CONCENTRATION PHENOMENA FOR A LOGARITHMIC FRACTIONAL SCHRÖDINGER-POISSON SYSTEM

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Abstract. In this paper, we deal with the concentration of positive solutions for the fractional Schrödinger-Poisson system involving a logarithmic nonlinearity given in the form

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u - \phi u = u\log u^2 & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $s, t \in (0,1)$ satisfy $4s + 2t > 3$, $(-\Delta)^\nu$, with $\nu \in \{s,t\}$, is the fractional Laplace operator, and the potential V is continuous satisfying only a local condition. By applying suitable variational arguments, we analyze the existence and concentration behavior of solutions as $\varepsilon \to 0$ for the above problem.

1. Introduction

In recent years, the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

has received more and more attention. As a model describing the interaction of a charge particle with an electromagnetic field, it arises in many mathematical physics contexts, and is usually known as the Schrödinger-Poisson system. In the case of $f(x,u) = 0$, the wave function u represents the steady-state solution of the quantum system proposed by Benci-Fortunato [11], describing the interaction between charged particles and electromagnetic fields. According to Maxwell's equation, the time-independent ϕ is an electrostatic potential and is related to u. Therefore, the system is also known as the Schrödinger-Maxwell system. In 1998, Benci-Fortunato [11] first introduced the system (1.1) on a bounded domain and proposed it as a model to describe the interaction of a charged particle with the electrostatic field by using deduction and variational methods. For the following stationary Schrödinger-Poisson system

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

numerous scholars were dedicated to proving the existence of semiclassical solutions for this system and the concentration phenomena with respect to the parameter ε. When considering the existence of solutions of system (1.2), the following global conditions on V are often required which were introduced by Rabinowitz [38]:

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(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V_\infty = \lim_{|x|\to\infty} V(x) > V_* = \inf_{x\in\mathbb{R}^3} V(x) > 0$.

In 2011, He [23] established the existence of multiple positive solutions by using the Ljusternik-Schnirelmann theory and proved that these positive solutions concentrate around the global minimum of V. He-Zou [24] obtained the existence and concentration of ground states when the potential V satisfies the global condition (V) and the nonlinearity has critical growth. The following local conditions on V were introduced by del Pino-Felmer [20]:

(V_1') There exists $V_1 > 0$ such that $V_1 = \inf_{x\in\mathbb{R}^3} V(x)$.

(V_2') There exists an open bounded set $\Omega \subset \mathbb{R}^3$ such that

$$V_1 < \min_{\partial\Omega} V \quad \text{and} \quad M = \{x \in \Omega \colon V(x) = V_1\} \neq \emptyset.$$

He-Zou [25] applied Szulkin and Weth's generalized Nehari manifold methods, penalization techniques and the Ljusternik-Schnirelmann theory to prove the existence of multiple solutions when the potential V satisfies just the local conditions above. For more results in the direction of the Schrödinger-Poisson system, we refer the interested reader to the works of Azzollini [9], Chen-Shu-Tang-Wen [15], Cingolani-Jeanjean [16], Peng [34], Ruiz [39], Zhao-Zhao [49] and the references therein.

In the setting of the fractional Laplacian the system (1.1) becomes the fractional Schrödinger-Poisson system. Teng [45] studied the existence of ground state solutions for the fractional Schrödinger-Poisson system with critical Sobolev exponent while Murcia-Siciliano [33] considered the semiclassical state of the system

$$\begin{cases} \varepsilon^{2s} (-\Delta)^s u + V(x)u + K(x)\phi u = f(u) & \text{in } \mathbb{R}^3, \\ \varepsilon^{\theta} (-\Delta)^{\alpha/2} \phi = \gamma_\alpha u^2 & \text{in } \mathbb{R}^3 \end{cases}$$

and established the multiplicity of positive solutions that concentrate on the minima of V as $\varepsilon \to 0$ by using the Ljusternik-Schnirelmann category theory. Yang-Yu-Zhao [47] were concerned with the existence and concentration behavior of ground state solutions for the fractional Schrödinger-Poisson system with critical nonlinearity. Ambrosio [6] used penalization techniques and the Ljusternik-Schnirelmann theory to deal with the multiplicity and concentration of positive solutions for fractional Schrödinger-Poisson type system with critical growth of type

$$\begin{cases} \varepsilon^{2s} (-\Delta)^s u + V(x)u + \phi(x)u = f(u) + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t} (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0,1)$ satisfy $4s+2t > 3$, $2_s^* = \frac{6}{3-2s}$ is the critical Sobolev exponent and the potential V satisfies the local conditions (V_1') and (V_2'). Meng-Zhang-He [31] dealt with the existence of a positive and a sign-changing least energy solution for a class of fractional Schrödinger-Poisson systems with critical growth and vanishing potentials. Other interesting results in this direction can be found in the papers of Chen-Li-Peng [14], Ji [27], Qu-He [37], Yang-Zhang-Zhao [48]. Here, $(-\Delta)^\nu$, with $\nu \in \{s, t\}$, is the fractional Laplace operator which is defined for any $u \colon \mathbb{R}^3 \to \mathbb{R}$ belonging to the Schwartz class by

$$(-\Delta)^\nu u(x) = C(3,\nu)\,\text{P.V.} \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2\nu}}\,\mathrm{d}y, \quad x \in \mathbb{R}^3,$$

where P.V. stands for the Cauchy principal value and $C(3,\nu)$ is a normalizing constant, see Di Nezza-Palatucci-Valdinoci [21]. Recently, there is a large interest in the study of partial differential equations involving nonlocal fractional Laplace operators. This type of nonlocal operator comes up naturally in many different applications, such as phase transitions, game theory, finance, image processing, Lévy processes, and optimization. For more details and applications, we refer the interested reader to the works of Applebaum [8], Bahrouni-Rădulescu-Winkert [10], Caffarelli-Silvestre [12], Di Nezza-Palatucci-Valdinoci [21], Molica Bisci-Rădulescu-Servadei [32], Pucci-Xiang-Zhang [35, 36] and the references therein.

Recently, the following time-dependent logarithmic Schrödinger equation given by

$$i\varepsilon\frac{\partial\Phi}{\partial t}=-\varepsilon^2\Delta\Phi+W(x)\Phi-\Phi\log|\Phi|^2,\quad N\geq 3 \tag{1.3}$$

where $\Phi\colon[0,+\infty)\times\mathbb{R}^N\to\mathbb{C}$, has also received a lot of attention due to its physical influence, such as quantum mechanics, quantum optics, nuclear physics, effective quantum and Bose-Einstein condensation. Standing wave solutions for (1.3) have the ansatz form $\Phi(t,x)=u(x)e^{-i\omega t/\varepsilon}$ with $\omega\in\mathbb{R}$, which leads to the system

$$-\varepsilon^2\Delta u+V(x)u=u\log u^2\quad\text{in }\mathbb{R}^N, \tag{1.4}$$

where $V(x)=W(x)-\omega$. From a mathematical point of view, (1.4) is very interesting because many difficulties arise when using variational methods to find solutions. Alves-de Morais Filho [2] considered semiclassical state solutions for the logarithmic elliptic equation (1.4) when V satisfies the following global condition

$$V(x)\in C(\mathbb{R}^N,\mathbb{R})\quad\text{and}\quad V_\infty=\lim_{|x|\to\infty}V(x)>V_*=\inf_{x\in\mathbb{R}^N}V(x)>-1.$$

They obtained the existence of solutions of (1.4) as well as the concentration behavior of solutions as $\varepsilon\to 0$. Alves-Ji [3] continued to study (1.4) where $V(x)$ satisfies the following local conditions

(V_1'') $V\in C(\mathbb{R}^N,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^N}V(x)=V_0>-1$;

(V_2'') There exists an open bounded set $\Omega\subset\mathbb{R}^N$ such that

$$-1<V_0=\inf_{x\in\Omega}V(x)<\min_{\partial\Omega}V\quad\text{and}\quad M=\{x\in\Omega\colon V(x)=V_0\}\neq\emptyset.$$

They used the penalization method in order to prove the existence of positive solutions of (1.4) as well as the concentration behavior under (V_1'') and (V_2''). We also mention the works of Alves-Ambrosio [1], Alves-Ji [4, 5], d'Avenia-Montefusco-Squassina [17], Ji-Szulkin [28], Tanaka-Zhang [44] and the references therein.

Motivated by the above works, in this paper we consider the existence and concentration behavior of solutions for the following logarithmic fractional Schrödinger-Poisson system

$$\begin{cases}\varepsilon^{2s}\left(-\Delta\right)^s u+V(x)u-\phi u=u\log u^2 & \text{in }\mathbb{R}^3,\\ \varepsilon^{2t}\left(-\Delta\right)^t\phi=u^2 & \text{in }\mathbb{R}^3,\end{cases} \tag{1.5}$$

where $\varepsilon>0$ is a small parameter, $s,t\in(0,1)$ are such that $4s+2t>3$, and V satisfies the local conditions

(V_1) $V\in C(\mathbb{R}^3,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^3}V(x)=V_0>-1$;

(V_2) There exists an open bounded set $\Omega\subset\mathbb{R}^3$ such that

$$-1<V_0=\inf_{x\in\Omega}V(x)<\min_{\partial\Omega}V\quad\text{and}\quad M=\{x\in\Omega\colon V(x)=V_0\}\neq\emptyset.$$

The main result in our paper reads as follows.

Theorem 1.1. *Suppose that V satisfies* (V_1) *and* (V_2). *Then, there exists* $\varepsilon_0 > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon_0)$, *the system* (1.5) *has a positive solution* $u \in H^s(\mathbb{R}^3)$ *satisfying* $u^2 \log u^2 \in L^1(\mathbb{R}^3)$. *Moreover, if* u_ε *is a solution of* (1.5) *and* $y_\varepsilon \in \mathbb{R}^3$ *denotes a global maximum point of* u_ε, *then*

$$\lim_{\varepsilon \to 0^+} V(y_\varepsilon) = V_0.$$

Our approach is mainly based on variational methods. Nevertheless, due to the combination of the nonlinear and nonlocal operator $(-\Delta)^\nu$ along with the logarithmic nonlinearity, several estimates will be more delicate and completely different from the references mentioned above. Some important points are highlighted below:

(i) Due to the logarithmic term, the associated energy functional of system (1.5) may take the value $+\infty$, since there is a function $u \in H^s(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} u^2 \log u^2 \, \mathrm{d}x = -\infty$. Thus, the energy functional is not well defined in $H^s(\mathbb{R}^3)$ and the classical variational methods cannot be applied here. In order to find solutions of system (1.5), we will make a technical decomposition to obtain a functional which is a sum of a lower semicontinuous convex functional and of a C^1-functional. Then, a new version of the Mountain Pass Theorem can be applied to get a (PS) sequence.

(ii) Because of the appearance of two fractional Laplace operators in (1.5), our analysis becomes more complex and interesting as well as new arguments have to be taken into account to solve our problem.

(iii) Since V only satisfies the local conditions in (V_1) and (V_2), we have no information on the behavior of the potential V at infinity. So it is complicated to prove the (PS) condition. Due to the presence of the nonlocal term, we modify the nonlinearity in a special penalization way which is different from Alves-Ji [3]. Then we can prove that the modified functional satisfies the (PS) condition. Furthermore, we show that the solution of the modified problem is in fact a solution of the original problem (1.5) provided $\varepsilon > 0$ is sufficiently small.

The paper is organized as follows. In Section 2, we recall some lemmas which will be needed in the paper. Then, in Section 3, we study the auxiliary problem which is a key point in our approach. Finally, Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section we present the main tools and notions that will occur in Sections 3 and 4. For $A \subset \mathbb{R}^3$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u \colon \mathbb{R}^3 \to \mathbb{R}$ and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. Let us define $D^{s,2}(\mathbb{R}^3)$ as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Then, we consider the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2\left(\mathbb{R}^3\right) : [u] < \infty \right\}$$

endowed with the norm

$$\|u\|^2 = [u]^2 + |u|_2^2.$$

Now, we recall the following main embeddings for the fractional Sobolev spaces, see Di Nezza-Palatucci-Valdinoci [21].

Lemma 2.1. *Let $s \in (0,1)$. Then $H^s(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2, 2_s^*]$ and compactly embedded in $L^p_{\text{loc}}(\mathbb{R}^3)$ for any $p \in [1, 2_s^*)$.*

In addition, we define by S_* the best constant of the Sobolev embedding $H^s(\mathbb{R}^3)$ into $L^{2_s^*}(\mathbb{R}^3)$. We also recall a version of the well-known concentration-compactness principle, see Felmer-Quaas-Tan [22].

Lemma 2.2. *If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^s(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2_s^*)$.

By Lemma 2.1, we have

$$H^s(\mathbb{R}^3) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{2.1}$$

For any fixed $u \in H^s(\mathbb{R}^3)$, let $L_u \colon D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ be the functional given by

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v \, \mathrm{d}x,$$

which is continuous in view of Hölder's inequality and (2.1). Indeed, it holds

$$|L_u(v)| \le \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} \, \mathrm{d}x \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2_t^*} \, \mathrm{d}x \right)^{\frac{1}{2_t^*}} \le C\|u\|^2 \|v\|_{D^{t,2}},$$

where

$$\|v\|^2_{D^{t,2}} = \iint_{\mathbb{R}^6} \frac{|v(x) - v(y)|^2}{|x-y|^{3+2t}} \, \mathrm{d}x \, \mathrm{d}y.$$

Then, by the Lax-Milgram Theorem, there is a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $\langle \phi_u^t, v \rangle$ for each $v \in D^{t,2}(\mathbb{R}^3)$, where $\langle \cdot, \cdot \rangle$ is the inner product on $D^{t,2}(\mathbb{R}^3)$. Thus, we obtain the t-Riesz formula

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \, \mathrm{d}y, \quad \text{where} \quad c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)},$$

is the only weak solution of the problem

$$(-\Delta)^t \phi_u^t = u^2 \quad \text{in } \mathbb{R}^3.$$

In order to study the system (1.5), we use the change of variable $x \to \varepsilon x$ and we will look for the solutions of the problem

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x) u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

Substituting $\phi^t = \phi_u^t$ into system (2.2), we can rewrite (2.2) as a single equation

$$(-\Delta)^s u + V(\varepsilon x) u - \phi_u^t u = u \log u^2 \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

Next, we can state the following useful properties whose proofs can be found in Liu-Zhang [30] and Teng [45].

Lemma 2.3. *For all $u \in H^s(\mathbb{R}^3)$, then the following properties are valid:*

(1) $\|\phi_u^t\|_{D^{t,2}} \leq C|u|^2_{\frac{12}{3+2t}} \leq C\|u\|^2$ *and* $\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x \leq C_t|u|^4_{\frac{12}{3+2t}}$. *Moreover* $\phi_u^t \colon H^s(\mathbb{R}^3) \to D^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*
(2) $\phi_u^t \geq 0$ *in* $\mathbb{R}^3$*;*
(3) *if* $y \in \mathbb{R}^3$ *and* $\bar{u}(x) = u(x+y)$*, then* $\phi_{\bar{u}}^t(x) = \phi_u^t(x+y)$ *and* $\int_{\mathbb{R}^3} \phi_{\bar{u}}^t \bar{u}^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x$*;*
(4) $\phi_{ru}^t = r^2 \phi_u^t$ *for all* $r \in \mathbb{R}$*;*
(5) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$*, then* $\phi_{u_n}^t \rightharpoonup \phi_u^t$ *in* $D^{t,2}(\mathbb{R}^3)$*;*
(6) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$*, then* $\int_{\mathbb{R}^3} \phi_{u_n}^t u^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_{(u_n-u)}^t (u_n-u)^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x + o_n(1)$*;*
(7) *if* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$*, then* $\phi_{u_n}^t \to \phi_u^t$ *in* $D^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x$.

We define $N \colon H^s(\mathbb{R}^3) \to \mathbb{R}$ by setting

$$N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x. \tag{2.4}$$

Similar to the Brézis-Lieb lemma, N possesses the following properties whose proofs can be found in Teng [45].

Lemma 2.4. *Let* N *be defined by* (2.4)*. If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^3$*, then*

$$N(u_n - u) = N(u_n) - N(u).$$

Note that a weak solution of (2.3) in $H^s(\mathbb{R}^3)$ is a critical point of the associated energy functional

$$\mathcal{I}_\varepsilon(u) := \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \,\mathrm{d}x,$$

defined for all $u \in \mathcal{H}_\varepsilon$, where

$$\mathcal{H}_\varepsilon := \left\{ u \in H^s(\mathbb{R}^3) \colon \int_{\mathbb{R}^3} V(\varepsilon x) u^2 \,\mathrm{d}x < \infty \right\}$$

is endowed with the norm

$$\|u\|_\varepsilon^2 := [u]^2 + \int_{\mathbb{R}^3} (V(\varepsilon x) + 1)\, u^2 \,\mathrm{d}x.$$

Obviously, $\mathcal{H}_\varepsilon$ is a Hilbert space with inner product

$$(u,v)_\varepsilon = \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} (V(\varepsilon x) + 1)\, uv \,\mathrm{d}x.$$

Definition 2.5. *A solution of problem* (2.3) *is a function* $u \in H^s(\mathbb{R}^3)$ *such that* $u^2 \log u^2 \in L^1(\mathbb{R}^3)$ *and*

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} V(\varepsilon x) uv \,\mathrm{d}x \\ &- \int_{\mathbb{R}^3} \phi_u^t uv \,\mathrm{d}x = \int_{\mathbb{R}^3} uv \log u^2 \,\mathrm{d}x, \quad \textit{for all } u,v \in C_0^\infty(\mathbb{R}^3). \end{aligned}$$

Due to the lack of smoothness of $\mathcal{I}_\varepsilon$, we shall use the approach explored in Ji-Szulkin [28] and Squassina-Szulkin [41]. Let us decompose $\mathcal{I}_\varepsilon$ into a sum of a C^1 functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, we define the functions

$$F_1(\xi) = \begin{cases} 0, & \text{if } \xi = 0 \\ -\frac{1}{2}\xi^2 \log \xi^2 & \text{if } 0 < |\xi| < \delta \\ -\frac{1}{2}\xi^2 \left(\log \delta^2 + 3\right) + 2\delta|\xi| - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta \end{cases}$$

and

$$F_2(\xi) = \begin{cases} 0, & \text{if } |\xi| < \delta \\ \frac{1}{2}\xi^2 \log \left(\xi^2/\delta^2\right) + 2\delta|\xi| - \frac{3}{2}\xi^2 - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta. \end{cases}$$

Then,

$$F_2(\xi) - F_1(\xi) = \frac{1}{2}\xi^2 \log \xi^2 \quad \text{for all } \xi \in \mathbb{R},$$

and the functional $\mathcal{I}_\varepsilon \colon \mathcal{H}_\varepsilon \to (-\infty, +\infty]$ may be rewritten as

$$\mathcal{I}_\varepsilon(u) = \Phi_\varepsilon(u) + \Psi(u) \quad \text{for } u \in \mathcal{H}_\varepsilon, \tag{2.5}$$

where

$$\Phi_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F_2(u)\,\mathrm{d}x,$$

and

$$\Psi(u) = \int_{\mathbb{R}^3} F_1(u)\,\mathrm{d}x.$$

As proved in Ji-Szulkin [28] and Squassina-Szulkin [41], $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$. If $\delta > 0$ is small enough, F_1 is convex, even,

$$F_1(\xi) \geq 0 \quad \text{and} \quad 0 \leq \frac{1}{2}F_1'(\xi)\xi \leq F_1(\xi) \leq F_1'(\xi)\xi \quad \text{for all } \xi \in \mathbb{R}. \tag{2.6}$$

For each fixed $p \in (2, 2_s^*)$, there exists a constant $C > 0$ such that

$$|F_2'(\xi)| \leq C|\xi|^{p-1} \quad \text{for all } \xi \in \mathbb{R}. \tag{2.7}$$

Note that $\Phi_\varepsilon \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and Ψ is convex and lower semicontinuous in $H^s(\mathbb{R}^3)$, but Ψ is not a C^1 functional due to the unboundedness of $\mathbb{R}^3$. Then, we will seek a critical point in the sense of a sub-differential. Next, we recall some definitions that can be found in Szulkin [43].

Definition 2.6. *Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality paring between E' and E. Let $J \colon E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous.*

(i) *The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the set*

$$\{w \in E' \colon \langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u\rangle \text{ for all } v \in E\};$$

(ii) *A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.*

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0 \quad \text{for all } v \in E; \tag{2.8}$$

(iii) *A Palais-Smale sequence at level d ($(\mathrm{PS})_d$ sequence for short) for J is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that $J(u_n) \to d$ and there exists a numerical sequence $\tau_n \to 0^+$ with*

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v-u_n\| \quad \textit{for all } v \in E;$$

(iv) *The functional J satisfies the Palais-Smale condition at level d (($(\mathrm{PS})_d$ condition for short)) if all $(\mathrm{PS})_d$ sequences have a convergent subsequence;*

(v) *The effective domain of J is the set $D(J) = \{u \in E \colon J(u) < +\infty\}$.*

In what follows, for each $u \in D(\mathcal{I}_\varepsilon)$, we set the functional $\mathcal{I}'_\varepsilon(u) \colon \mathcal{H}_{\varepsilon,c} \to \mathbb{R}$ given by

$$\langle \mathcal{I}'_\varepsilon(u), z\rangle = \langle \Phi'_\varepsilon(u), z\rangle - \int F'_1(u) z \,\mathrm{d}x \quad \text{for all } z \in \mathcal{H}_{\varepsilon,c},$$

where

$$\mathcal{H}_{\varepsilon,c} = \{u \in \mathcal{H}_\varepsilon \colon u \text{ has compact support}\}.$$

Further, we define

$$\|\mathcal{I}'_\varepsilon(u)\| = \sup\left\{\langle \mathcal{I}'(u), z\rangle : z \in \mathcal{H}_{\varepsilon,c} \text{ and } \|z\|_\varepsilon \leq 1\right\}.$$

If $\|\mathcal{I}'_\varepsilon(u)\|$ is finite, then $\mathcal{I}'_\varepsilon(u)$ may be extended to a bounded operator in $\mathcal{H}_\varepsilon$, and so, it can be seen as an element of $\mathcal{H}'_\varepsilon$.

Lemma 2.7. *Let $\mathcal{I}_\varepsilon$ satisfies (2.5).*

(i) *If $u \in D(\mathcal{I}_\varepsilon)$ is a critical point of $\mathcal{I}_\varepsilon$. Then, it holds*

$$\langle \Phi'_\varepsilon(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0 \quad \textit{for all } v \in \mathcal{H}_\varepsilon,$$

or equivalently

$$(u, v-u)_\varepsilon + \int_{\mathbb{R}^3} F_1(v)\,\mathrm{d}x - \int_{\mathbb{R}^3} F_1(u)\,\mathrm{d}x \geq \int_{\mathbb{R}^3} F'_2(u)(v-u)\,\mathrm{d}x$$

for all $v \in \mathcal{H}_\varepsilon$;

(ii) *For each $u \in D(\mathcal{I}_\varepsilon)$ such that $\|\mathcal{I}'_\varepsilon(u)\| < +\infty$, we have $\partial\mathcal{I}_\varepsilon(u) \neq \emptyset$, that is, there exists $w \in \mathcal{H}'_\varepsilon$, which is denoted by $w = \mathcal{I}'_\varepsilon(u)$, such that*

$$\langle \Phi'_\varepsilon(u), v-u\rangle + \int_{\mathbb{R}^3} F_1(v)\,\mathrm{d}x - \int_{\mathbb{R}^3} F_1(u)\,\mathrm{d}x \geq \langle w, v-u\rangle$$

for all $v \in \mathcal{H}_\varepsilon$;

(iii) *If a function $u \in D(\mathcal{I}_\varepsilon)$ is a critical point of $\mathcal{I}_\varepsilon$, then u is a solution of (2.3);*

(iv) *If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon$ is a Palais-Smale sequence, then*

$$\langle \mathcal{I}'_\varepsilon(u_n), z\rangle = o_n(1)\|z\|_\varepsilon \quad \textit{for all } z \in \mathcal{H}_{\varepsilon,c};$$

(v) *If Ω is a bounded domain with regular boundary, then Ψ (and hence $\mathcal{I}_\varepsilon$) is of class C^1 in $H^s(\Omega)$. More precisely, the functional*

$$\Psi(u) = \int_\Omega F_1(u)\,\mathrm{d}x \quad \textit{for all } u \in H^s(\Omega)$$

belongs to $C^1(H^s(\Omega), \mathbb{R})$.

Proof. (i) This follows from (2.8).

(ii) This can be obtained arguing as in the proof of Squassina-Szulkin [42] and recalling that $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}_\varepsilon$.

(iii) and (iv) This can be shown by following the same lines of the proofs of Ji-Szulkin [28].

(v) Since $|F_1'(t)| \leq C(1+|t|^{q-1})$ with $q \in (2, 2_s^*)$, it is enough to proceed as in the proof of Willem [46]. □

As a consequence of the above properties, we have the following result.

Lemma 2.8. *If $u \in D(\mathcal{I}_\varepsilon)$ and $\|\mathcal{I}_\varepsilon'(u)\| < +\infty$, then $F_1'(u)u \in L^1(\mathbb{R}^3)$.*

Proof. Let $\varpi \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \leq \varpi \leq 1$ in $\mathbb{R}^3$, $\varpi(x) = 1$ for $|x| \leq 1$ and $\varpi(x) = 0$ for $|x| \geq 2$. For $R > 0$ and $u \in D(\mathcal{I}_\varepsilon)$, let $\varpi_R(x) = \varpi(\frac{x}{R})$ and $u_R(x) = \varpi_R(x)u(x)$. Let us prove that

$$\lim_{R\to\infty} \|u_R - u\|_\varepsilon = 0. \tag{2.9}$$

Clearly, $u_R \to u$ in $L^2(\mathbb{R}^3)$. On the other hand,

$$\begin{aligned}
[u_R - u]^2 &\leq 2\left[\iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}} |u(x)|^2 \,\mathrm{d}x\,\mathrm{d}y \right.\\
&\qquad \left. + \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} |\varpi_R(x) - 1|^2 \,\mathrm{d}x\,\mathrm{d}y\right]\\
&= 2\left[\mathcal{A}_R + \mathcal{B}_R\right].
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{A}_R &= \iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}} |u(x)|^2 \,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}} \,\mathrm{d}x \right.\\
&\qquad \left. + \int_{|x-y|\leq R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}} \,\mathrm{d}x\right) \mathrm{d}y\\
&\leq \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{4\|\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}} \,\mathrm{d}x \right.\\
&\qquad \left. + R^{-2} \int_{|x-y|\leq R} \frac{\|\nabla\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}} \,\mathrm{d}x\right) \mathrm{d}y\\
&\leq C \int_{\mathbb{R}^3} |u(x)|^2 \,\mathrm{d}y \left(\int_R^\infty \frac{1}{r^{2s+1}} \,\mathrm{d}r + R^{-2} \int_0^R \frac{1}{r^{2s-1}} \,\mathrm{d}r\right)\\
&\leq \frac{C}{R^{2s}}.
\end{aligned}$$

Thus, $0 \leq \mathcal{A}_R \to 0$. Moreover, $\mathcal{B}_R \to 0$ by the dominated convergence theorem. Then, (2.9) holds.

From Lemma 2.7 (ii),

$$\langle \Phi_\varepsilon'(u), u_R\rangle + \int_{\mathbb{R}^3} F_1'(u)u_R \,\mathrm{d}x = \langle w, u_R\rangle \quad \text{for all } w \in \mathcal{H}_\varepsilon'. \tag{2.10}$$

Then, combining (2.9), (2.10) with Lemma 2.7 (v), we can see that $\int_{\mathbb{R}^3} F_1'(u)u_R \,\mathrm{d}x \leq C$ for large $R > 0$. From $u_R \to u$ a.e. in $\mathbb{R}^3$ as $R \to \infty$ and Fatou's lemma, we derive that

$$\int_{\mathbb{R}^3} F_1'(u)u \,\mathrm{d}x \leq \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)u_R \,\mathrm{d}x \leq \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)u\varpi_R \,\mathrm{d}x \leq C.$$

This finishes the proof. □

An immediate consequence of the last lemma are the following results.

Corollary 2.9. *For each $u \in D(\mathcal{I}_\varepsilon) \setminus \{0\}$ with $\|\mathcal{I}_\varepsilon'(u)\| < +\infty$, we have that*

$$\mathcal{I}_\varepsilon'(u)u = [u]^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} u^2 \log u^2 \,\mathrm{d}x$$

and

$$\mathcal{I}_\varepsilon(u) - \frac{1}{2}\mathcal{I}_\varepsilon'(u)u = \frac{1}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x.$$

Corollary 2.10. *If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon$ is a (PS) sequence for $\mathcal{I}_\varepsilon$, then $\mathcal{I}_\varepsilon'(u_n)u_n = o_n(1)\|u_n\|_\varepsilon$. If $\{u_n\}_{n\in\mathbb{N}}$ is bounded, we have*

$$\begin{aligned}\mathcal{I}_\varepsilon(u_n) &= \mathcal{I}_\varepsilon(u_n) - \frac{1}{2}\mathcal{I}_\varepsilon'(u_n)u_n + o_n(1)\|u_n\|_\varepsilon \\ &= \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \,\mathrm{d}x + o_n(1)\|u_n\|_\varepsilon\end{aligned}$$

for all $n \in \mathbb{N}$.

Corollary 2.11. *If $u \in \mathcal{H}_\varepsilon$ is a critical point of $\mathcal{I}_\varepsilon$ and $v \in \mathcal{H}_\varepsilon$ verifies $F_1'(u)v \in L^1(\mathbb{R}^3)$, then $\mathcal{I}_\varepsilon'(u)v = 0$.*

Finally, we give a version of the Mountain Pass Theorem without supposing the (PS) condition for the functional $I(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1$ and Ψ is convex and lower semicontinuous. The theorem was first introduced by Alves-de Morais Filho [2].

Proposition 2.12. *Let E be a Banach space and $I\colon E \to (-\infty, +\infty]$ be a functional satisfying*

(i) *$I(u) = I_1(u) + I_2(u)$ with $I_1 \in C^1(E, \mathbb{R})$, and $I_2\colon E \to (-\infty, +\infty]$ is convex, lower semicontinuous and $I_2(u) \not\equiv +\infty$;*
(ii) *$I(0) = 0$ and $I|_{\partial B_{r_0}(0)} \geq b_0$ for some $r_0, b_0 > 0$;*
(iii) *$I(e) \leq 0$ for some $e \notin \bar{B}_{r_0}(0)$.*

Then for given $\tau > 0$, there exists $u_\tau \in E$ such that

$$\langle I_1'(u_\tau), w - u_\tau\rangle + I_2(w) - I_2(u_\tau) \geq -3\tau\|w - u_\tau\| \quad \text{for all } w \in E,$$

and

$$I(u_\tau) \in [c - \tau, c + \tau],$$

where

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], E)\colon \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Corollary 2.13. *Under the conditions of Proposition 2.12, there exists a* (PS) *sequence* $\{u_n\}_{n\in\mathbb{N}} \subset E$ *for* I, *i.e., for any* $w \in E$,

$$I(u_n) \to c \quad \textit{and} \quad \langle I_1'(u_n), w - u_n\rangle + I_2(w) - I_2(u_n) \geq -\sigma_n \|w - u_n\|, \tag{2.11}$$

with $\sigma_n \to 0^+$.

3. An auxiliary problem

In order to prove our main theorem, we will modify problem (2.3) and then consider the existence of solutions to the auxiliary problem. For our problem, since the logarithmic nonlinearity fulfills $t\log t^2 + t \neq o(t)$ as $t \to 0$ and due to the occurrence of the nonlocal term ϕ_u^t, we cannot apply directly the penalization argument employed in Del Pino-Felmer [20]. Here, we give a new scheme, which is proposed in Peng [34].

For $u \in \mathcal{H}_\varepsilon$ with $u > 0$, let

$$\tilde{F}_2'(u) = \phi_u^t u + F_2'(u).$$

Choose suitable $\theta \in \left(0, \frac{2}{3}\right)$ satisfying $V_0 + 1 \geq 4\theta$ and define

$$\tilde{F}_*'(u) = \begin{cases} \tilde{F}_2'(u) & \text{if } \tilde{F}_2'(u) \leq \theta u, \\ \theta u & \text{if } \tilde{F}_2'(u) > \theta u, \end{cases}$$

where V_0 is given in (V_1). We put

$$G_2'(x,u) = \chi_\Omega(x)\tilde{F}_2'(u) + (1 - \chi_\Omega(x))\,\tilde{F}_*'(u),$$

where χ_Ω is the characteristic function of the set Ω.

For $\varepsilon > 0$, we define the auxiliary problem

$$(-\Delta)^s + (V(\varepsilon x) + 1)\,u = G_2'\left(\varepsilon x, u^+\right) - F_1'\left(u^+\right) \quad \text{for } x \in \mathbb{R}^3, \tag{3.1}$$

where $u^+ = \max\{u(x), 0\}$.

Remark 3.1. *Note that if* u_ε *is a positive solution of* (3.1) *satisfying* $\tilde{F}_2'(u) \leq \theta u$ *for each* $x \in \Omega_\varepsilon^c$, *then it is also a positive solution of* (2.3) *and consequently, the pair* $(u_\varepsilon, \phi_{u_\varepsilon}) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ *is a positive solution of system* (1.5), *where* $\Omega_\varepsilon^c = \{x \in \mathbb{R}^3 \colon \varepsilon x \notin \Omega\}$.

It is clear that weak solutions of (3.1) are positive critical points of the following energy functional

$$\mathcal{J}_\varepsilon(u) := \frac{1}{2}\|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} F_1(u^+)\,\mathrm{d}x - \int_{\mathbb{R}^3} G_2\left(\varepsilon x, u^+\right)\,\mathrm{d}x,$$

in the sub-differential sense while $G_2\left(\varepsilon x, u^+\right)$ is given by

$$G_2\left(\varepsilon x, u^+\right) = \begin{cases} \frac{1}{4}\phi_{u^+}^t \left|u^+\right|^2 + F_2\left(u^+\right), & \text{if } x \in \Omega_\varepsilon \cup \left\{\tilde{F}_2'(u) \leq \theta u\right\}, \\ \frac{\theta}{2}\left|u^+\right|^2, & \text{if } x \in \Omega_\varepsilon^c \cap \left\{\tilde{F}_2'(u) > \theta u\right\}. \end{cases}$$

Let $\mathcal{H}_\varepsilon^+$ be the open subset of $\mathcal{H}_\varepsilon$ given by

$$\mathcal{H}_\varepsilon^+ = \left\{u \in \mathcal{H}_\varepsilon \colon \left|\operatorname{supp}\left(u^+\right) \cap \Omega_\varepsilon\right| > 0\right\}.$$

The next lemma implies that $\mathcal{J}_\varepsilon$ possesses the Mountain Pass geometry.

Lemma 3.2. *Assume that* (V_1) *and* (V_2) *hold. Then we have the following:*

(i) *There exist* $\alpha, \rho > 0$ *such that* $\mathcal{J}_\varepsilon(u) \geq \rho$ *with* $\|u\|_\varepsilon = \alpha$;

(ii) *There exists $e \in \mathcal{H}_\varepsilon$ such that $\|u\|_\varepsilon > \alpha$ and $\mathcal{J}_\varepsilon(e) < 0$.*

Proof. Noting that $G_2(\varepsilon x, u) = \frac{\theta}{2}u^2$ for $\Omega_\varepsilon^c \cap \left\{\tilde{F}_2'(u) > \theta u\right\}$, the fact that $\theta \leq \frac{V_0+1}{4}$ and $F_1(u^+) \geq 0$, we have

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{4}\|u\|_\varepsilon^2 - \frac{1}{4}\int_{\Omega_\varepsilon \cup \left\{\tilde{F}_2' \leq \theta u\right\}} \phi_{u^+}^t \left|u^+\right|^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon \cup \left\{\tilde{F}_2' \leq \theta u\right\}} F_2\left(u^+\right) \,\mathrm{d}x,$$

which follows from Lemma 2.3 and (2.7) for $p \in (2, 2_s^*)$ that

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{4}\|u\|_\varepsilon^2 - C_0\|u\|_\varepsilon^4 - C_1\|u\|_\varepsilon^p.$$

The claim follows if we choose ρ and $\|u\|_\varepsilon = \alpha$ small enough.

On the other hand, fixing $\varphi \in C_c^\infty(\Omega_\varepsilon) \setminus \{0\}$, by Lemma 2.3, we have

$$\begin{aligned}
\mathcal{J}_\varepsilon(\tau\varphi) &= \frac{\tau^2}{2}\|\varphi\|_\varepsilon^2 - \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_\varphi^t |\varphi|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} \tau^2\varphi^2 \log(|\tau\varphi|^2) \,\mathrm{d}x \\
&\leq \frac{\tau^2}{2}\|\varphi\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^3} \varphi^2 \log(|\tau\varphi|^2) \,\mathrm{d}x \\
&\leq \tau^2 \left(\mathcal{J}_\varepsilon(\varphi) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_\varphi^t |\varphi|^2 \,\mathrm{d}x - \log(\tau)\int_{\Omega_\varepsilon} \varphi^2 \,\mathrm{d}x\right).
\end{aligned}$$

As $\tau \to +\infty$, then

$$\mathcal{J}_\varepsilon(\tau\varphi) \to -\infty,$$

and the proof of the lemma is now complete. □

By Proposition 2.12 and Corollary 2.13, there exists a (PS) sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon$ of $\mathcal{J}_\varepsilon$ at the level $c_\varepsilon > 0$, where

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \mathcal{J}_\varepsilon(\gamma(t)),$$

and $\Gamma_\varepsilon := \left\{\gamma \in C^1([0,1], \mathcal{H}_\varepsilon) \colon \gamma(0) = 0, \mathcal{J}_\varepsilon(\gamma(1)) < 0\right\}$.

Now, we will prove some results that will be useful in the proof of Theorem 1.1.

Lemma 3.3. *For any $\varepsilon > 0$, all (PS) sequences of $\mathcal{J}_\varepsilon$ are bounded in $\mathcal{H}_\varepsilon$.*

Proof. We define

$$\Lambda_\varepsilon^1 = \left\{\tilde{F}_2'(u) \leq \theta u\right\}, \quad \Lambda_\varepsilon^2 = \Omega_\varepsilon^c \cap \left\{\tilde{F}_2'(u) > \theta u\right\} \quad \text{and} \quad \Lambda_\varepsilon = \Lambda_\varepsilon^1 \cup \Lambda_\varepsilon^2.$$

Then, it holds

$$\mathbb{R}^3 = \Omega_\varepsilon \cup \Lambda_\varepsilon^1 \cup \Lambda_\varepsilon^2 = \Omega_\varepsilon \cup \Lambda_\varepsilon,$$

and

$$\mathcal{J}_\varepsilon(u_n) = \mathcal{J}_\varepsilon|_{\Omega_\varepsilon}(u_n) + \mathcal{J}_\varepsilon|_{\Lambda_\varepsilon}(u_n).$$

Without any loss of generality we will assume that $u_n^+ \neq 0$, because otherwise, we have the inequality

$$\|u_n\|_\varepsilon \leq 2M.$$

In the following, we analyze the two cases:

Case I: For any $x \in \Omega_\varepsilon$, inspired by Alves-Ji [5], we have that $\mathcal{J}_\varepsilon|_{\Omega_\varepsilon} = \mathcal{I}_\varepsilon|_{\Omega_\varepsilon} \in C^1(\Omega_\varepsilon, \mathbb{R})$. Then there is $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \,\mathrm{d}x &= 2\mathcal{I}_\varepsilon|_{\Omega_\varepsilon}(u_n) - \mathcal{I}_\varepsilon'|_{\Omega_\varepsilon}(u_n)u_n \\ &= 2c_\varepsilon + o_n(1) + o_n(1)\left\|u_n\right\|_\varepsilon \\ &\le C + o_n(1)\left\|u_n\right\|_\varepsilon .\end{aligned}$$

Consequently,

$$\|u_n\|_\varepsilon^2 \le C + o_n(1)\|u_n\|_\varepsilon. \tag{3.2}$$

Instead of the classical logarithmic Sobolev inequality of Lieb-Loss [29], we use the fractional logarithmic Sobolev inequality recently established by Chatzakou-Ruzhansky [13]. According to Theorem 1.1 in [13], for any $a > 0$ and $u \in H^s(\mathbb{R}^3)$, there holds

$$\int_{\mathbb{R}^3} u^2 \log\left(\frac{u^2}{|u|_2^2}\right) dx + \frac{3}{s}(1+\log a)|u|_2^2 \le C(3,s,a)|(-\Delta)^{s/2}u|_2^2.$$

Noting that $|(-\Delta)^{s/2}u|_2^2 = C_{n,s}[u]^2$ and $\log(u^2/|u|_2^2) = \log u^2 - \log|u|_2^2$, we can rewrite this inequality as

$$\int_{\mathbb{R}^3} u^2 \log u^2 dx \le \kappa[u]^2 + C_\kappa |u|_2^2 + |u|_2^2 \log|u|_2^2, \tag{3.3}$$

where κ can be chosen arbitrarily small by adjusting the parameter a. Choosing $\kappa \in (0,1)$, $\xi \in (0,1)$, and using the fact that $|u_n|_2^2 \le \|u_n\|_\varepsilon^2$ along with (3.2), the inequality (3.3) yields

$$\begin{aligned}\int_{\Omega_\varepsilon} u_n^2 \log u_n^2 \,\mathrm{d}x &\le \kappa[u_n]^2 + C_\kappa|u_n|_2^2 + |u_n|_2^2\log|u_n|_2^2 \\ &\le \kappa\left\|u_n\right\|_{\Omega_\varepsilon}^2 + C_1\left(1 + \left\|u_n\right\|_{\Omega_\varepsilon}\right)^{1+\xi},\end{aligned} \tag{3.4}$$

where

$$\|u\|_{\Omega_\varepsilon}^2 = \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_\varepsilon} (V(\varepsilon x)+1)\, u^2 \,\mathrm{d}x.$$

Then by (3.4), we have that

$$\begin{aligned}c_\varepsilon + o_n(1) &= \mathcal{I}_\varepsilon|_{\Omega_\varepsilon}(u_n) - \frac{1}{2}\mathcal{I}_\varepsilon'|_{\Omega_\varepsilon}(u_n)u_n \\ &\ge \frac{1}{4}\|u_n\|_{\Omega_\varepsilon}^2 - \frac{1}{4}\int_{\Omega_\varepsilon} u_n^2 \log u_n^2 \,\mathrm{d}x \\ &\ge \frac{1-\kappa}{4}\|u_n\|_{\Omega_\varepsilon}^2 - C_2\left(1+\|u_n\|_{\Omega_\varepsilon}\right)^{1+\xi},\end{aligned}$$

which implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded for any $x \in \Omega_\varepsilon$ since $\kappa < 1$ and $1 + \xi < 2$.

Case II: For any $x \in \Lambda_\varepsilon$. By the assumption, there exists $M > 0$ such that for any $n \in \mathbb{N}$,

$$\begin{aligned}M &\ge \mathcal{J}_\varepsilon|_{\Lambda_\varepsilon}(u_n) \\ &= \frac{1}{2}\left\|u_n\right\|_{\Lambda_\varepsilon}^2 + \int_{\Lambda_\varepsilon} F_1\left(u_n^+\right) \mathrm{d}x - \int_{\Lambda_\varepsilon} G_2\left(\varepsilon x, u_n^+\right) \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|u_n\|^2_{\Lambda_\varepsilon} + \int_{\Lambda_\varepsilon} F_1\left(u_n^+\right) \mathrm{d}x - \int_{\Lambda_\varepsilon^1}\left(\frac{1}{4}\phi^t_{u_n^+}\left|u_n^+\right|^2 \mathrm{d}x + F_2\left(u_n^+\right)\right)\mathrm{d}x \\
&\quad - \frac{\theta}{2}\int_{\Lambda_\varepsilon^2}\left|u_n^+\right|^2 \mathrm{d}x \\
&\geq \frac{1-\theta}{2}\|u_n\|^2_{\Lambda_\varepsilon} + \int_{\Lambda_\varepsilon} F_1\left(u_n^+\right) \mathrm{d}x - \int_{\Lambda_\varepsilon} F_2\left(u_n^+\right)\mathrm{d}x - \frac{1}{4}\int_{\Lambda_\varepsilon^1}\phi^t_{u_n^+}\left|u_n^+\right|^2 \mathrm{d}x \\
&= \frac{1-\theta}{2}\|u_n\|^2_{\Lambda_\varepsilon} - \frac{1}{2}\int_{\Lambda_\varepsilon}\left|u_n^+\right|^2\log\left|u_n^+\right|^2 \mathrm{d}x - \frac{1}{4}\int_{\Lambda_\varepsilon^1}\phi^t_{u_n^+}\left|u_n^+\right|^2 \mathrm{d}x,
\end{aligned}
$$

where

$$
\|u\|^2_{\Lambda_\varepsilon} = \iint_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Lambda_\varepsilon}\left(V(\varepsilon x)+1\right)u^2\,\mathrm{d}x.
$$

Then, for any $n\in\mathbb{N}$,

$$
(1-\theta)||u_n||^2_{\Lambda_\varepsilon} \leq 2M + \int_{\Lambda_\varepsilon}\left|u_n^+\right|^2\log\left|u_n^+\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\Lambda_\varepsilon^1}\phi^t_{u_n^+}\left|u_n^+\right|^2 \mathrm{d}x. \tag{3.5}
$$

Recalling that for any $x\in\Lambda_\varepsilon^1$, there holds $\phi^t_u u \leq \theta u$, and so

$$
\int_{\Lambda_\varepsilon^1}\phi^t_u u^2\,\mathrm{d}x \leq \int_{\Lambda_\varepsilon^1}\theta u^2\,\mathrm{d}x. \tag{3.6}
$$

Applying the fractional logarithmic Sobolev inequality (3.3) and the estimate (3.6) to (3.5), we obtain

$$
\begin{aligned}
(1-\theta)\|u_n\|^2_{\Lambda_\varepsilon} &\leq 2M + \kappa[u_n]^2 + C_\kappa|u_n|_2^2 + |u_n|_2^2\log|u_n|_2^2 \\
&\quad + \frac{\theta}{2}\int_{\Lambda_\varepsilon^1}|u_n|^2\,\mathrm{d}x \\
&\leq 2M + \kappa\|u_n\|^2_\varepsilon + C_3\left(1+\|u_n\|_\varepsilon\right)^{1+\xi} + \frac{\theta}{2}\|u_n\|^2_{\Lambda_\varepsilon},
\end{aligned}
$$

where we have used the fact that $[u_n]^2 \leq \|u_n\|^2_\varepsilon$ and $|u_n|_2^2 \leq \|u_n\|^2_\varepsilon$, along with (3.6). Recalling that $\|u_n\|^2_\varepsilon = \|u_n\|^2_{\Omega_\varepsilon} + \|u_n\|^2_{\Lambda_\varepsilon}$ and using the boundedness of $\|u_n\|_{\Omega_\varepsilon}$ derived in **Case I**, we have

$$
\|u_n\|^2_\varepsilon \leq C_4 + \|u_n\|^2_{\Lambda_\varepsilon}.
$$

Substituting this into the inequality above, we get

$$
\left(1-\frac{3}{2}\theta-\varepsilon\right)\|u_n\|^2_{\Lambda_\varepsilon} \leq C_5\left(1+\|u_n\|_{\Lambda_\varepsilon}\right)^{1+\xi}.
$$

Since $\theta\in(0,\frac{2}{3})$, we have $1-\frac{3}{2}\theta>0$. By choosing $\varepsilon>0$ sufficiently small such that $1-\frac{3}{2}\theta-\kappa>0$, and noting that $1+\xi<2$, we conclude that $\|u_n\|_{\Lambda_\varepsilon}$ is bounded.

Therefore, combining the results from **Case I** and **Case II**, we have that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. □

Lemma 3.4. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a* (PS) *sequence for $\mathcal{J}_\varepsilon$. Then for any $\zeta>0$, there exists $r=r(\zeta)>R>0$ such that*

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_r}\left(\int_{\mathbb{R}^3}\frac{\left|u_n(x)-u_n(y)\right|^2}{|x-y|^{3+2s}}\,\mathrm{d}y + \left(V(\varepsilon x)+1\right)u_n^2\right)\mathrm{d}x < \zeta.
$$

Proof. We consider $r > R$ and a function $\psi = \psi_r \in C_0^\infty(\mathbb{R}^3)$ such that $\psi \equiv 0$ if $x \in B_r(0)$, $\psi \equiv 1$ if $x \notin B_{2r}(0)$ with $0 \leq \psi(x) \leq 1$, and $|\nabla\psi(x)| \leq \frac{C}{r}$, where C is a constant independent of r, for all $x \in \mathbb{R}^N$. As $\{u_n\}_{n\in\mathbb{N}}$ is bounded, the sequence $\{\psi u_n\}_{n\in\mathbb{N}}$ is also bounded. This shows that $\mathcal{I}_\varepsilon'(u_n)(\psi u_n) = o_n(1)$, namely,

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} (V(\varepsilon x)+1)\,u_n^2\psi(x)\,\mathrm{d}x \\ &= \int_{\Omega_\varepsilon} F_2'(u^+)u_n\psi(x)\,\mathrm{d}x + \int_{\Omega_\varepsilon^c} \tilde{F}_*'(u^+)u_n\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} F_1'(u^+)u_n\psi(x)\,\mathrm{d}x \\ &\quad - \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))\,(\psi(x)-\psi(y))}{|x-y|^{3+2s}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y + o_n(1).\end{aligned}$$

We take $r > R$ such that $\Omega_\varepsilon \subset B_r$. Then, by (2.6) and the definitions of $\tilde{F}_*'$, we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} (V(\varepsilon x)+1)\,u_n^2\psi(x)\,\mathrm{d}x \\ &\leq \theta \int_{\mathbb{R}^3} u_n^2\psi(x)\,\mathrm{d}x - \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))\,(\psi(x)-\psi(y))}{|x-y|^{3+2s}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + o_n(1).\end{aligned} \tag{3.7}$$

Now, we will prove that

$$\lim_{r\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))\,(\psi(x)-\psi(y))}{|x-y|^{3+2s}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.8}$$

By Hölder's inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, we have

$$\begin{aligned}&\left|\iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))\,(\psi(x)-\psi(y))}{|x-y|^{3+2s}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leq \left(\iint_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq C\left(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}.\end{aligned}$$

It is enough to prove

$$\lim_{r\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,|u_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{3.9}$$

in order to show that (3.8) is true. Note that $\mathbb{R}^6$ can be written as

$$\begin{aligned}\mathbb{R}^6 &= \big((\mathbb{R}^3 \setminus B_{2r}) \times (\mathbb{R}^3 \setminus B_{2r})\big) \cup \big((\mathbb{R}^3 \setminus B_{2r}) \times B_{2r}\big) \cup \big(B_{2r} \times \mathbb{R}^3\big) \\ &=: \mathbb{X}_r^1 \cup \mathbb{X}_r^2 \cup \mathbb{X}_r^3,\end{aligned}$$

so,

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} |u_n(x)|^2 \, \mathrm{d}x \, \mathrm{d}y \\ &= \iint_{\mathbb{X}_r^1} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} |u_n(x)|^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \iint_{\mathbb{X}_r^2} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} |u_n(x)|^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \iint_{\mathbb{X}_r^3} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} |u_n(x)|^2 \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{3.10}$$

Since $\psi = 1$ in $\mathbb{R}^3 \setminus B_{2r}$, we have

$$\iint_{\mathbb{X}_r^1} \frac{|\psi(x)-\psi(y)|^2 |u_n(x)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y = 0. \tag{3.11}$$

Let $k > 4$. Then

$$\mathbb{X}_r^2 = \left(\mathbb{R}^3 \setminus B_{2r}\right) \times B_{2r} \subset \left(\left(\mathbb{R}^3 \setminus B_{kr}\right) \times B_{2r}\right) \cup \left((B_{kr} \setminus B_{2r}) \times B_{2r}\right).$$

Let us observe that, if $(x,y) \in \left(\mathbb{R}^3 \setminus B_{kr}\right) \times B_{2r}$, then

$$|x-y| \geq |x| - |y| \geq |x| - 2r > \frac{|x|}{2}.$$

Taking into account that $0 \leq \psi \leq 1, |\nabla\psi| \leq \frac{C}{r}$ and using Hölder inequality, we have

$$\begin{aligned}
&\iint_{\mathbb{X}_r^2} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^3\setminus B_{kr}} \mathrm{d}x \int_{B_{2r}} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y \\
&\quad + \int_{B_{kr}\setminus B_{2r}} \mathrm{d}x \int_{B_{2r}} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y \\
&\leq 2^{5+2s} \int_{\mathbb{R}^3\setminus B_{kr}} \mathrm{d}x \int_{B_{2r}} \frac{|u_n(x)|^2}{|x|^{3+2s}} \mathrm{d}y \\
&\quad + \frac{C}{r^2} \int_{B_{kr}\setminus B_{2r}} \mathrm{d}x \int_{B_{2r}} \frac{|u_n(x)|^2}{|x-y|^{3+2(s-1)}} \mathrm{d}y \\
&\leq Cr^3 \int_{\mathbb{R}^3\setminus B_{kr}} \frac{|u_n(x)|^2}{|x|^{3+2s}} \mathrm{d}x + \frac{C}{r^2}(kr)^{2(1-s)} \int_{B_{kr}\setminus B_{2r}} |u_n(x)|^2 \,\mathrm{d}x \\
&\leq Cr^3 \left(\int_{\mathbb{R}^3\setminus B_{kr}} |u_n(x)|^{2_s^*} \,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \left(\int_{\mathbb{R}^3\setminus B_{kr}} \frac{1}{|x|^{\frac{9}{2s}+3}} \mathrm{d}x\right)^{\frac{2s}{3}} \\
&\quad + \frac{Ck^{2(1-s)}}{r^{2s}} \int_{B_{kr}\setminus B_{2r}} |u_n(x)|^2 \,\mathrm{d}x \\
&\leq \frac{C}{k^3} \left(\int_{\mathbb{R}^3\setminus B_{kr}} |u_n(x)|^{2_s^*} \,\mathrm{d}x\right)^{\frac{2}{2_s^*}} + \frac{Ck^{2(1-s)}}{r^{2s}} \int_{B_{kr}\setminus B_{2r}} |u_n(x)|^2 \,\mathrm{d}x \\
&\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{r^{2s}} \int_{B_{kr}\setminus B_{2r}} |u_n(x)|^2 \,\mathrm{d}x.
\end{aligned} \tag{3.12}$$

Fixing $\delta \in (0,1)$, and we have that

$$\begin{aligned}
&\iint_{\mathbb{X}_r^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y \\
&\leq \int_{B_{2r}\setminus B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y \\
&\quad + \int_{B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y.
\end{aligned} \tag{3.13}$$

Let us estimate (3.13). We have

$$\begin{aligned}
&\int_{B_{2r}\setminus B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3\cap\{y:\,|x-y|<r\}} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y \\
&\leq \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u_n(x)|^2 \,\mathrm{d}x,
\end{aligned}$$

and

$$\int_{B_{2r}\setminus B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3\cap\{y:\,|x-y|\geq r\}} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \mathrm{d}y$$

$$\leq \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x,$$

which shows that

$$\int_{B_{2r}\setminus B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y \leq \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x. \tag{3.14}$$

By the definition of $\psi, \delta \in (0,1)$, and $0 \leq \psi \leq 1$,

$$\begin{aligned}
&\int_{B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y \\
&= \int_{B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3\setminus B_r} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y \\
&\leq 4 \int_{B_{\delta r}} \mathrm{d}x \int_{\mathbb{R}^3\setminus B_r} \frac{|u_n(x)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y \\
&\leq C \int_{B_{\delta r}} |u_n|^2 \, \mathrm{d}x \int_{(1-\delta)r}^{\infty} \frac{1}{r^{1+2s}} \, \mathrm{d}r \\
&= \frac{C}{[(1-\delta)r]^{2s}} \int_{B_{\delta r}} |u_n|^2 \, \mathrm{d}x,
\end{aligned} \tag{3.15}$$

where we have used the fact that if $(x,y) \in B_{\delta r} \times (\mathbb{R}^3 \setminus B_r)$, then $|x-y| > (1-\delta)r$. From (3.13), (3.14) and (3.15), one has

$$\begin{aligned}
&\iint_{\mathbb{X}_r^3} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x + \frac{C}{[(1-\delta)r]^{2s}} \int_{B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x.
\end{aligned} \tag{3.16}$$

From (3.10), (3.11), (3.12) and (3.16), we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{r^{2s}} \int_{B_{kr}\setminus B_{2r}} |u_n(x)|^2 \, \mathrm{d}x \\
&\quad + \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x + \frac{C}{[(1-\delta)r]^{2s}} \int_{B_{\delta r}} |u_n(x)|^2 \, \mathrm{d}x.
\end{aligned} \tag{3.17}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded, by Lemma 2.1, we may assume that $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ for some $u \in H^s(\mathbb{R}^3)$. Taking the limit as $n \to \infty$ in (3.17) and applying Hölder's inequality, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{r^{2s}} \int_{B_{kr}\setminus B_{2r}} |u(x)|^2 \, \mathrm{d}x \\
&\quad + \frac{C}{r^{2s}} \int_{B_{2r}\setminus B_{\delta r}} |u(x)|^2 \, \mathrm{d}x + \frac{C}{[(1-\delta)r]^{2s}} \int_{B_{\delta r}} |u(x)|^2 \, \mathrm{d}x
\end{aligned}$$

$$\leq \frac{C}{k^3} + Ck^2 \left(\int_{B_{kr} \setminus B_{2r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}}$$
$$+ C \left(\int_{B_{2r} \setminus B_{\delta r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}} + C \left(\frac{\delta}{1-\delta} \right)^{2s} \left(\int_{B_{\delta r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}}.$$

Since $u \in L^{2_s^*}(\mathbb{R}^3)$, $k > 4$ and $\delta \in (0,1)$, we have

$$\limsup_{r \to \infty} \int_{B_{kr} \setminus B_{2r}} |u(x)|^{2_s^*} \, \mathrm{d}x = \limsup_{r \to \infty} \int_{B_{2r} \setminus B_{\delta r}} |u(x)|^{2_s^*} \, \mathrm{d}x = 0.$$

Let $\delta = \frac{1}{k}$. We get

$$\limsup_{r \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 \, |\psi(x) - \psi(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq \lim_{k \to \infty} \limsup_{r \to \infty} \left[\frac{C}{k^3} + Ck^2 \left(\int_{B_{kr} \setminus B_{2r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}} \right.$$
$$\left. + C \left(\int_{B_{2r} \setminus B_{\frac{1}{k} r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}} + C \left(\frac{1}{k-1} \right)^{2s} \left(\int_{B_{\frac{1}{k} r}} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}} \right]$$
$$\leq \lim_{k \to \infty} \frac{C}{k^3} + C \left(\frac{1}{k-1} \right)^{2s} \left(\int_{\mathbb{R}^3} |u(x)|^{2_s^*} \, \mathrm{d}x \right)^{\frac{2}{2_s^*}} = 0,$$

which shows that (3.9) holds. By (3.7), (3.8), the definition of ψ, the boundedness of $\{u_n\}_{n \in \mathbb{N}}$ and $\theta \leq \frac{V_0+1}{4}$, we infer that

$$\lim_{r \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^3 \setminus B_r} \left(\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y + (V(\varepsilon x) + 1) \, u_n^2 \right) \mathrm{d}x = 0,$$

which completes the proof. □

Lemma 3.5. *The functional $\mathcal{J}_\varepsilon$ verifies the* (PS) *condition in $\mathcal{H}_\varepsilon$ at any level $c_\varepsilon \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}}$ be a (PS) sequence for $\mathcal{J}_\varepsilon$ at the level c_ε. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, see Lemma 3.3, up to a subsequence, we may assume that

$$u_n \rightharpoonup u \quad \text{in } \mathcal{H}_\varepsilon.$$

By Lemma 3.4, for each $\zeta > 0$, there is $r = r(\zeta) > 0$ such that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3 \setminus B_r} \left(\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y + (V(\varepsilon x) + 1) \, u_n^2 \right) \mathrm{d}x < \zeta. \tag{3.18}$$

By (3.18) and the fact that $\mathcal{H}_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$, we can deduce that $u_n \to u$ in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$. Furthermore, from the definition of G_2', Lemma 2.3 and (2.7), one has

$$\int_{\mathbb{R}^3} G_2'(\varepsilon x, u_n^+) \omega \, \mathrm{d}x \to \int_{\mathbb{R}^3} G_2'(\varepsilon x, u^+) \omega \, \mathrm{d}x \quad \text{for all } \omega \in C_0^\infty(\mathbb{R}^3),$$

$$\int_{\mathbb{R}^3} G_2'(\varepsilon x, u_n^+)u_n^+ \,\mathrm{d}x \to \int_{\mathbb{R}^3} G_2'(\varepsilon x, u^+)u^+ \,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^3} G_2(\varepsilon x, u_n^+) \,\mathrm{d}x \to \int_{\mathbb{R}^3} G_2(\varepsilon x, u^+) \,\mathrm{d}x.$$

Since $\mathcal{J}_\varepsilon'(u_n)\omega = o_n(1)$ for all $\omega \in C_c^\infty(\mathbb{R}^3)$, we infer that $\mathcal{J}_\varepsilon'(u)\omega = o_n(1)$ for all $\omega \in C_c^\infty(\mathbb{R}^3)$, and thus $\mathcal{J}_\varepsilon(u)u = 0$, that is,

$$\|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} F_1'(u^+)u^+ \,\mathrm{d}x = \int_{\mathbb{R}^3} G_2'(\varepsilon x, u^+)u^+ \,\mathrm{d}x.$$

On the other hand, using the fact that $\mathcal{J}_\varepsilon(u_n)u_n = o_n(1)$, namely

$$\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3} F_1'(u_n^+)u_n^+ \,\mathrm{d}x = \int_{\mathbb{R}^3} G_2'(\varepsilon x, u_n^+)u_n^+ \,\mathrm{d}x + o_n(1),$$

we can deduce that

$$\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3} F_1'(u_n^+)u_n^+ \,\mathrm{d}x = \|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} F_1'(u^+)u^+ \,\mathrm{d}x + o_n(1).$$

Therefore, $u_n \to u$ in $\mathcal{H}_\varepsilon$ and $F_1'(u_n^+)u_n^+ \to F_1'(u^+)u^+$ in $L^1(\mathbb{R}^3)$. Applying the Dominated Convergence Theorem and (2.6), it is easy to check that $F_1(u_n^+) \to F_1(u^+)$ in $L^1(\mathbb{R}^3)$. Consequently, $0 \in \partial \mathcal{J}_\varepsilon(u)$ and $\mathcal{J}_\varepsilon(u) = c_\varepsilon$. □

Lemma 3.6. *The functional $\mathcal{J}_\varepsilon$ has a positive critical point $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon$.*

Proof. In view of Lemmas 3.2–3.5, there exists a critical point $u_\varepsilon \in \mathcal{H}_\varepsilon$ of $\mathcal{J}_\varepsilon$ such that $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon$ for each $\varepsilon > 0$. In particular, taking $v = u_\varepsilon \pm \tau\omega$ with $\tau > 0$ and $\omega \in C_c^\infty(\mathbb{R}^3)$ in the definition of a critical point of $\mathcal{J}_\varepsilon$, we can prove that $\mathcal{J}_\varepsilon'(u_\varepsilon)\omega = 0$ for all $\omega \in C_c^\infty(\mathbb{R}^3)$ and by the density $\mathcal{J}_\varepsilon'(u_\varepsilon)v = 0$ for all $v \in \mathcal{H}_\varepsilon$. Let us show that $u_\varepsilon \geq 0$ in $\mathbb{R}^3$. Let $u_\varepsilon^- := \min\{u_\varepsilon, 0\}$, since $\mathcal{J}_\varepsilon'(u_\varepsilon)u_\varepsilon^- = 0$ and the definitions of F_1 and G_2, we derive that $u_\varepsilon^- = 0$. By performing a Moser iteration argument (see Lemma 3.13 below) and Silvestre [40], we can prove that $u_\varepsilon \in L^p(\mathbb{R}^3) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $p \in [2, \infty)$. From the strong maximum principle in Silvestre [40], we can conclude that $u_\varepsilon > 0$ in $\mathbb{R}^3$. □

Let us introduce

$$\mathcal{N}_\varepsilon = \{u \in D(\mathcal{J}_\varepsilon) \setminus \{0\} \colon \mathcal{J}_\varepsilon'(u)u = 0\}.$$

Lemma 3.7. *For each $u \in \mathcal{H}_\varepsilon^+$, let $h_u \colon \mathbb{R}^+ \to \mathbb{R}$ be defined as $h_u(\tau) = \mathcal{J}_\varepsilon(\tau u)$. Then, there exists a unique $\tau_u > 0$ such that*

$$\begin{aligned} h_u'(\tau) &> 0 \quad \text{for } \tau \in (0, \tau_u),\\ h_u'(\tau) &< 0 \quad \text{for } \tau \in (\tau_u, \infty). \end{aligned}$$

Proof. Arguing as in the proof of Lemma 3.2, we can see that $h_u(0) = 0$, $h_u(\tau) > 0$ for $\tau > 0$ small and $h_u(\tau) < 0$ for $\tau > 0$ large. Hence, $\max_{\tau \in [0,\infty)} h_u(\tau)$ is achieved at some $\tau = \tau_u > 0$ such that $h_u'(\tau_u) = 0$ and $\tau_u u \in \mathcal{N}_\varepsilon$. Next, we prove the uniqueness of τ_u. Suppose by contradiction that there exist $\tau_2 > \tau_1 > 0$ such that $h_u'(\tau_1) = h_u'(\tau_2) = 0$. Thus, for $i = 1, 2$, we have

$$\tau_i\|u\|_\varepsilon^2 - \int_{\Omega_\varepsilon} \tilde{F}_2'(\tau_i u^+)\, u^+ \,\mathrm{d}x - \int_{\Omega_\varepsilon^c} \tilde{F}_*'(\tau_i u^+)\, u^+ \,\mathrm{d}x + \int_{\mathbb{R}^N} F_1'(\tau_i u^+)\, u^+ \,\mathrm{d}x = 0,$$

from which we get

$$\|u\|_{\varepsilon}^{2}=\int_{\Omega_{\varepsilon}}\frac{\tilde{F}_{2}'\left(\tau_{i}u^{+}\right)u^{+}}{\tau_{i}}\,\mathrm{d}x+\int_{\Omega_{\varepsilon}^{c}}\frac{\tilde{F}_{*}'\left(\tau_{i}u^{+}\right)u^{+}}{\tau_{i}}\,\mathrm{d}x-\int_{\mathbb{R}^{N}}\frac{F_{1}'\left(\tau_{i}u^{+}\right)u^{+}}{\tau_{i}}\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}&\int_{\Omega_{\varepsilon}}\left(\frac{\tilde{F}_{2}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{\tilde{F}_{2}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&+\int_{\Omega_{\varepsilon}^{c}}\left(\frac{\tilde{F}_{*}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{\tilde{F}_{*}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&=\int_{\mathbb{R}^{N}}\left(\frac{F_{1}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{F_{1}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x.\end{aligned}$$

Since $u\in\mathcal{H}_{\varepsilon}^{+}$, the left-hand side of the above identity is positive. Concerning the right-hand side, one has

$$\begin{aligned}&\int_{\mathbb{R}^{N}}\left(\frac{F_{1}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{F_{1}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&=\int_{\left\{u^{+}<\frac{\delta}{\tau_{2}}\right\}}\left(\frac{F_{1}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{F_{1}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&\quad+\int_{\left\{\frac{\delta}{\tau_{2}}<u^{+}<\frac{\delta}{\tau_{1}}\right\}}\left(\frac{F_{1}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{F_{1}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&\quad+\int_{\left\{u^{+}>\frac{\delta}{\tau_{1}}\right\}}\left(\frac{F_{1}'\left(\tau_{2}u^{+}\right)u^{+}}{\tau_{2}}-\frac{F_{1}'\left(\tau_{1}u^{+}\right)u^{+}}{\tau_{1}}\right)\mathrm{d}x\\&=\int_{\left\{u^{+}<\frac{\delta}{\tau_{2}}\right\}}\left(u^{+}\right)^{2}\log\left(\frac{\tau_{1}}{\tau_{2}}\right)^{2}\mathrm{d}x\\&\quad+\int_{\left\{\frac{\delta}{\tau_{2}}<u^{+}<\frac{\delta}{\tau_{1}}\right\}}\left(\left(u^{+}\right)^{2}\log\frac{\tau_{1}^{2}\left(u^{+}\right)^{2}}{\delta^{2}}+2u^{+}\left(\frac{\delta}{\tau_{2}}-\left(u^{+}\right)\right)\right)\mathrm{d}x\\&\quad+\int_{\left\{u^{+}>\frac{\delta}{\tau_{1}}\right\}}\left(\frac{1}{\tau_{2}}-\frac{1}{\tau_{1}}\right)2\delta u^{+}\,\mathrm{d}x.\end{aligned}$$

As the right-hand side of the above identity is negative, we get a contradiction and the proof of the lemma is finished. □

Remark 3.8. *In view of Lemma 3.7, for any $u\in\mathcal{H}_{\varepsilon}^{+}$, there exists a unique $\tau_u>0$ such that $\tau_u u\in\mathcal{N}_{\varepsilon}$. On the other hand, if $u\in\mathcal{N}_{\varepsilon}$, then $u\in\mathcal{H}_{\varepsilon}^{+}$. Otherwise, $|\operatorname{supp}\left(u^{+}\right)\cap\Omega_{\varepsilon}|=0$ and*

$$\begin{aligned}[u]^{2}+\int_{\mathbb{R}^{N}}\left(V_{0}+1\right)u^{2}\,\mathrm{d}x\leq\|u\|_{\varepsilon}^{2}&=\int_{\Omega_{\varepsilon}^{c}}\tilde{F}_{*}'\left(u^{+}\right)u^{+}\,\mathrm{d}x-\int_{\mathbb{R}^{N}}F_{1}'\left(u^{+}\right)u^{+}\,\mathrm{d}x\\&\leq\theta\int_{\mathbb{R}^{N}}u^{2}\,\mathrm{d}x\end{aligned}$$

which implies $[u]^{2}+\int_{\mathbb{R}^{N}}\left(V_{0}+1-\theta\right)u^{2}\,\mathrm{d}x\leq0$ and this gives a contradiction because $V_{0}+1-\theta\geq3\theta>0$ and $u\not\equiv0$.

Next, we introduce the following system which is called the limit case of (1.5), that is,

$$\begin{cases} \varepsilon^{2s} (-\Delta)^s u + V_0 u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t} (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.19}$$

Let us consider the energy functional $\mathcal{I}_0 \colon H^s(\mathbb{R}^N) \to (-\infty, \infty]$ given by

$$\begin{aligned} \mathcal{I}_0(u) &= \frac{1}{2}\|u\|_{Y_{V_0}}^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \,\mathrm{d}x \\ &= \frac{1}{2}\|u\|_{Y_{V_0}}^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F_2(u)\,\mathrm{d}x + \int_{\mathbb{R}^N} F_1(u)\,\mathrm{d}x, \end{aligned}$$

where $Y_\mu = H^s(\mathbb{R}^N)$, with $\mu > -1$, is endowed with the norm

$$\|u\|_{Y_\mu}^2 = [u]^2 + (\mu+1)|u|_2^2.$$

Next, we prove the existence of a ground state for system (3.19).

Lemma 3.9. *The system (3.19) has a positive ground state solution u_0 which fulfills*

$$\mathcal{I}_0(u_0) = c_0 = \inf_{u \in \mathcal{N}_0} \mathcal{I}_0(u) = \inf_{u \in D(\mathcal{I}_0)\setminus\{0\}} \max_{\tau \geq 0} \mathcal{I}_0(\tau u),$$

where

$$\mathcal{N}_0 = \left\{ u \in D(\mathcal{I}_0) \setminus \{0\} \colon \mathcal{I}(u) = \frac{1}{2}|u|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \,\mathrm{d}x \right\},$$

and

$$D(\mathcal{I}_0) = \{ u \in Y_{V_0} \colon \mathcal{I}_0(u) < \infty \}.$$

Proof. We use some ideas found in Squassina-Szulkin [41, 42]. Define

$$\Phi_0(u) = \frac{1}{2}\|u\|_{Y_{V_0}}^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F_2(u)\,\mathrm{d}x,$$

and note that

$$\Psi(u) = \int_{\mathbb{R}^3} F_1(u)\,\mathrm{d}x,$$

then,

$$\mathcal{I}_0(u) = \Phi_0(u) + \Psi(u).$$

Let $u \in D(\mathcal{I}_0) \setminus \{0\}$.

Step 1: We claim that the map $\tau \mapsto \mathcal{I}_0(\tau u)$ admits a unique maximum point on $(0,\infty)$. Indeed, we consider the map $\psi_u \colon (0,\infty) \to \mathbb{R}$ given by

$$\psi_u(\tau) = \mathcal{I}_0(\tau u) = \frac{\tau^2}{2}\|u\|_{Y_{V_0}}^2 - \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \,\mathrm{d}x - \frac{\tau^2}{2}\int_{\mathbb{R}^3} u^2 \log(\tau u)^2 \,\mathrm{d}x$$

for $\tau > 0$. It is clear that $\psi_u(\tau) > 0$ for $\tau > 0$ small and $\psi_u(\tau) < 0$ for $\tau > 0$ large. Moreover, there exists a unique $\tau_* > 0$ such that $\psi_u'(\tau_*) = 0$. In fact, for given $u \in Y_{V_0}$, one can observe that

$$\psi_u(\tau) = C_1\tau^2 - C_2\tau^4 - C_3\tau^2 \log \tau^2 \quad \text{and} \quad \psi_u'(\tau) = C_4\tau - C_5\tau^3 - C_6\tau \log \tau^2.$$

We set

$$h(\tau) = C_4 - C_5\tau^2 - C_6 \log \tau^2,$$

then $\psi_u'(\tau) = \tau h(\tau)$. Moreover, it is easy to see that h is a decreasing function in τ and h possesses the unique zero point τ_*. To be more precise, $h(\tau) > 0$ for $\tau \in (0, \tau_*)$ and $h(\tau) < 0$ for $\tau \in (\tau_*, +\infty)$. Hence, one can conclude that there exists a unique point $\tau_* > 0$ such that $\psi_u'(\tau) > 0$ for $\tau \in (0, \tau_*)$ and $\psi_u'(\tau) < 0$ for $\tau \in (\tau_*, +\infty)$. This shows the claim.

Step 2: Since $\psi_u'(\tau) = \mathcal{I}_0'(\tau u)u$, the ray $\{\tau u \colon \tau > 0\}$ intersects the Nehari manifold $\mathcal{N}_0$ at exactly one point. Moreover, there is $\tau_0 > 0$ such that, for all $u \in D(\mathcal{I}_0) \cap \mathbb{S}_1, \tau \mapsto \Phi_0(\tau u)$ is increasing in $(0, \tau_0)$, where $\mathbb{S}_1$ is the unit sphere in Y_{V_0}. Since $\tau \mapsto \Psi(\tau u)$ is increasing for all $\tau > 0$ (by convexity), $\mathcal{N}_0$ is bounded away from the origin. Define

$$\Gamma_0 = \left\{\gamma \in C\left([0,1], Y_{V_0}\right) : \gamma(0) = 0, \mathcal{I}_0(\gamma(1)) < 0\right\},$$

and

$$\tilde{c} = \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} \mathcal{I}_0(\gamma(t)).$$

Since $\Psi \geq 0$ and using (2.7), it is easy to verify that there exist $\alpha, \rho > 0$ such that $\mathcal{I}_0(u) \geq \rho$ for all $u \in Y_{V_0}$ such that $\|u\|_{Y_{V_0}} = \alpha$. In particular, $\tilde{c} \geq \rho > 0$ (and $\tilde{c}$ is the mountain pass level). Clearly, $\tilde{c} \leq c_0$. Suppose by contradiction that there is $\varepsilon_0 \in \left(0, \frac{\tilde{c}}{2}\right)$ such that there are no (PS) sequences in

$$(\mathcal{I}_0)_{\tilde{c}-2\varepsilon_0}^{\tilde{c}+2\varepsilon_0} = \left\{u \in Y_{V_0} \colon \tilde{c} - 2\varepsilon_0 \leq \mathcal{I}_0(u) \leq \tilde{c} + 2\varepsilon_0\right\}.$$

Arguing as in the proof of Ji-Szulkin [28], we achieve a contradiction. Since ε_0 may be chosen arbitrarily small, we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\mathcal{I}_0(u_n) \to \tilde{c}$ and $\mathcal{I}_0'(u_n) \to 0$.

Step 3: We show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in Y_{V_0}. Assume that, for some $d \in \mathbb{R}$, it holds $\mathcal{I}_0(u_n) \leq d$ for all $n \in \mathbb{N}$. Then

$$\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \,\mathrm{d}x = 2\mathcal{I}_0(u_n) - \mathcal{I}_0'(u_n)u_n \leq 2d + o_n(1)\|u_n\|_{Y_{V_0}}.$$

Now, arguing as in the proof of Lemma 3.3 and using inequality (3.4), we see that

$$\begin{aligned} d + o_n(1) &= \mathcal{I}_0(u_n) - \frac{1}{4}\mathcal{I}_0'(u_n)u_n \\ &\geq \frac{1}{4}\|u_n\|_{Y_{V_0}}^2 - \frac{1}{4}\int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x \\ &\geq C_7\left(\|u_n\|_{Y_{V_0}}^2 - (1 + \|u_n\|_{Y_{V_0}})^{1+\xi}\right), \end{aligned}$$

which shows that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in Y_{V_0}.

Step 4: We prove that there exists a critical point $\tilde{u} \neq 0$ of $\mathcal{I}_0$ such that $\mathcal{I}_0(\tilde{u}) \leq \tilde{c}$. This fact implies that $u_0 \in \mathcal{N}_0$ and that $\mathcal{I}_0(u_0) = c_0 = \tilde{c}$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in Y_{V_0}, we may assume that $u_n \rightharpoonup u$ in Y_{V_0}, $u_n \to u$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ for all $p \in [1, 2_s^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Since Ψ is lower semicontinuous and convex, it is also weakly lower semicontinuous. So $\Psi(u) < \infty$ and $u \in D(\mathcal{I}_0)$. It is easy to check that

$$0 = \lim_{n\to\infty} \mathcal{I}_0'(u_n)v = \mathcal{I}_0'(u)v,$$

for all $v \in C_c^\infty(\mathbb{R}^3)$. Moreover, by Fatou's lemma

$$\tilde{c} = \liminf_{n\to\infty}\left[\mathcal{I}_0(u_n) - \frac{1}{2}\mathcal{I}_0'(u_n)u_n\right]$$

$$
\begin{aligned}
&= \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^3} u_n^2\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\,\mathrm{d}x\right]\\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,\mathrm{d}x\\
&= \mathcal{I}_0(u) - \frac{1}{2}\mathcal{I}_0'(u)u\\
&= \mathcal{I}_0(u),
\end{aligned}
$$

which yields $\mathcal{I}_0(u) \leq \tilde{c}$. Now, if $u \not\equiv 0$, we are done. Suppose $u \equiv 0$. If $|u_n|_p \to 0$, using Lemma 2.3, (2.6) and (2.7), then

$$
\begin{aligned}
o_n(1) &= \mathcal{I}_0'(u_n)\,u_n\\
&= \|u_n\|_{Y_{V_0}}^2 - \int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F_2'(u_n)\,u_n\,\mathrm{d}x + \int_{\mathbb{R}^3} F_1'(u_n)\,u_n\,\mathrm{d}x\\
&\geq \|u_n\|_{Y_{V_0}}^2 - C_8\|u_n\|_{Y_{V_0}}^4 - C_9\int_{\mathbb{R}^3}|u_n|^p\,\mathrm{d}x\\
&= \|u_n\|_{Y_{V_0}}^2 - C_8\|u_n\|_{Y_{V_0}}^4 + o_n(1),
\end{aligned}
$$

which yields $\|u_n\|_{Y_{V_0}} \to 0$ or $\|u_n\|_{Y_{V_0}}$ must be larger than some positive constants. If $\|u_n\|_{Y_{V_0}}$ must be larger than some positive constants, it contradicts the assumption. So, $\|u_n\|_{Y_{V_0}} \to 0$. Taking $w = 0$ in (2.11), we get $\Phi_0'(u_n)(-u_n) - \Psi(u_n) \geq -\sigma_n\|u_n\|_{Y_{V_0}}$, with $\sigma_n \to 0$, and thus $\Psi(u_n) \to 0$. Therefore, $\mathcal{I}_0(u_n) \to 0$ which contradicts $\mathcal{I}_0(u_n) \to \tilde{c} > 0$. Thus, $|u_n|_p \nrightarrow 0$ and using Lemma 2.2, we can find $(y_n) \subset \mathbb{R}^3$ and $\delta > 0$ such that

$$
\int_{B_1(y_n)} |u_n|^p\,\mathrm{d}x \geq \delta.
$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$. Clearly, $\mathcal{I}_0(\tilde{u}_n) \to \tilde{c}$ and $\mathcal{I}_0'(\tilde{u}_n) \to 0$. Moreover,

$$
\int_{B_1} |\tilde{u}_n|^p\,\mathrm{d}x = \int_{B_1(y_n)} |u_n|^p\,\mathrm{d}x \geq \delta,
$$

and thus, after passing to a subsequence if necessary, $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in Y_{V_0}. Arguing as before, we can see that $\tilde{u}$ is a nontrivial critical point of $\mathcal{I}_0$ such that $\mathcal{I}_0(\tilde{u}) \leq \tilde{c}$. Hence, we have proved that there exists $u_0 \in \mathcal{N}_0$ such that $\mathcal{I}_0(u_0) = c_0$.

Step 5: Finally, we show that $u_0 > 0$ in $\mathbb{R}^3$. Let $\tau_{u_0} > 0$ be such that

$$
\log \tau_{u_0}^2 = \frac{[|u_0|]^2 - [u_0]^2}{\int_{\mathbb{R}^3} u_0^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_0}^t u_0^2\,\mathrm{d}x},
$$

and so $\tau_{u_0}|u_0| \in \mathcal{N}_0$. Since $[|u_0|] \leq [u_0]$, and noting that the nonlocal term and the logarithmic term depend only on u^2, we have

$$
\mathcal{I}_0(\tau|u_0|) \leq \mathcal{I}_0(\tau u_0) \quad \text{for all } \tau > 0.
$$

Let $\tau_{|u_0|} > 0$ be the unique value such that $\tau_{|u_0|}|u_0| \in \mathcal{N}_0$. By the definition of the ground state level c_0, we have

$$
c_0 \leq \mathcal{I}_0(\tau_{|u_0|}|u_0|) \leq \mathcal{I}_0(\tau_{|u_0|}u_0) \leq \max_{\tau>0}\mathcal{I}_0(\tau u_0) = \mathcal{I}_0(u_0) = c_0.
$$

This chain of inequalities implies that $\mathcal{I}_0(\tau_{|u_0|}u_0) = \mathcal{I}_0(u_0)$. Since the map $\tau \mapsto \mathcal{I}_0(\tau u_0)$ admits a strict global maximum at $\tau = 1$, we deduce that $\tau_{|u_0|} = 1$. Consequently, $|u_0| \in \mathcal{N}_0$ and $\mathcal{I}_0(|u_0|) = c_0$. Therefore, replacing u_0 by $|u_0|$ if necessary, we may assume that $u_0 \geq 0$ in $\mathbb{R}^3$. Since the nonlinearity $f(u) = u\log u^2$ satisfies the subcritical growth condition, standard regularity arguments for fractional elliptic equations (see, for instance, the bootstrap method used in [1] or [21]) imply that $u_0 \in L^\infty(\mathbb{R}^3)$. Consequently, by the regularity results in Silvestre [40], we conclude that u_0 is continuous. Now, assume by contradiction that there exists $x_0 \in \mathbb{R}^3$ such that $u_0(x_0) = 0$. Since $u_0 \geq 0$ and $u_0 \not\equiv 0$, the definition of the fractional Laplacian implies that

$$(-\Delta)^s u_0\,(x_0) = -C_{3,s}\ \text{P.V.} \int_{\mathbb{R}^3} \frac{u_0(y)}{|x_0 - y|^{3+2s}}\,\mathrm{d}y < 0.$$

However, the equation (3.19) gives

$$(-\Delta)^s u_0(x_0) = -V_0 u_0(x_0) + \phi(x_0)u_0(x_0) + u_0(x_0)\log u_0^2(x_0) = 0,$$

where we used the fact that $\lim_{t\to 0^+} t\log t^2 = 0$. This leads to a contradiction. Thus, $u_0(x) > 0$ for all $x \in \mathbb{R}^3$. The proof is complete. $\square$

Next, we show that the mountain pass level c_ε is the ground state level of $\mathcal{I}_\varepsilon$ and we establish an interesting relation between c_ε and c_0.

Lemma 3.10. *The following hold:*

(a) $c_\varepsilon \geq \alpha > 0$ *for all* $\varepsilon > 0$.
(b) $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u)$ *for all* $\varepsilon > 0$.
(c) $\limsup_{\varepsilon\to 0} c_\varepsilon = c_0$.

Proof. Part (a) follows from Lemma 3.2 (i). In order to prove (b), we take $u \in \mathcal{N}_\varepsilon$ and let $\mathcal{J}_\varepsilon(\bar\tau u) < 0$ for some $\bar\tau > 0$. Consider $\gamma_\varepsilon\colon [0,1] \to \mathcal{H}_\varepsilon$ given by $\gamma_\varepsilon(\tau) = \tau\cdot\bar\tau u$. Then

$$c_\varepsilon \leq \max_{\tau\in[0,1]} \mathcal{J}_\varepsilon\left(\gamma_\varepsilon(\tau)\right) \leq \max_{\tau\in[0,1]} \mathcal{J}_\varepsilon(\tau u) = \mathcal{J}_\varepsilon(u), \tag{3.20}$$

and by the arbitrariness of $u \in \mathcal{N}_\varepsilon$, we get $c_\varepsilon \leq \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u)$. Next, we prove the reverse inequality. By Lemma 3.6, we know that there exists $u_\varepsilon \in \mathcal{H}_\varepsilon$ with $u_\varepsilon > 0$ in $\mathbb{R}^3$ such that

$$\mathcal{J}_\varepsilon\left(u_\varepsilon\right) = c_\varepsilon \quad \text{and} \quad 0 \in \partial\mathcal{J}_\varepsilon\left(u_\varepsilon\right).$$

Then, $u_\varepsilon \in \mathcal{N}_\varepsilon$ and it holds

$$c_\varepsilon = \mathcal{J}_\varepsilon\left(u_\varepsilon\right) \geq \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

Let us prove (c). To this end, let $\varphi \in C_c^\infty\left(\mathbb{R}^3\right)$ be such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in B_1 and $\varphi = 0$ in B_2^c. For $R > 0$, we set $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$ and $u_R(x) = \varphi_R(x)u_0(x)$, where u_0 is given by Lemma 3.9. Then, we have $u_R \to u_0$ in $H^s(\mathbb{R}^3)$ as $R \to \infty$. For fixed $R > 0$, we can proceed as in the proof of (3.20) to obtain that, for a fixed $\varepsilon > 0$,

$$c_\varepsilon \leq \max_{\tau\in[0,1]} \mathcal{J}_\varepsilon\left(\tau u_R\right) = \mathcal{J}_\varepsilon(\tau_\varepsilon u_R),$$

and

$$\|u_R\|_\varepsilon^2 = \int_{\Omega_\varepsilon} \frac{\tilde F_2'\left(\tau_\varepsilon u_R\right)u_R}{\tau_\varepsilon}\,\mathrm{d}x + \int_{\Omega_\varepsilon^c} \frac{\tilde F_*'\left(\tau_\varepsilon u_R\right)u_R}{\tau_\varepsilon}\,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{F_1'\left(\tau_\varepsilon u_R\right)u_R}{\tau_\varepsilon}\,\mathrm{d}x.$$

Since $V(\varepsilon x) \to V_0$ as $\varepsilon \to 0$, by the Lebesgue Dominated Convergence Theorem, we have from the left-hand side of the above equality that

$$\lim_{\varepsilon\to 0} \|u_R\|_\varepsilon^2 = \|u_R\|_{Y_{V_0}}^2.$$

Assuming $\tau_\varepsilon \to +\infty$ as $\varepsilon \to 0$, since $\Omega_\varepsilon \to \mathbb{R}^3$ as $\varepsilon \to 0$, it is easy to verify that the right-hand side of the above equality goes to $+\infty$ as $\varepsilon \to 0$, which is a contradiction. Thus, τ_ε is bounded in $\mathbb{R}$ for ε small enough. Now, we notice that

$$\begin{aligned}\mathcal{J}_\varepsilon(\tau_\varepsilon u_R) &= \mathcal{I}_0(\tau_\varepsilon u_R) + \frac{\tau_\varepsilon^2}{2}\int_{\mathbb{R}^3}(V(\varepsilon x) - V_0)\, u_R^2 \,\mathrm{d}x \\ &\quad + \int_{\Omega_\varepsilon} \tilde{F}_2(\tau_\varepsilon u_R)\,\mathrm{d}x + \int_{\Omega_\varepsilon^c} \tilde{F}_*(\tau_\varepsilon u_R)\,\mathrm{d}x - \int_{\mathbb{R}^3} \tilde{F}_2(\tau_\varepsilon u_R)\,\mathrm{d}x \\ &\le \mathcal{I}_0(\tau_R u_R) + \frac{\tau_\varepsilon^2}{2}\int_{\mathbb{R}^3}(V(\varepsilon x) - V_0)\, u_R^2\,\mathrm{d}x,\end{aligned}$$

where $\tau_R > 0$ is such that

$$\mathcal{I}_0(\tau_R u_R) = \max_{\tau\ge 0} \mathcal{I}_0(\tau u_R).$$

Since $\sup_{x\in B_R}|V(\varepsilon x) - V_0| \to 0$ as $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\tau_\varepsilon u_R) \le \mathcal{I}_0(\tau_R u_R). \tag{3.21}$$

Similarly, one can verify that $\{\tau_R\}$ is also bounded for R large, and therefore $|\tau_R| \le k$ for some $k > 0$ and $\tau_R \to \tau_0$ for some $\tau_0 > 0$. Recalling that F_1 is increasing on $(0+\infty)$, we have $F_1(\tau_R u_R) \le F_1(k u_0)$. Moreover, since $u_0 \in \mathcal{N}_0$, we can infer that $F_1(k u_0) \in L^1(\mathbb{R}^3)$. Thus, if $R \to \infty$ and $\tau_R \to \tau_0$, by Lebesgue Dominated Convergence theorem, one has

$$F_1(\tau_R u_R) \to F_1(\tau_0 u_0) \quad \text{in } L^1(\mathbb{R}^3),$$

and

$$F_1'(\tau_R u_R)\tau_R u_R \to F_1'(\tau_0 u_0)\tau_0 u_0 \quad \text{in } L^1(\mathbb{R}^3).$$

We claim that $\tau_0 = 1$. In fact, using $\tau_R u_R \to \tau_0 u_0$ in $H^s(\mathbb{R}^3)$, we reach

$$c_0 \le \lim_{R\to\infty} \mathcal{I}_0(\tau_R u_R) = \mathcal{I}_0(\tau_0 u_0) \le \max_{\tau>0} \mathcal{I}_0(\tau u_0) = \mathcal{I}_0(u_0) = c_0,$$

which implies that $\tau_0 = 1$. Combining this with (3.21), one gets

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \mathcal{I}_0(u_0) = c_0.$$

Insofar as

$$\mathcal{J}_\varepsilon(u) \ge \mathcal{I}_\varepsilon(u) \ge \mathcal{I}_0(u) \quad \text{for all } \varepsilon > 0, u \in D(\mathcal{J}_\varepsilon),$$

and by part (b) with $\varepsilon = 0$, the reverse inequality holds, namely

$$\limsup_{\varepsilon\to 0} c_\varepsilon \ge c_0.$$

Hence,

$$\lim_{\varepsilon\to 0^+} c_\varepsilon = c_0,$$

which concludes the proof. □

Lemma 3.11. *Let $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ be a sequence satisfying $\mathcal{I}_0(\omega_n) \to c_0$, $\omega_n \rightharpoonup \omega \neq 0$ and $\mathcal{I}_0'(\omega)\omega \le 0$, then $\omega_n \to \omega$ in $H^s(\mathbb{R}^3)$.*

Proof. Since $\mathcal{I}_0'(\omega)\omega \leq 0$, there is $\tau \in (0,1]$ such that $\tau\omega \in \mathcal{N}_0$ and so

$$\begin{aligned} c_0 &\leq \mathcal{I}_0(\tau\omega) \\ &= \frac{\tau^2}{2}\int_{\mathbb{R}^3} \omega^2 \,\mathrm{d}x + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_\omega^t \omega^2 \,\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3} \omega_n^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{4}\phi_{\omega_n}^t \omega_n^2 \,\mathrm{d}x\right) \\ &\leq \limsup_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3} \omega_n^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{4}\phi_{\omega_n}^t \omega_n^2 \,\mathrm{d}x\right) \\ &= \lim_{n\to\infty} \mathcal{I}_0(\omega_n) \\ &= c_0. \end{aligned}$$

The above argument yields $\tau = 1$ and $\omega_n \to \omega$ in $L^2(\mathbb{R}^3)$. Since $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $L^{2_s^*}(\mathbb{R}^3)$, by interpolation on the Lebesgue spaces, it follows that

$$\omega_n \to \omega \quad \text{in } L^p(\mathbb{R}^3) \text{ for all } p \in [2, 2_s^*),$$

and therefore $\int \tilde{F}_2'(\omega_n)\,\omega_n \,\mathrm{d}x \to \int \tilde{F}_2'(\omega)\omega \,\mathrm{d}x$. Finally, by the equalities $\mathcal{I}_0'(\omega)\omega = 0$ and

$$\|\omega_n\|_{Y_{V_0}} + \int F_1'(\omega_n)\,\omega_n \,\mathrm{d}x = \int \tilde{F}_2'(\omega_n)\,\omega_n \,\mathrm{d}x,$$

we get

$$\|\omega_n\|_{Y_{V_0}} \to \|\omega\|_{Y_{V_0}} \quad \text{in } H^s(\mathbb{R}^3),$$

from which the desired result follows. □

The next lemma plays an important role in describing the concentration phenomena of solutions with respect to ε.

Lemma 3.12. *Let $\varepsilon_n \to 0$ and $u_n \in \mathcal{H}_{\varepsilon_n}$ satisfying $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ and $\mathcal{J}_{\varepsilon_n}'(u_n) = 0$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $\tilde{u}_n = u_n(x + x_n)$ possesses a convergent subsequence in $H^s(\mathbb{R}^3)$. Moreover, there is $x_0 \in \Omega$ such that*

$$\lim_{n\to\infty} \varepsilon_n x_n = x_0 \quad \text{and} \quad V(x_0) = V_0.$$

Proof. From Lemma 3.10 (c), we observe that $c_{\varepsilon_n} \to c_0 > 0$ as $n \to \infty$. Using the same method in Lemma 3.3, we can prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\varepsilon_n}$.

Step 1: We claim there exist $r, \beta > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \int_{B_r(x_n)} |u_n(x)|^2 \,\mathrm{d}x \geq \beta.$$

Supposing that there is r_0 such that

$$\limsup_{n\to\infty} \int_{B_{r_0}(x_n)} |u_n(x)|^2 \,\mathrm{d}x = 0.$$

Then, by Lemma 2.2, one obtains

$$u_n \to 0 \quad \text{in } L^p(\mathbb{R}^3) \text{ for } p \in (2, 2_s^*).$$

Thus, we have $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n} \to 0$ as $n \to \infty$, which is a contradiction.

Step 2: Set $\tilde{u}_n = u_n(x + x_n)$, then $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded and there exists $\tilde{u} \in H^s(\mathbb{R}^3)$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } H^s(\mathbb{R}^3)$$

and

$$\int_{B_r(0)} |\tilde{u}|^2 \,\mathrm{d}x \geq \beta. \tag{3.22}$$

Hence, $\tilde{u} \neq 0$.

Step 3: In what follows, we prove that the sequence $\{\varepsilon_n x_n\}_{n\in\mathbb{N}} \subset \overline{\Omega}$. Arguing by contradiction and suppose there exist $\varsigma > 0$ and a subsequence of $\{\varepsilon_n x_n\}_{n\in\mathbb{N}}$, not relabeled, such that

$$\operatorname{dist}\left(\varepsilon_n x_n, \overline{\Omega}\right) \geq \varsigma \quad \text{for all } n \in \mathbb{N}.$$

Then, we can find $r > 0$ such that

$$B_r(\varepsilon_n x_n) \subset \Omega^c \quad \text{for all } n \in \mathbb{N}. \tag{3.23}$$

Choosing a nonnegative sequence $\{w_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^3)$ such that

$$w_j \to \tilde{u} \quad \text{in } H^s(\mathbb{R}^3). \tag{3.24}$$

For fixed $j \in \mathbb{N}$, $\mathcal{J}'_{\varepsilon_n}(\tilde{u}_n) w_j = 0$, that is,

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(w_j(x) - w_j(y))}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^3} (V(\varepsilon_n x + \varepsilon_n x_n) + 1)\tilde{u}_n w_j \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} G'_2(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n) w_j \,\mathrm{d}x - \int_{\mathbb{R}^3} F'_1(\tilde{u}_n) w_j \,\mathrm{d}x.
\end{aligned}$$

In view of (3.23) and the definition of G'_2, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} G'_2(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n) w_j \,\mathrm{d}x \\
&= \int_{B_{\frac{r}{\varepsilon_n}}(0)} G'_2(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n) w_j \,\mathrm{d}x + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} G'_2(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n) w_j \,\mathrm{d}x \\
&\leq \theta \int_{B_{\frac{r}{\varepsilon_n}}(0)} \tilde{u}_n w_j \,\mathrm{d}x + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F'_2(\tilde{u}_n) w_j \,\mathrm{d}x + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} \phi^t_{\tilde{u}_n} \tilde{u}_n w_j \,\mathrm{d}x,
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(w_j(x) - w_j(y))}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^3} (V(\varepsilon_n x + \varepsilon_n x_n) + 1)\tilde{u}_n w_j \,\mathrm{d}x \\
&\leq \theta \int_{B_{\frac{r}{\varepsilon_n}}(0)} \tilde{u}_n w_j \,\mathrm{d}x + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F'_2(\tilde{u}_n) w_j \,\mathrm{d}x \\
&\quad + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} \phi^t_{\tilde{u}_n} \tilde{u}_n w_j \,\mathrm{d}x - \int_{\mathbb{R}^3} F'_1(\tilde{u}_n) w_j \,\mathrm{d}x.
\end{aligned}$$

Since $\theta \leq \frac{V_0+1}{4}$ and $F_1'(\tilde{u}_n) w_j > 0$, we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(w_j(x)-w_j(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \tilde{V}\int_{\mathbb{R}^3} \tilde{u}_n w_j \,\mathrm{d}x \\ &\leq \int_{B^c_{\frac{r}{\varepsilon_n}}} F_2'(\tilde{u}_n) w_j \,\mathrm{d}x + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} \phi^t_{\tilde{u}_n} \tilde{u}_n w_j \,\mathrm{d}x,\end{aligned}$$

where $\tilde{V} = V_0 + 1 - \theta$. By Lemma 2.3, (2.7) and the boundedness of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, we can infer that

$$\int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F_2'(\tilde{u}_n) w_j \,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} \phi^t_{\tilde{u}_n} \tilde{u}_n w_j \,\mathrm{d}x \to 0 \quad \text{as } n\to\infty,$$

and

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(w_j(x)-w_j(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \tilde{V}\int_{\mathbb{R}^3} \tilde{u}_n w_j \,\mathrm{d}x \\ &\longrightarrow \iint_{\mathbb{R}^6} \frac{(\tilde{u}(x)-\tilde{u}(y))(w_j(x)-w_j(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \tilde{V}\int_{\mathbb{R}^3} \tilde{u} w_j \,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\iint_{\mathbb{R}^6} \frac{(\tilde{u}(x)-\tilde{u}(y))(w_j(x)-w_j(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \tilde{V}\int_{\mathbb{R}^3} \tilde{u} w_j \,\mathrm{d}x \leq 0.$$

Let $j\to\infty$ in the above inequality, it follows from (3.24) that

$$[\tilde{u}]^2 + \tilde{V}\int_{\mathbb{R}^3} \tilde{u}^2 \,\mathrm{d}x = 0,$$

which contradicts (3.22). Therefore, we conclude that $\{\varepsilon_n x_n\}_{n\in\mathbb{N}} \subset \overline{\Omega}$.

Step 4: From this fact, we can find $x_0 \in \overline{\Omega}$ such that

$$\varepsilon_n x_n \to x_0 \quad \text{as } n\to\infty.$$

Following the spirit of (c) in Lemma 3.10, we can prove that $x_0 \in \Omega$. Indeed, since $\tilde{u}_n$ is a solution, by the definition of G_2', for any $\psi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(\psi(x)-\psi(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} (V(\varepsilon_n x + \varepsilon_n x_n)+1)\tilde{u}_n \psi \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} F_2'(\tilde{u}_n)\psi \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi^t_{\tilde{u}_n}\tilde{u}_n\psi \,\mathrm{d}x - \int_{\mathbb{R}^3} F_1'(\tilde{u}_n) w_j \,\mathrm{d}x.\end{aligned}$$

Using $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ and $\varepsilon_n x_n \to x_0$, there holds

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(\psi(x)-\psi(y))}{|x-y|^{3+2s}} \mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} (V(x_0)+1)\tilde{u}\psi \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} F_2'(\tilde{u})\psi \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi^t_{\tilde{u}}\tilde{u}\psi \,\mathrm{d}x - \int_{\mathbb{R}^3} F_1'(\tilde{u}) w_j \,\mathrm{d}x,\end{aligned}$$

which deduces

$$\begin{aligned}&[\tilde{u}] + \int_{\mathbb{R}^3} (V(x_0)+1)\tilde{u}^2 \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} F_2'(\tilde{u})\tilde{u} \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi^t_{\tilde{u}}\tilde{u}^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F_1'(\tilde{u})\tilde{u} \,\mathrm{d}x.\end{aligned} \tag{3.25}$$

Hence, there exists $\tau_1 \in (0,1]$ such that

$$\tau_1 \tilde{u} \in \mathcal{N}_{V(x_0)} = \left\{ u \in H^s\left(\mathbb{R}^3\right) \setminus \{0\} \colon \mathcal{I}'_{V(x_0)}(u)u = 0 \right\},$$

where

$$\mathcal{I}_{V(x_0)} = \frac{1}{2}\|u\|^2_{Y_{V(x_0)}} - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \, \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \, \mathrm{d}x.$$

Denoted by $c_{V(x_0)}$ the least energy of $\mathcal{I}_{V(x_0)}$, then

$$c_{V(x_0)} \le \mathcal{I}_{V(x_0)}\left(\tau_1 \tilde{u}\right) \le \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}\left(u_n\right) = \liminf_{n\to\infty} c_{\varepsilon_n} = c_0,$$

which implies that $c_{V(x_0)} \le c_0$. Thus we have $V\left(x_0\right) \le V_0$. Since $V(x) \ge V_0$ for any $x \in \Omega$ and $V(x) > V_0$ for any $x \in \partial\Omega$, we infer that $V\left(x_0\right) = V_0$ and $x_0 \in \Omega$.

Step 5: In the sequel, we are going to prove that

$$\tilde{u}_n \to \tilde{u} \quad \text{in } H^s(\mathbb{R}^3).$$

For each $n \in \mathbb{N}$, there exists $\zeta_n \in \mathbb{R}$ such that $\bar{u}_n = \zeta_n \tilde{u}_n \in \mathcal{N}_0$. Then,

$$c_0 \le \mathcal{I}_0\left(\bar{u}_n\right) \le \max_{\tau\ge 0} \mathcal{J}_{\varepsilon_n}\left(\tau \tilde{u}_n\right) = \mathcal{J}_{\varepsilon_n}\left(u_n\right) = c_{\varepsilon_n}.$$

Then (c) of Lemma 3.10 gives that $\mathcal{I}_0\left(\bar{u}_n\right) \to c_0$. Moreover, one can easily prove that $\{\bar{u}_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ are both bounded. Thus, we can assume $\zeta_n \to \zeta_0$ and $\bar{u}_n \rightharpoonup \bar{u} = \zeta_0 \tilde{u}$ for some $\zeta_0 > 0$. Similar to (3.25), we can also prove $\mathcal{I}'_0(\bar{u})\bar{u} \le 0$. Then, by applying Lemma 3.11, we have

$$\bar{u}_n \to \bar{u} \quad \text{in } H^s(\mathbb{R}^3),$$

or equivalently,

$$\tilde{u}_n \to \tilde{u} \quad \text{in } H^s(\mathbb{R}^3),$$

which finishes the proof. □

The following lemma is crucial in proving that the solutions of the auxiliary problem (3.1) are the solutions of the original problem (2.3). We shall adopt some ideas found in Ambrosio [6].

Lemma 3.13. *Let $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ be given as in Lemma 3.12. Then $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset L^\infty(\mathbb{R}^3)$ and there exists $K > 0$ such that*

$$|\tilde{u}_n|_\infty \le K \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there holds

$$\tilde{u}_n(x) \to 0 \quad \text{as } |x| \to +\infty.$$

Proof. For each $L > 0$, let $\tilde{u}_{L,n} := \min\{\tilde{u}_n, L\}$ and denote the function

$$\ell(\tau) := \ell_{L,\sigma}(\tau) = \tau\tau_L^{2(\sigma-1)} \in \mathcal{H}_\varepsilon,$$

with $\sigma > 1$ to be determined later. Note that ℓ is increasing, thus we have

$$(a-b)(\ell(a)-\ell(b)) \ge 0 \quad \text{for any } a, b \in \mathbb{R}.$$

Consider the functions

$$\mathcal{Q}(t) := \frac{|t|^2}{2} \quad \text{and} \quad \mathcal{L}(t) := \int_0^t \left(\ell'(\tau)\right)^{\frac{1}{2}} \mathrm{d}\tau,$$

and note that

$$\mathcal{L}(\tau) \geq \frac{1}{\sigma}\tau\tau_L^{\sigma-1}.$$

Hence, from Lemma 2.1, we obtain

$$[\mathcal{L}(\tilde{u}_n)]^2 \geq S_*^{-1}\,|\mathcal{L}(\tilde{u}_n)|_{2_s^*}^2 \geq S_*^{-1}\frac{1}{\sigma^2}|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{2_s^*}^2. \tag{3.26}$$

In addition, for any $a, b \in \mathbb{R}$, it holds

$$\mathcal{Q}'(a-b)(\ell(a)-\ell(b)) \geq |\mathcal{L}(a)-\mathcal{L}(b)|^2.$$

In fact, suppose that $a > b$, it follows from Jensen's inequality that

$$\begin{aligned}\mathcal{Q}'(a-b)(\ell(a)-\ell(b)) &= (a-b)(\ell(a)-\ell(b))\\ &= (a-b)\int_b^a \ell'(\tau)\,\mathrm{d}\tau\\ &= (a-b)\int_b^a (\mathcal{L}'(\tau))^2\,\mathrm{d}\tau\\ &\geq \left(\int_b^a \mathcal{L}'(\tau)\,\mathrm{d}\tau\right)^2\\ &= (\mathcal{L}(a)-\mathcal{L}(b))^2.\end{aligned}$$

A similar argument holds if $a \leq b$. Thus, we infer that

$$|\mathcal{L}(\tilde{u}_n)(x)-\mathcal{L}(\tilde{u}_n)(y)|^2 \leq (\tilde{u}_n(x)-\tilde{u}_n(y))\left(\tilde{u}_n(x)\tilde{u}_{L,n}^{2(\sigma-1)}(x)-\tilde{u}_n(y)\tilde{u}_{L,n}^{2(\sigma-1)}(y)\right).$$

Using $\ell(\tilde{u}_n)$ as test function in (3.1), in view of the above inequality and $\phi_{\tilde{u}_n}^t \geq 0$, we get that

$$\begin{aligned}&[\mathcal{L}(\tilde{u}_n)]^2 + \int_{\mathbb{R}^3}(V(\varepsilon_n x+\varepsilon_n x_n)+1)\,\tilde{u}_n^2\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x + \int_{\mathbb{R}^3}F_1'(u)\tilde{u}_n\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x\\ &\leq \iint_{\mathbb{R}^6}\frac{(\tilde{u}_n(x)-\tilde{u}_n(y))\left(\tilde{u}_n(x)\tilde{u}_{L,n}^{2(\sigma-1)}(x)-\tilde{u}_n(y)\tilde{u}_{L,n}^{2(\sigma-1)}(y)\right)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{\mathbb{R}^3}(V_\varepsilon(x)+1)\,\tilde{u}_n^2\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x + \int_{\mathbb{R}^3}F_1'(\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^3}G_2'(\varepsilon_n x+\varepsilon_n x_n,\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x.\end{aligned}$$

By the definition of G_2', fixed $p \in (2, 2_s^*)$, there exists $C > 0$ such that

$$0 \leq G_2'(x,\tau) \leq \theta\tau + C\tau^{p-1} \quad \text{for } (x,\tau) \in \mathbb{R}^3\times[0,\infty).$$

The above estimates, (3.26), (V_1) and $V_0 + 1 - \theta \geq 3\theta > 0$ provide

$$|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{2_s^*}^2 \leq \sigma^2 S_*\,[\mathcal{L}(\tilde{u}_n)]^2 \leq \sigma^2 S_*\int_{\mathbb{R}^3}\tilde{u}_n^p\tilde{u}_{L,n}^{2(\sigma-1)}\,\mathrm{d}x. \tag{3.27}$$

Since $\tilde{u}_n^p\tilde{u}_{L,n}^{2(\sigma-1)} = \tilde{u}_n^{p-2}(\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1})^2$, we can use (3.27) and Hölder's inequality to deduce that

$$|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{2_s^*}^2 \leq \sigma^2 S_*|\tilde{u}_n|_{2_s^*}^{p-2}|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{\alpha_s^*}^2,$$

where

$$\alpha_s^* = \frac{22_s^*}{2_s^*-(p-2)} \in (2, 2_s^*).$$

Since u is bounded, we conclude that

$$|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{2_s^*}^2 \leq \sigma^2 S_* |\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{\alpha_s^*}^2.$$

Note that, if $\tilde{u}_n \in L^{\sigma\alpha_s^*}(\mathbb{R}^3)$, using the fact that $\tilde{u}_{L,n} \leq \tilde{u}_n$, then

$$|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{2_s^*}^2 \leq \sigma^2 S_* |\tilde{u}_n|_{\sigma\alpha_s^*}^{2\sigma} < \infty,$$

which together with Fatou's lemma implies

$$|\tilde{u}_n|_{\sigma 2_s^*}^{2\sigma} \leq C\sigma^2 |\tilde{u}_n|_{\sigma\alpha_s^*}^{2\sigma},$$

as $L \to \infty$. Now, taking $\sigma = 2_s^*/\alpha_s^* > 0$, we have

$$|\tilde{u}_n|_{\sigma 2_s^*}^{2\sigma} \leq C\sigma^2 |\tilde{u}_n|_{2_s^*}^{2\sigma},$$

and replacing σ by σ^j, $j \in \mathbb{N}$, in the above inequality, we obtain that

$$|\tilde{u}_n|_{\sigma^j 2_s^*}^{2\sigma^j} \leq C(\sigma^j)^2 |\tilde{u}_n|_{2_s^*}^{2\sigma^j}.$$

Then, by an argument of induction, we may verify that

$$|\tilde{u}_n|_{2_s^*\sigma^j} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\cdots+\frac{j}{\sigma^j}} (2C)^{\frac{1}{2}(\frac{1}{\sigma}+\frac{1}{\sigma^2}+\cdots+\frac{1}{\sigma^j})} |\tilde{u}_n|_{2_s^*}, \tag{3.28}$$

for every $j \in \mathbb{N}$. Note that

$$\sum_{j=1}^{\infty} \frac{1}{\sigma^j} = \frac{1}{\sigma-1} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{i}{\sigma^j} = \frac{\sigma}{(\sigma-1)^2}.$$

Since $\sigma > 1$, passing to the limit as $j \to \infty$ in (3.28), we may infer that $u \in L^\infty(\mathbb{R}^N)$ and

$$|\tilde{u}_n|_\infty \leq \sigma^{\frac{\sigma}{(\sigma-1)^2}} (2C)^{\frac{1}{\sigma-1}} |\tilde{u}_n|_{2_s^*}.$$

Using $|\tilde{u}_n|_{2_s^*} \leq M$, it shows that $\tilde{u}_n \in L^\infty(\mathbb{R}^3)$ and

$$|\tilde{u}_n|_\infty \leq K.$$

In particular, by interpolation, $|\tilde{u}_n|_r \leq K$ for all $n \in \mathbb{N}$ and $r \in [2,\infty]$, and $\tilde{u}_n \to \tilde{u}$ in $L^r(\mathbb{R}^3)$ for all $r \in [2,\infty)$. Now, we notice that $\tilde{u}_n$ satisfies

$$(-\Delta)^s \tilde{u}_n + \theta\tilde{u}_n \leq C\tilde{u}_n^{p-1} \quad \text{in } \mathbb{R}^3.$$

Let $z_n \in H^s(\mathbb{R}^3)$ be such that

$$(-\Delta)^s z_n + \theta z_n = C\tilde{u}_n^{p-1} \quad \text{in } \mathbb{R}^3. \tag{3.29}$$

Applying Del Pezzo-Quaas [18], we can see $0 \leq \tilde{u}_n \leq z_n$ in $\mathbb{R}^3$. By (3.29), we derive that

$$[z_n]^2 + \theta|z_n|_2^2 = C\int_{\mathbb{R}^3} \tilde{u}_n^{p-1} z_n \, dx,$$

and applying the Young's inequality we get, for all $\eta \in (0,\theta)$,

$$[z_n]^2 + \theta|z_n|_2^2 \leq \eta|z_n|_2^2 + C\int_{\mathbb{R}^3} \tilde{u}_n^{2(p-1)} \, dx.$$

Since $\int_{\mathbb{R}^3} \tilde{u}_n^{2(p-1)} \, dx \leq C$ for all $n \in \mathbb{N}$ (note that $2(p-1) > 2$), we deduce that $\|z_n\| \leq C$ for all $n \in \mathbb{N}$. Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in $H^s(\mathbb{R}^3)$. Since $\tilde{u}_n \to \tilde{u}$ in $L^{2(p-1)}(\mathbb{R}^3)$, it follows from the weak formulation of the equation solved by z_n that z solves

$$(-\Delta)^s z + \theta z = C\tilde{u}^{p-1} \quad \text{in } \mathbb{R}^3.$$

Then, observing that

$$[z_n]^2 + \theta |z_n|_2^2 = C \int_{\mathbb{R}^3} \tilde{u}_n^{p-1} z_n \, \mathrm{d}x,$$

and

$$[z]^2 + \theta |z|_2^2 = C \int_{\mathbb{R}^3} \tilde{u}^{p-1} z \, \mathrm{d}x,$$

and that

$$\int_{\mathbb{R}^3} \tilde{u}_n^{p-1} z_n \, \mathrm{d}x \to \int_{\mathbb{R}^3} \tilde{u}^{p-1} z \, \mathrm{d}x,$$

we deduce that

$$[z_n]^2 + \theta |z_n|_2^2 = [z]^2 + \theta |z|_2^2 + o_n(1).$$

Recalling that $H^s(\mathbb{R}^3)$ is a uniformly convex space, we see that $z_n \to z$ in $H^s(\mathbb{R}^3)$. Since $C\tilde{u}_n^{p-1} \leq C z_n^{p-1}$, we can see that if we test (3.29) by $z_n z_{L,n}^{2(\sigma-1)}$, we can proceed as before to obtain the estimate

$$|z_n z_{L,n}^{\sigma-1}|_{2_s^*}^2 \leq \sigma^2 C_* \int_{\mathbb{R}^3} z_n^p z_{L,n}^{2(\sigma-1)} \, \mathrm{d}x,$$

instead of (3.27). Using $|z_n|_{2_s^*} \leq C$ for all $n \in \mathbb{N}$, we can perform the same iteration argument given above to deduce that

$$|z_n|_\infty \leq K \quad \text{for all } n \in \mathbb{N}. \tag{3.30}$$

By Iannizzotto-Mosconi-Squassina [26] and using (3.30), we deduce that

$$\|z_n\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C \quad \text{for all } n \in \mathbb{N},$$

for some $\alpha \in (0,1)$ independent of $n \in \mathbb{N}$. Combining this fact with $z_n \to z$ in $L^2(\mathbb{R}^3)$, we can apply Ambrosio [7] to conclude that $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |z_n(x)| = 0$. This and $0 \leq \tilde{u}_n \leq z_n$ in $\mathbb{R}^3$ yield $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |\tilde{u}_n(x)| = 0$. □

4. Proof of Theorem 1.1

Lemma 4.1. *Let $\varepsilon_n \to 0$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ be a solution of (3.1). Then there exists $n^* \in \mathbb{N}$ such that*

$$\frac{\tilde{F}_2'(u_n)}{u_n} \leq \theta \quad \textit{for all } x \in \Omega_{\varepsilon_n}^c \textit{ and for all } n \geq n^*.$$

Proof. By Lemma 3.12, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\tilde{u}_n(x) = u_n(x + x_n)$ has a convergent subsequence and $\varepsilon_n x_n \to x_0 \in \Omega$ as $n \to \infty$. Then, there exists $r > 0$ such that for any $n \in \mathbb{N}$, $B_r(\varepsilon_n x_n) \subset \Omega$, which implies $B_{\frac{r}{\varepsilon_n}}(x_n) \subset \Omega_{\varepsilon_n}$ and $\Omega_{\varepsilon_n}^c \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(x_n)$. Furthermore, we observe that

$$\begin{aligned}
\left|\phi_{\tilde{u}_n}^t(x)\right| &= \left| \int_{\mathbb{R}^3} \frac{|\tilde{u}_n(y)|^2}{|x-y|^{3-2t}} \, \mathrm{d}y \right| \\
&\leq \left| \int_{|x-y|\leq 1} \frac{|\tilde{u}_n(y)|^2}{|x-y|^{3-2t}} \, \mathrm{d}y \right| + \left| \int_{|x-y|>1} \frac{|\tilde{u}_n(y)|^2}{|x-y|^{3-2t}} \, \mathrm{d}y \right| \\
&\leq C|\tilde{u}_n|_\infty^2 + |\tilde{u}_n|_2^2,
\end{aligned}$$

which, together with Lemma 3.13 and (2.7), implies that for $\theta > 0$ given above, there exist $R > 0$ and $n_1 \in \mathbb{N}$ such that

$$\frac{\tilde{F}_2'(\tilde{u}_n)}{\tilde{u}_n} = \frac{F_2'(\tilde{u}_n)}{\tilde{u}_n} + \phi_{\tilde{u}_n}^t \leq \theta \quad \text{for all } |x| \geq R \text{ and } n \geq n_1.$$

Recall $u_n(x) = \tilde{u}_n(x + x_n)$. Then we have

$$\frac{\tilde{F}_2'(\tilde{u}_n)}{\tilde{u}_n} \leq \theta \quad \text{for all } x \in \mathbb{R}^3 \setminus B_R(x_n) \text{ and } n \geq n_1.$$

On the other hand, for given $R > 0$ above, there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$

$$\Omega_{\varepsilon_n}^c \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(x_n) \subset \mathbb{R}^3 \setminus B_R(x_n).$$

Taking $n^* = \max\{n_1, n_2\}$, we can conclude that

$$\frac{\tilde{F}_2'(\tilde{u}_n)}{\tilde{u}_n} \leq \theta \quad \text{for all } x \in \Omega_{\varepsilon_n}^c \text{ and } n \geq n^*.$$

□

Now we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.6, problem (3.1) has a positive solution u_ε for each $\varepsilon > 0$. Furthermore, Lemma 4.1 implies that there exists $\varepsilon_* > 0$ small such that

$$\frac{\tilde{F}_2'(u_\varepsilon)}{u_\varepsilon} \leq \theta \quad \text{for } x \in \Omega_\varepsilon^c \text{ and } \varepsilon \in (0, \varepsilon_*).$$

This implies that u_ε is a positive solution of (2.3). Letting

$$w_\varepsilon(x) = u_\varepsilon\left(\frac{x}{\varepsilon}\right),$$

we know that $w_\varepsilon(x)$ is a positive solution of the original problem (1.5).

Now, we study the concentration behavior of the solutions. Let $\varepsilon_n \to 0$ and $u_n \in \mathcal{H}_{\varepsilon_n}$ be a positive solution with $\varepsilon = \varepsilon_n$ in (3.1). Let $\tilde{u}_n = u_n(x + x_n)$, where x_n is given in Lemma 3.12. First, we claim that there exists $\tilde{\varrho} > 0$ such that

$$|\tilde{u}_n|_\infty \geq \tilde{\varrho} \quad \text{for any } n \in \mathbb{N}. \tag{4.1}$$

Indeed, if $|\tilde{u}_n|_\infty \to 0$, then there is $n_3 \in \mathbb{N}$ such that, for all $n \geq n_3$,

$$G_2'(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n) \leq \frac{V_0}{2} \tilde{u}_n.$$

Then,

$$\|\tilde{u}_n\|_{Y_{V_0}}^2 \leq \|\tilde{u}_n\|_{\varepsilon_n}^2 \leq \int_{\mathbb{R}^3} G_2'(\varepsilon_n x + \varepsilon_n x_n, \tilde{u}_n)\, \tilde{u}_n \, \mathrm{d}x \leq \frac{V_0}{2} \int_{\mathbb{R}^3} \tilde{u}_n^2 \, \mathrm{d}x.$$

Thus $\tilde{u}_n \equiv 0$ for all $n \geq n_3$, which contradicts the fact that $\tilde{u}_n \to \tilde{u} \neq 0$. Hence, (4.1) holds, and we can infer that $\tilde{u}_n$ admits a global maximum point $q_n \in B_{R_*}(0)$ for some $R_* > 0$. Set $y_n := \varepsilon_n(q_n + x_n)$ which is the maximum point of $w_n(x)$. Moreover, regarding from Lemma 3.12, we have that $\{y_n\}_{n\in\mathbb{N}}$ is bounded and

$$y_n \to y_0 \in \Omega \quad \text{as } n \to \infty.$$

Therefore, we can conclude that

$$\lim_{n\to\infty} V(y_n) = V(x_0) = V_0.$$

□

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