,\mu}^q\right)^2 + (p_*-1)\|v_n\|_{p_*,\beta,\partial\Omega}^{p_*} \\ &+ \lambda\gamma\int_\Omega |v_n|^{1-\gamma}\,\mathrm{d}x - (r-1)\|v_n\|_r^r.\end{aligned}$$

Because of $v_n \in \mathcal{N}_\lambda^-$, it holds $\psi'_{v_n}(1) = 0$ and $\psi''_{v_n}(1) < 0$ for all $n \in \mathbb{N}$. Hence, taking the limit as $n \to \infty$ in the above equations of ψ'_{v_n} and ψ''_{v_n}, we infer, similar to the proof of $\lim_{n\to\infty} J_\lambda(v_n) = J_\lambda(v_\lambda)$, that $\psi'_{v_\lambda}(1) = 0$ and $\psi''_{v_\lambda}(1) \leq 0$. Then, applying $\lambda < \Lambda_1$ and Lemma 3.3 gives $\mathcal{N}_\lambda^\circ = \emptyset$ which implies $\psi''_{v_\lambda}(1) < 0$. Thus, $v_\lambda \in \mathcal{N}_\lambda^-$. Furthermore, we have $|v_\lambda| \in W^{1,\mathcal{H}}(\Omega)$ and because of

$$J_\lambda(|v_\lambda|) = J_\lambda(v_\lambda) = \Theta_\lambda^-, \quad \psi'_{|v_\lambda|}(1) = \psi'_{v_\lambda}(1) = 0, \quad \text{and} \quad \psi''_{|v_\lambda|}(1) = \psi''_{v_\lambda}(1) < 0,$$

we have $|v_\lambda| \in \mathcal{N}_\lambda^-$. Therefore, we can assume $v_\lambda \geq 0$ a.e. in Ω. □

Now, we can prove that v_λ is indeed a weak solution of problem (K_λ).

Proposition 4.8. *Suppose hypothesis* (H) *is satisfied and let* $\lambda \in (0, \hat{\Lambda})$, *where* $\hat{\Lambda}$ *is given in* (4.19). *Then,* v_λ *from Proposition 4.7 is a weak solution of problem* (K_λ).

Proof. From Proposition 4.7 we know that $v_\lambda \geq 0$ a.e. in Ω, so it is sufficient to prove $v_\lambda \neq 0$ a.e. in Ω. Arguing by contradiction, suppose there is a measurable set $K \subseteq \Omega$ with $|K| > 0$ and $v_\lambda = 0$ in K. Taking $h \in W^{1,\mathcal{H}}(\Omega)$ with $h > 0$, as well as $v_\lambda \in \mathcal{N}_\lambda^-$, from Lemma 4.3 we can find $\varepsilon > 0$ and a continuous mapping $\zeta\colon B_\varepsilon(0) \to (0,\infty)$ such that

$$\zeta(0\cdot h) = 1 \quad \text{and} \quad \zeta(th)(v_\lambda + th) \in \mathcal{N}_\lambda^- \quad \text{for all } t \in [0,\delta_h),$$

where $\delta_h > 0$ is a number, chosen small enough such that $th \in B_\varepsilon(0)$ for all $t \in [0,\delta_h)$. This implies

$$J_\lambda(v_\lambda) = \Theta_\lambda^- \leq J_\lambda(\zeta(th)(v_\lambda + th)) \quad \text{for all } t \in [0,\delta_h). \tag{4.24}$$

Note that a similar argumentation as in (4.23) can be done for v_λ in order to derive from Remark 3.5 that

$$\max_{t\geq 0} \psi_{v_\lambda}(t) = \psi_{v_\lambda}(1). \tag{4.25}$$

Because of $h > 0$ and $\gamma < 1$, we infer that $(\zeta(th)(v_\lambda + th))^{1-\gamma} > (\zeta(th)(v_\lambda))^{1-\gamma}$ a.e. in $\Omega \setminus K$ and therefore

$$\int_{\Omega\setminus K} \left((\zeta(th)(v_\lambda + th))^{1-\gamma} - (\zeta(th)(v_\lambda))^{1-\gamma}\right) \mathrm{d}x \geq 0 \quad \text{for all } t \geq 0. \tag{4.26}$$

By applying (4.24), (4.25), and (4.26) together with $v_\lambda = 0$ in K, for all $t \in [0,\delta_h)$ we obtain

$$\begin{aligned}
0 &\leq J_\lambda(\zeta(th)(v_\lambda + th)) - J_\lambda(v_\lambda) = J_\lambda(\zeta(th)(v_\lambda + th)) - \psi_{v_\lambda}(1)\\
&\leq J_\lambda(\zeta(th)(v_\lambda + th)) - \psi_{v_\lambda}(\zeta(th)) = J_\lambda(\zeta(th)(v_\lambda + th)) - J_\lambda(\zeta(th)v_\lambda)\\
&= M\left(\Phi(\zeta(th)(v_\lambda + th))\right) - M\left(\Phi(\zeta(th)v_\lambda)\right)\\
&\qquad - \frac{\lambda}{1-\gamma}\int_\Omega \left((\zeta(th)(v_\lambda + th))^{1-\gamma} - (\zeta(th)(v_\lambda))^{1-\gamma}\right) \mathrm{d}x\\
&\qquad + \frac{1}{p_*}\left(\|\zeta(th)(v_\lambda + th)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(th)(v_\lambda)\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\
&\qquad - \frac{1}{r}\left(\|\zeta(th)(v_\lambda + th)\|_r^r - \|\zeta(th)(v_\lambda)\|_r^r\right)\\
&= M\left(\Phi(\zeta(th)(v_\lambda + th))\right) - M\left(\Phi(\zeta(th)v_\lambda)\right)\\
&\qquad - \frac{\lambda}{1-\gamma}\int_{\Omega\setminus K} \left((\zeta(th)(v_\lambda + th))^{1-\gamma} - (\zeta(th)(v_\lambda))^{1-\gamma}\right) \mathrm{d}x\\
&\qquad - \frac{\lambda}{1-\gamma}\int_K \left((\zeta(th)(v_\lambda + th))^{1-\gamma} - (\zeta(th)(v_\lambda))^{1-\gamma}\right) \mathrm{d}x\\
&\qquad + \frac{1}{p_*}\left(\|\zeta(th)(v_\lambda + th)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(th)(v_\lambda)\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\
&\qquad - \frac{1}{r}\left(\|\zeta(th)(v_\lambda + th)\|_r^r - \|\zeta(th)(v_\lambda)\|_r^r\right)\\
&\leq M\left(\Phi(\zeta(th)(v_\lambda + th))\right) - M\left(\Phi(\zeta(th)v_\lambda)\right) - \frac{\lambda(\zeta(th))^{1-\gamma}t^{1-\gamma}}{1-\gamma}\int_K h^{1-\gamma}\,\mathrm{d}x\\
&\qquad + \frac{1}{p_*}\left(\|\zeta(th)(v_\lambda + th)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(th)(v_\lambda)\|_{p_*,\beta,\partial\Omega}^{p_*}\right)\\
&\qquad - \frac{1}{r}\left(\|\zeta(th)(v_\lambda + th)\|_r^r - \|\zeta(th)(v_\lambda)\|_r^r\right).
\end{aligned}$$

We divide this estimate by $t > 0$, take the limit as $t \searrow 0$ and from Lebesgue's dominated convergence theorem together with $1-\gamma > 0$, $\lim_{t\searrow 0} \zeta(th) = 1$, $\lim_{t\searrow 0} t^{-\gamma} =$

∞, and $\int_K h^{1-\gamma}\,\mathrm{d}x > 0$ we obtain

$$0 \leq \lim_{t\searrow 0} \frac{J_\lambda(\zeta(th)(v_\lambda + th)) - J_\lambda(v_\lambda)}{t} = -\infty,$$

that is a contradiction. We conclude that $v_\lambda > 0$ a.e. in Ω.

In the next step we will show that, for all $h \in W^{1,\mathcal{H}}(\Omega)$ with $h \geq 0$ in Ω, we have $v_\lambda^{-\gamma} h \in L^1(\Omega)$ and

$$\begin{aligned} m(\Phi(v_\lambda)) \left(\int_\Omega \mathcal{L}(v_\lambda) \cdot \nabla h \,\mathrm{d}x + \int_\Omega \alpha(x) v_\lambda^{p-1} h \,\mathrm{d}x \right) + \int_{\partial\Omega} \beta(x) v_\lambda^{p_*-1} h \,\mathrm{d}\sigma \\ \geq \lambda \int_\Omega v_\lambda^{-\gamma} h \,\mathrm{d}x + \int_\Omega v_\lambda^{r-1} h \,\mathrm{d}x. \end{aligned} \tag{4.27}$$

We take $h \in W^{1,\mathcal{H}}(\Omega)$ with $h \geq 0$ and consider a decreasing sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = 0$. We define a sequence of measurable functions $\{\varphi_n\}_{n\in\mathbb{N}}$ by

$$\varphi_n \colon \Omega \to \mathbb{R}, \quad \varphi_n(x) := \frac{(\zeta(t_n h)(v_\lambda(x) + t_n h(x)))^{1-\gamma} - (\zeta(t_n h) v_\lambda(x))^{1-\gamma}}{t_n}.$$

Since $\zeta > 0$ and $h \geq 0$, it is clear that φ_n is nonnegative for all $n \in \mathbb{N}$. Furthermore, applying $\lim_{t\searrow 0} \zeta(th) = 1$ and the mean value theorem, for a.a. $x \in \Omega$, we observe that

$$\lim_{n\to\infty} \varphi_n(x) = (1-\gamma) v_\lambda(x)^{-\gamma} h(x). \tag{4.28}$$

Similar as in the first part of this proof, by recalling the properties of ζ, for $n \in \mathbb{N}$ large enough we get

$$\begin{aligned} 0 &\leq \frac{J_\lambda(\zeta(t_n h)(v_\lambda + t_n h)) - J_\lambda(v_\lambda)}{t_n} \\ &\leq \frac{M\left(\Phi(\zeta(t_n h)(v_\lambda + t_n h))\right) - M\left(\Phi(\zeta(t_n h) v_\lambda)\right)}{t_n} - \frac{\lambda}{1-\gamma} \int_\Omega \varphi_n \,\mathrm{d}x \\ &\quad + \frac{1}{p_*} \frac{\|\zeta(t_n h)(v_\lambda + t_n h)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(t_n h) v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t_n} \\ &\quad - \frac{1}{r} \frac{\|\zeta(t_n h)(v_\lambda + t_n h)\|_r^r - \|\zeta(t_n h) v_\lambda\|_r^r}{t_n}. \end{aligned}$$

Taking the limit superior of this estimate as $n \to \infty$ and using Lebesgue's dominated convergence theorem as well as $\lim_{t\searrow 0} \zeta(th) = 1$ results in

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \left(\frac{M\left(\Phi(\zeta(t_n h)(v_\lambda + t_n h))\right) - M\left(\Phi(\zeta(t_n h) v_\lambda)\right)}{t_n} - \frac{\lambda}{1-\gamma} \int_\Omega \varphi_n \,\mathrm{d}x \right. \\ &\quad + \frac{1}{p_*} \frac{\|\zeta(t_n h)(v_\lambda + t_n h)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(t_n h)(v_\lambda)\|_{p_*,\beta,\partial\Omega}^{p_*}}{t_n} \\ &\quad \left. - \frac{1}{r} \frac{\|\zeta(t_n h)(v_\lambda + t_n h)\|_r^r - \|\zeta(t_n h)(v_\lambda)\|_r^r}{t_n} \right) \\ &\leq \lim_{n\to\infty} \frac{M\left(\Phi(\zeta(t_n h)(v_\lambda + t_n h))\right) - M\left(\Phi(\zeta(t_n h) v_\lambda)\right)}{t_n} \\ &\quad + \limsup_{n\to\infty} \left(-\frac{\lambda}{1-\gamma} \int_\Omega \varphi_n \,\mathrm{d}x \right) \end{aligned}$$

$$
\begin{aligned}
&+ \lim_{n\to\infty} \frac{1}{p_*} \frac{\|\zeta(t_nh)(v_\lambda+t_nh)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\zeta(t_nh)(v_\lambda)\|_{p_*,\beta,\partial\Omega}^{p_*}}{t_n}\\
&- \lim_{n\to\infty} \frac{1}{r} \frac{\|\zeta(t_nh)(v_\lambda+t_nh)\|_{r}^{r} - \|\zeta(t_nh)(v_\lambda)\|_{r}^{r}}{t_n}\\
= {}& m(\Phi(v_\lambda)) \left(\int_\Omega \mathcal{L}(v_\lambda)\cdot\nabla h \,\mathrm{d}x + \int_\Omega \alpha(x) v_\lambda^{p-1} h\,\mathrm{d}x \right) - \frac{\lambda}{1-\gamma} \liminf_{n\to\infty} \int_\Omega \varphi_n \,\mathrm{d}x\\
&+ \int_{\partial\Omega} \beta(x) v_\lambda^{p_*-1} h \,\mathrm{d}\sigma - \int_\Omega v_\lambda^{r-1} h\,\mathrm{d}x.
\end{aligned}
$$

From Fatou's Lemma and (4.28) it follows that

$$
\begin{aligned}
\lambda \int_\Omega v_\lambda^{-\gamma} h \,\mathrm{d}x &\le \frac{\lambda}{1-\gamma} \liminf_{n\to\infty} \int_\Omega \varphi_n\,\mathrm{d}x\\
&\le m(\Phi(v_\lambda)) \left(\int_\Omega \mathcal{L}(v_\lambda)\cdot\nabla h\,\mathrm{d}x + \int_\Omega \alpha(x) v_\lambda^{p-1} h\,\mathrm{d}x\right)\\
&\quad + \int_{\partial\Omega} \beta(x) v_\lambda^{p_*-1} h\,\mathrm{d}\sigma - \int_\Omega v_\lambda^{r-1} h\,\mathrm{d}x.
\end{aligned}
$$

Therefore, $v_\lambda^{-\gamma} h \in L^1(\Omega)$ and (4.27) holds. For arbitrary $\varphi \in W^{1,\mathcal{H}}(\Omega)$ consider $\varphi = \varphi^+ - \varphi^-$ which gives $v_\lambda^{-\gamma}\varphi \in L^1(\Omega)$.

The rest of this proof can be done repeating the same arguments for v_λ as we have seen in the last part in the proof of Proposition 4.5 □

The proof of Theorem 1.1 follows now from Propositions 4.5 and 4.8.

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