BOUNDEDNESS OF WEAK SOLUTIONS TO MULTIVALUED ANISOTROPIC DOUBLE PHASE PROBLEMS WITH LOGARITHMIC PERTURBATION

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Abstract. In this paper we consider multivalued mixed boundary value problems driven by the variable exponent double phase operator with ω-logarithmic perturbation of the form

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,u,\nabla u) \in \mathcal{F}(x,u,\nabla u) & \text{in } \Omega,\\ \dfrac{-\partial u}{\partial \nu_{\mathcal{A}}} \in \mathcal{G}(x,u) & \text{on } \Gamma_1,\\ u=0, & \text{on } \Gamma_2. \end{cases}$$

First, we prove new continuous and compact (trace) embedding results related to the considered Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_L}$. Based on these embedding results, we show the boundedness of solutions to the multivalued mixed boundary problem above by applying De Giorgi's method and localization arguments. Finally, we consider several special cases of the problem above and establish boundedness results of them.

1. Introduction

Double phase problems are getting more and more attention these days due to its wide range in application, see, for example, the works of Arora–Crespo-Blanco–Winkert [3] Colombo–Mingione [16, 17], Crespo-Blanco–Gasiński–Harjulehto–Winkert [18], Liu–Dai [40], Lu–Vetro–Zeng [43], Zeng–Bai–Gasiński–Winkert [59], Zeng–Rădulescu–Winkert [63, 64] and the references therein. Especially in [43], a series of useful properties related to a kind of variable exponent double phase operator with ω-logarithmic perturbation are established. This operator is defined as

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right) \tag{1.1}$$

for $u \in W^{1,\mathcal{H}_L}(\Omega)$ (the precise definition is given in Section 2), where the function $\mathcal{H}_L$ is defined as the double phase function with a new ω-logarithmic perturbation term of the form

$$\mathcal{H}_L(x,t) = [t^{p(x)} + \mu(x)t^{q(x)}]\log(e+\omega t) \tag{1.2}$$

for all $x \in \Omega$ and for all $t \geq 0$, where $\omega \geq 0$, $p,q \in C(\overline{\Omega})$ such that $1 < p(x) < N$, $p(x) \leq q(x)$ for all $x \in \overline{\Omega}$, and $0 \leq \mu(\cdot) \in L^1(\Omega)$ with $\Omega_1 := \{x \in \Omega \colon p(x) < q(x)\} \subsetneq \Omega_0 := \{x \in \Omega \colon \mu(x) = 0\}$. A natural question is to ask how to get sharp embedding results for the related Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$ similar to the results obtained by Cianchi [13, 14], Cianchi–Diening [15], and Ho–Winkert [33]. Indeed, a first main result of this paper is the proof for the embedding

$$W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega). \tag{1.3}$$

where

$$\mathcal{B}(x,t) = t^{\tau(x)}\log^{\frac{\tau(x)}{p(x)}}(e+\omega t) + \mu(x)^{\frac{\pi(x)}{q(x)}} t^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e+\omega t)$$

for all $x \in \overline{\Omega}$ and for all $t \in [0,\infty)$ with $\tau,\pi \in C(\overline{\Omega})$ such that $1 < \tau(x) \leq p^*(x)$ and $1 < \pi(x) \leq q^*(x)$ for all $x \in \overline{\Omega}$. Here we use the notation $p^*(\cdot) := \frac{Np(\cdot)}{N-p(\cdot)}$ and $q^*(\cdot) := \frac{Nq(\cdot)}{N-q(\cdot)}$ being the critical Sobolev exponents of $p(\cdot)$ and $q(\cdot)$, respectively. The proof of the embedding (1.3) uses ideas of the work of Cianchi–Diening [15], who recently obtained a sharp embedding theorem for Musielak-Orlicz Sobolev spaces into Musielak-Orlicz spaces. Moreover, we also obtain a trace embedding of the form

$$W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{B}_\Gamma}(\Gamma_1). \tag{1.4}$$

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where

$$\mathcal{B}_\Gamma(x,t)=t^{\theta(x)}\log^{\frac{\theta(x)}{p(x)}}(e+\omega t)+\mu(x)^{\frac{\vartheta(x)}{q(x)}}t^{\vartheta(x)}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega t)$$

for all $x\in\overline{\Omega}$ and for all $t\in[0,\infty)$ with $\theta,\vartheta\in C(\overline{\Omega})$ such that $1<\theta(x)\le(p_*)^-$ and $1<\vartheta(x)\le(q_*)^-$ for all $x\in\overline{\Omega}$ with $p_*(\cdot):=\frac{(N-1)p(\cdot)}{N-p(\cdot)}$ and $q_*(\cdot):=\frac{(N-1)q(\cdot)}{N-q(\cdot)}$ denoting the critical exponents on the boundary of $p(\cdot)$ and $q(\cdot)$, respectively. Note that for any $r\in C(\overline{\Omega})$, we define $r^-:=\min_{x\in\overline{\Omega}}r(x)$. The proof of the trace embedding (1.4) is mainly based on the results taken from Cianchi [14], where a sharp trace embedding theorem for Orlicz-Sobolev spaces into Orlicz spaces on the boundary is obtained. However, the corresponding trace embedding for Musielak-Orlicz Sobolev spaces like $W^{1,\mathcal{H}_L}(\Omega)$ into suitable Musielak-Orlicz spaces has not been proved yet. So, for the trace embeddings we take $(p_*)^-$ and $(q_*)^-$ as the critical exponents instead of $p_*(\cdot)$ and $q_*(\cdot)$. Based on these embedding results together with De Giorgi's iteration technique, we are then going to prove the boundedness of weak solutions $u\in W^{1,\mathcal{H}_L}(\Omega)$ of the multivalued mixed boundary value problem given by

$$\begin{cases}-\operatorname{div}\mathcal{A}(x,u,\nabla u)\in\mathcal{F}(x,u,\nabla u) & \text{in }\Omega,\\ \dfrac{-\partial u}{\partial\nu_{\mathcal{A}}}\in\mathcal{G}(x,u) & \text{on }\Gamma_1,\\ u=0, & \text{on }\Gamma_2,\end{cases}\tag{P}$$

where $\Omega\subset\mathbb{R}^N(N\ge 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, Γ_1,Γ_2 are two disjoint parts of $\partial\Omega$ with $\Gamma_1\cup\Gamma_2=\partial\Omega$, $\mathcal{A}\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function and $\mathcal{F}\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ as well as $\mathcal{G}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ are two multivalued functions defined in the domain and on its boundary, respectively. Moreover, $\frac{-\partial u}{\partial\nu_{\mathcal{A}}}=\mathcal{A}(x,u,\nabla u)\cdot\nu$ with ν being the outward unit normal at Γ_1.

The variable exponent double phase operator with ω-logarithmic perturbation given in (1.1) is developed from the classical double phase operator

$$\operatorname{div}\left(|\nabla w|^{p-2}\nabla w+\mu(x)|\nabla w|^{q-2}\nabla w\right),$$

with its related energy functional

$$w\mapsto\int_\Omega\left(|\nabla w|^p+\mu(x)|\nabla w|^q\right)\mathrm{d}x.\tag{1.5}$$

Functionals of the form (1.5) have been first occurred by a work of Zhikov [66] in order to describe models of strongly anisotropic materials. It has also applications in the duality theory and the research of the Lavrentiev phenomenon, see Zhikov [67]. Also, in elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents q and p. From the mathematical point of view, the modulating coefficient $\mu(\cdot)$ can change the behavior of the integral functional (1.5). In particular, on the set $\Omega_{>0}:=\{x\in\Omega:\mu(x)>0\}$, it has ellipticity in the gradient of order q and of order p on the points where μ vanishes. Moreover, first regularity properties of local minimizers of functionals like (1.5) have been proved in the papers by Baroni–Colombo–Mingione [6, 7] and Colombo–Mingione [16, 17]. We also mention the pioneering works of Marcellini [44, 45] for integral functionals with nonstandard growth condition, see also the recent paper of Beck–Mingione [8]. After these fundamental works, several papers for existence of solutions related to double phase equations with different right-hand sides and various techniques appeared. See for example, Crespo-Blanco–Gasiński–Harjulehto–Winkert [18], Liu–Dai [40], Zeng–Bai–Gasiński–Winkert [59] and Zeng-Rădulescu-Winkert [63]. Especially, double phase problems involving logarithmic perturbation occur in the context of generalized Newtonian fluids and can be applied in the theory of plasticity with logarithmic hardening law. For related papers we refer to Arora–Crespo-Blanco–Winkert [3], Fuchs–Mingione [23], Lu–Vetro–Zeng [43] Marcellini–Papi [46] and Vetro–Winkert [55]. We also point out that there are several applications of the double phase operator, for example in, transonic flow problems, nonlinear theory of composite materials, nonlinear Derrick's problem as well as image processing that can be found in [4, 9, 64, 65].

Another main feature of problem (P) is the occurrence of the multivalued mixed boundary value conditions. Such conditions do apply in several problems in engineering and economics, such as fluid mechanics problems with nonmonotone friction, nonsmooth contact mechanics problems and aeronautics. Related works studying various multivalued problems with mixed boundary can be found in the papers by Han [28], Kalita–Kowalski [34], Li–Huang [38], Liu–Zeng–Gasiński–Kim [41], Migórski–Dudek [48], Migórski–Khan–Zeng [50, 49], Zeng–Gasiński–Winkert–Bai [60], Zeng–Migórski–Khan [61], Zeng–Migórski–Tarzia [62] and Zeng–Rădulescu–Winkert [63].

In this paper, in order to show the boundedness of weak solutions of problem (P), the main tool to be applied is the so-called De Giorgi–Nash–Moser theory. Its early research can be dated back to the works by De Giorgi [20], Nash [53] as well as Moser [51]. It is well known that the De Giorgi–Nash–Moser theory is not only useful for showing the local and global boundedness of weak solutions but also plays an important role in proving the (weak) Harnack inequality and the Hölder continuity for weak solutions, see for example, the monographs of Gilbarg–Trudinger [27], Ladyženskaja–Solonnikov–Ural'ceva [36], Ladyženskaja–Ural'ceva [37] and Lieberman [39]. Furthermore, we also mention some recent results concerning the boundedness of weak solutions established with the De Giorgi iteration or the Moser iteration can be found in Alonso–Morimoto–Sun–Yang [1], Amoroso–Crespo-Blanco–Pucci–Winkert [2], Barletta–Cianchi–Marino [5], Crespo-Blanco–Winkert [19], Gasiński–Winkert [25, 26] and Marino–Winkert [47].

In this paper, we extend boundedness results to the multivalued mixed boundary problem (P) via a modified De Giorgi iteration, covering both subcritical and critical growth. The method is mainly based on ideas of the works by Ho–Kim [31], Ho–Kim–Winkert–Zhang [32], Ho–Winkert [33], and Winkert–Zacher [57, 58]. It is worth to mention that Ho-Winkert [33] considered the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

and Neumann problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) & \text{in } \Omega,\\ \mathcal{A}(x,u,\nabla u)\cdot \nu = \mathcal{C}(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

and obtained the L^∞-bounds for weak solutions $u \in W^{1,\mathcal{H}}(\Omega)$ of problems (1.6) as well as (1.7) where they allow general growth on the nonlinearity based on new embedding results similar to (1.3) and (1.4) but without the log-term. On this basis, we extend the boundedness results to the multivalued mixed boundary problem (P), where the function space is the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$ generated by the function $\mathcal{H}_L$ given in (1.2) with $p(\cdot), q(\cdot)$-growth and ω-logarithmic perturbation. To the best of our knowledge, this is the first work dealing with boundedness results to this type of multivalued mixed boundary value problems as given in (P).

Furthermore, we point out that there are many interesting special cases of problem (P).

Remark 1.1.

(i) *If $\omega = 0$ in $\mathcal{H}_L$, then $\mathcal{H}_L$ equals to $\mathcal{H}$ without logarithmic perturbation, and problem* (P) *becomes* (P1)*;*

(ii) *If $\mu(\cdot) = 1$ and $\omega = 0$, then the operator related to $\mathcal{H}_L$ is the $p(\cdot), q(\cdot)$-Laplacian, and problem* (P) *becomes* (P2)*;*

(iii) *If $\mu(\cdot) = 0$, then $\mathcal{H}_L$ exhibits the $L^{p(\cdot)} \log L$ growth, and problem* (P) *becomes* (P3)*;*

(iv) *If p, q are constant functions, that is $1 < p(x) \equiv p$ and $1 < q(x) \equiv q$ for all $x \in \overline{\Omega}$, then $\mathcal{H}_L$ is with constant exponents, and problem* (P) *becomes* (P4)*;*

(v) *If $|\Gamma_1| = 0$, then problem* (P) *becomes a Dirichlet boundary value problem* (P5)*;*

(vi) *If $|\Gamma_2| = 0$, then problem* (P) *becomes a Neumann boundary value problem* (P6)*;*

(vii) *Let $\mathcal{F}$ or $\mathcal{G}$ be single-valued Carathéodory functions. In particular, let $\mathcal{F}$ and $\mathcal{G}$ be two single-valued Carathéodory functions, and $\omega = 0$, then, if $|\Gamma_1| = 0$ problem* (P) *becomes the Dirichlet problem* (1.6)*, denoted by* (P7)*, and if $|\Gamma_2| = 0$ problem* (P) *becomes the Neumann problem* (1.7)*, denoted by* (P8)*.*

(viii) *If $\mathcal{F}$ is independent on the gradient of the unknown function, that is, $\mathcal{F}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$, then we can choose $\mathcal{F}$ and $\mathcal{G}$ to be Clarke's generalized gradient of two locally Lipschitz continuous functions. So, problem* (P) *becomes problem* (P9)*, which leads to a generalized hemivariational inequality, see Section 5 for more information on this.*

This paper is organized as follows. In Section 2, as the preliminaries of the remaining sections, we introduce some basic definitions and results concerning the mapping $\mathcal{H}_L$ with unbalanced growth and ω-logarithmic perturbation, as well as the related Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$. Section 3 concentrates on proving the new continuous and compact embeddings with respect to the space $W^{1,\mathcal{H}_L}(\Omega)$, see Propositions 3.4 and 3.5. In Section 4, based on the embedding results shown in Section 3, we will show our main theorems, that is, the L^∞-bounds of weak solutions to problem (P), by employing De Giorgi's iteration method in the subcritical case (see Subsection 4.1) and the critical case (see Subsection 4.2). Moreover, in Section 5, we deal with the special cases of problem (P) that are mentioned in Remark 1.1.

2. Preliminaries

In this section we recall some notations and results mainly taken from Lu-Vetro-Zeng [43], where a type of ω-logarithmic perturbed double phase operator with variable exponents and its related Musielak-Orlicz spaces are studied, see also the work of Arora–Crespo-Blanco–Winkert [3] for a different type of logarithmic double phase operator. For more information with respect to Musielak-Orlicz spaces we refer to the contributions of Diening–Harjulehto–Hästö–Růžička [21], Fan–Zhao [22], Harjulehto–Hästö [30], Kováčik–Rákosník [35] and Rădulescu–Repovš [54].

In the sequel, for any $r \in C(\overline{\Omega})$ we define

$$r^- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ := \max_{x\in\overline{\Omega}} r(x),$$

and introduce the conjugate variable exponent of $r > 1$ denoted by $r' \in C(\overline{\Omega})$ satisfying $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for all $x \in \overline{\Omega}$.

Next, we give the definition of a N-function.

Definition 2.1.

(i) *A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be a* N*-function, if it is continuous, convex, $\varphi(t) = 0$ if and only if $t = 0$,*

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \quad \textit{and} \quad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty.$$

(ii) *A function $\varphi\colon \Omega\times[0,\infty) \to [0,\infty)$ is said to be a generalized* N*-function, denoted by $\varphi \in \mathrm{N}(\Omega)$, if $\varphi(\cdot,t)$ is measurable for all $t \geq 0$ and $\varphi(x,\cdot)$ is a* N*-function for a.a. $x \in \Omega$.*

Definition 2.2.

(i) *A function $\varphi\colon \Omega\times[0,\infty) \to [0,\infty)$ is said to be locally integrable if $\varphi(\cdot,t) \in L^1(\Omega)$ for all $t > 0$.*

(ii) *Let $\varphi, \psi \in \mathrm{N}(\Omega)$. Then φ is weaker than ψ, denoted by $\varphi \prec \psi$, if there exist $c_1, c_2 > 0$ satisfying*

$$\varphi(x,t) \leq c_1\psi(x,c_2t) + h(x) \quad \textit{for a.a. } x \in \Omega \textit{ and for all } t \geq 0,$$

with $h \in L^1(\Omega)$ being a nonnegative function. In addition, φ, ψ are equivalent, denoted by $\varphi \sim \psi$, if $\varphi \prec \psi$ and $\psi \prec \varphi$.

(iii) *Let $\varphi, \psi \in \mathrm{N}(\Omega)$. Then φ increases essentially slower than ψ near infinity, denoted by $\varphi \ll \psi$, if for all $k > 0$ it holds that*

$$\lim_{t\to\infty} \frac{\varphi(x,kt)}{\psi(x,t)} = 0 \quad \textit{uniformly for a.a. } x \in \Omega.$$

Now, we are able to introduce the specific definition of a Musielak-Orlicz space. If $\varphi \in \mathrm{N}(\Omega)$, then

$$\rho_\varphi(u) := \int_\Omega \varphi(x,|u|)\,\mathrm{d}x,$$

is the related modular function. In the sequel, we denote by $M(\Omega)$ the space of measurable functions from Ω to $\mathbb{R}$. The Musielak-Orlicz space of φ is then given by

$$L^\varphi(\Omega) := \{u \in M(\Omega)\colon \text{ there exists } \lambda > 0 \text{ such that } \rho_\varphi(\lambda u) < +\infty\}$$

endowed with the Luxemburg norm

$$\|u\|_{\varphi,\Omega} := \inf\left\{\lambda > 0\colon \rho_\varphi\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

For the convenience, we may write the norm for the domain Ω as $\|u\|_\varphi$ instead of $\|u\|_{\varphi,\Omega}$ in the rest of this paper. Moreover, the associated Musielak-Orlicz Sobolev space is defined by

$$W^{1,\varphi}(\Omega) := \{u \in L^\varphi(\Omega)\colon |\nabla u| \in L^\varphi(\Omega)\}$$

equipped with the norm

$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi,$$

where $\|\nabla u\|_\varphi := \|\,|\nabla u|\,\|_\varphi$. Analogously, we write $\rho_\varphi(\nabla u) = \rho_\varphi(|\nabla u|)$. The completion of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$ is the space $W_0^{1,\varphi}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,\varphi}(\Omega)}$.

The following embedding result is due to Musielak [52, Theoreem 8.5] and will be important for us to establish the continuous and compact embedding results in Section 3.

Proposition 2.3. *If $\varphi \in \mathrm{N}(\Omega)$ and $\psi \in N(\Omega)$ satisfying $\varphi \prec \psi$, then $L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega)$.*

Throughout this paper we will assume the following hypotheses:

(H0) $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ with $\Omega_1 := \{x \in \Omega\colon p(x) < q(x)\} \nsubseteq \Omega_0 := \{x \in \Omega\colon \mu(x) = 0\}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

Under hypotheses (H0), it is easy to check that $\mathcal{H}_L$ given in (1.2) is a locally integrable N-function. The modular function related to $\mathcal{H}_L$ is given by

$$\rho_{\mathcal{H}_L}(u) = \int_\Omega \mathcal{H}_L(x,|u|)\,\mathrm{d}x$$

while the corresponding Musielak-Orlicz space is

$$L^{\mathcal{H}_L}(\Omega) = \{u \in M(\Omega)\colon \rho_{\mathcal{H}_L}(u) < +\infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}_L} = \inf\left\{\lambda > 0\colon \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Furthermore, $W^{1,\mathcal{H}_L}(\Omega)$ is defined as

$$W^{1,\mathcal{H}_L}(\Omega) := \{u \in L^{\mathcal{H}_L}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_L}(\Omega)\},$$

and $W^{1,\mathcal{H}_L}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}_L}(\Omega)$. Both $W^{1,\mathcal{H}_L}_0(\Omega)$ and $W^{1,\mathcal{H}_L}(\Omega)$ are endowed with the norm

$$\|u\|_{1,\mathcal{H}_L} = \|u\|_{\mathcal{H}_L} + \|\nabla u\|_{\mathcal{H}_L}.$$

The following proposition can be found in the paper of Lu–Vetro–Zeng [43, Proposition 2.13].

Proposition 2.4. *Let hypotheses* (H0) *be satisfied, then the space $L^{\mathcal{H}_L}(\Omega)$ endowed with the Luxemburg norm $\|\cdot\|_{\mathcal{H}_L}$ is a separable and reflexive Banach space. In addition, $W^{1,\mathcal{H}_L}(\Omega)$ and $W^{1,\mathcal{H}_L}_0(\Omega)$ are separable and reflexive Banach spaces.*

Next, we introduce the function $f_\sigma\colon \Omega \times [0,\infty) \to [0,\infty)$ given as

$$f_\sigma(x,t) = \frac{t^\sigma}{\log(e+\omega(x)t)} \quad \text{with } \sigma > 0 \text{ and } 0 \le \omega(\cdot) \in L^\infty(\Omega).$$

It was shown that there exists σ^* such that for $\sigma \ge \sigma^*$, we have that $f_\sigma(x,t) > 0$ is increasing with respect to t. As for $0 < \sigma < \sigma^*$, we can find t_1, t_2 satisfying the following: if $0 < t < t_1$ and $t > t_2$, then $f_\sigma(x,\cdot)$ is increasing, and if $t_1 \le t \le t_2$ then $f_\sigma(x,\cdot)$ is decreasing. Thus, $f_\sigma(x,a) \le C_\sigma \cdot f_\sigma(x,b)$ for any $0 < a \le b$ where $C_\sigma = \frac{f_\sigma(x,t_1)}{f_\sigma(x,t_2)} > 1$.

The next proposition shows the relation between the norm and its modular, see Theorem 2.21 in the work by Lu–Vetro–Zeng [43].

Proposition 2.5. *Let hypotheses* (H0) *be satisfied, $u \in L^{\mathcal{H}_L}(\Omega)$ and the modular is defined by*

$$\rho_{\mathcal{H}_L}(u) = \int_\Omega \left[|u|^{p(x)} + \mu(x)|u|^{q(x)}\right] \log(e+\omega|u|)\,\mathrm{d}x \quad \text{for all } u \in L^{\mathcal{H}_L}(\Omega).$$

The the following hold:

- (i) $\|u\|_{\mathcal{H}_L} = \lambda \Leftrightarrow \rho_{\mathcal{H}_L}(\frac{u}{\lambda}) = 1$ *with* $u \neq 0$;
- (ii) $\|u\|_{\mathcal{H}_L} < 1 (resp. = 1, > 1) \Leftrightarrow \rho_{\mathcal{H}_L}(u) < 1 (resp. = 1, > 1)$;
- (iii) *if* $\|u\|_{\mathcal{H}_L} < 1$ *then* $C_\sigma^{-1}\|u\|^{q^+ + \sigma}_{\mathcal{H}_L} \le \rho_{\mathcal{H}_L}(u) \le \|u\|^{p^-}_{\mathcal{H}_L}$;
- (iv) *if* $\|u\|_{\mathcal{H}_L} > 1$ *then* $\|u\|^{p^-}_{\mathcal{H}_L} \le \rho_{\mathcal{H}_L}(u) \le C_\sigma\|u\|^{q^+ + \sigma}_{\mathcal{H}_L}$;
- (v) $\|u\|_{\mathcal{H}_L} \to 0 \Leftrightarrow \rho_{\mathcal{H}_L}(u) \to 0$;
- (vi) $\|u\|_{\mathcal{H}_L} \to \infty \Leftrightarrow \rho_{\mathcal{H}_L}(u) \to \infty$;
- (vii) $\|u\|_{\mathcal{H}_L} \to 1 \Leftrightarrow \rho_{\mathcal{H}_L}(u) \to 1$;
- (viii) *if* $u_n \to u$ *in* $L^{\mathcal{H}_L}(\Omega)$ *then* $\rho_{\mathcal{H}_L}(u_n) \to \rho_{\mathcal{H}_L}(u)$.

The space $W^{1,\mathcal{H}_L}(\Omega)$ can be equipped with the equivalent norm

$$\|u\|_{\hat{\rho}_{\mathcal{H}_L}} := \inf\left\{\lambda > 0\colon \hat{\rho}_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) \le 1\right\}, \tag{2.1}$$

where

$$\begin{aligned}\hat{\rho}_{\mathcal{H}_L}(u) :=& \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right) \log(e+\omega|\nabla u|)\,\mathrm{d}x\\ &+ \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right) \log(e+\omega|u|)\,\mathrm{d}x\end{aligned}$$

for $u \in W^{1,\mathcal{H}_L}(\Omega)$.

Similar to Proposition 2.5 we have the following relations between $\|\cdot\|_{\hat{\rho}_{\mathcal{H}_L}}$ and $\hat{\rho}_{\mathcal{H}_L}(\cdot)$, see also Theorem 2.22 by Lu–Vetro–Zeng [43].

Proposition 2.6. *Let hypotheses* (H0) *be satisfied and* $u \in W^{1,\mathcal{H}_L}(\Omega)$. *Then the following hold:*

(i) $\|u\|_{\hat\rho_{\mathcal{H}_L}} = \lambda \Leftrightarrow \hat\rho_{\mathcal{H}_L}(\frac{u}{\lambda}) = 1$ *with* $u \neq 0$;
(ii) $\|u\|_{\hat\rho_{\mathcal{H}_L}} < 1 (resp. = 1, > 1) \Leftrightarrow \hat\rho_{\mathcal{H}_L}(u) < 1 (resp. = 1, > 1)$;
(iii) *if* $\|u\|_{\hat\rho_{\mathcal{H}_L}} < 1$ *then* $C_\sigma^{-1}\|u\|^{q^+ + \sigma}_{\hat\rho_{\mathcal{H}_L}} \leq \hat\rho_{\mathcal{H}_L}(u) \leq \|u\|^{p^-}_{\hat\rho_{\mathcal{H}_L}}$;
(iv) *if* $\|u\|_{\hat\rho_{\mathcal{H}_L}} > 1$ *then* $\|u\|^{p^-}_{\hat\rho_{\mathcal{H}_L}} \leq \hat\rho_{\mathcal{H}_L}(u) \leq C_\sigma\|u\|^{q^+ + \sigma}_{\hat\rho_{\mathcal{H}_L}}$;
(v) $\|u\|_{\hat\rho_{\mathcal{H}_L}} \to 0 \Leftrightarrow \hat\rho_{\mathcal{H}_L}(u) \to 0$;
(vi) $\|u\|_{\hat\rho_{\mathcal{H}_L}} \to \infty \Leftrightarrow \hat\rho_{\mathcal{H}_L}(u) \to \infty$;
(vii) $\|u\|_{\hat\rho_{\mathcal{H}_L}} \to 1 \Leftrightarrow \hat\rho_{\mathcal{H}_L}(u) \to 1$;
(viii) *if* $u_n \to u$ *in* $W^{1,\mathcal{H}_L}(\Omega)$ *then* $\hat\rho_{\mathcal{H}_L}(u_n) \to \hat\rho_{\mathcal{H}_L}(u)$.

For $1 < r \in C(\overline{\Omega})$, we define $\mathcal{H}_{r,L} \colon \Omega \times [0,\infty) \to [0,\infty)$ as

$$\mathcal{H}_{r,L}(x,t) = t^{r(x)} \log(e + \omega t)$$

for all $x \in \overline{\Omega}$, for all $t \geq 0$, and $\omega \geq 0$ is the same constant given in $\mathcal{H}_L$. The following embedding results are taken from Theorem 2.23 by Lu–Vetro–Zeng [43].

Proposition 2.7. *Let hypotheses* (H0) *be satisfied. Then the following hold:*

(i) $L^{\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}_{p,L}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W^{1,\mathcal{H}_{p,L}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}_L}_0(\Omega) \hookrightarrow W^{1,\mathcal{H}_{p,L}}_0(\Omega) \hookrightarrow W^{1,r(\cdot)}_0(\Omega)$, *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \Omega$;
(ii) *let* $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, *then* $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W^{1,\mathcal{H}_L}_0(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p^*(x)$ *for all* $x \in \overline{\Omega}$;
(iii) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{H}_{r,L}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega), W^{1,\mathcal{H}_L}_0(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{H}_{r,L}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;
(iv) *let* $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma > N$, *then* $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *and* $W^{1,\mathcal{H}_L}_0(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p_*(x)$ *for all* $x \in \overline{\Omega}$;
(v) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega), W^{1,\mathcal{H}_L}_0(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p_*(x)$ *for all* $x \in \overline{\Omega}$;

We point out that a function $h \colon \overline{\Omega} \to \mathbb{R} \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ is called log-Hölder continuous, if $|h(x) - h(y)| \leq \frac{C}{|\log|x-y||}$ for some $C > 0$ and for all $x, y \in \overline{\Omega}$ with $|x - y| < \frac{1}{2}$.

Remark 2.8. *If the domain* $\Omega \subset \mathbb{R}^N$ *is bounded and* $\gamma > N$, *then the following inclusions hold*

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

The next proposition and its proof can be found in Lu–Vetro–Zeng [43, Proposition 2.24].

Proposition 2.9. *Let hypotheses* (H0) *be satisfied.*

(i) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{H}_L}(\Omega)$;
(ii) *the Poincaré inequality holds, namely*

$$\|u\|_{\mathcal{H}_L} \leq C\|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W^{1,\mathcal{H}_L}_0(\Omega),$$

where the constant $C > 0$ *is independent of* u.

In the sequel, for all $t \in \mathbb{R}$, we set $t_+ := \max\{t, 0\}$ and $t_- := -\min\{t, 0\}$. Moreover, we write $u_\pm(\cdot) := [u(\cdot)]_\pm$ for any function $u \colon \Omega \to \mathbb{R}$. The following proposition is taken from Lu–Vetro–Zeng [43, Proposition 2.25].

Proposition 2.10. *Let hypotheses* (H0) *be satisfied and let* $u \in W^{1,\mathcal{H}_L}(\Omega)$, $v \in W^{1,\mathcal{H}_L}_0(\Omega)$ *and* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_L}(\Omega)$ *be a sequence.*

(i) $\pm u_\pm \in W^{1,\mathcal{H}_L}(\Omega)$, *and* $\nabla(\pm u_\pm) = \nabla u 1_{\{\pm u > 0\}}$;
(ii) *if* $u_n \to u$ *in* $W^{1,\mathcal{H}_L}(\Omega)$, *then* $\pm(u_n)_\pm \to \pm u_\pm$ *in* $W^{1,\mathcal{H}_L}(\Omega)$;
(iii) $\pm v_\pm \in W^{1,\mathcal{H}_L}_0(\Omega)$.

As we mentioned before, we can take the multivalued terms $\mathcal{F}$ and $\mathcal{G}$ as two Clarke's generalized gradient of locally Lipschitz functions. So, we recall the following definitions and results about nonsmooth analysis. For a given real Banach space X and corresponding dual space X^*, a function $F \colon X \to \mathbb{R}$ is locally Lipschitz continuous at $u \in X$, if

$$|F(u) - F(v)| \leq L_u\|u - v\|_X \quad \text{for all } u, v \in N(u),$$

where $L_u > 0$ is the Lipschitz constant and $N(u)$ denotes the neighborhood of u.

Definition 2.11. *For a locally Lipschitz continuous function $F\colon X \to \mathbb{R}$, let $F^\circ(u;v)$ be Clarke's generalized directional derivative of F at the point $u \in X$ in the direction $v \in X$, that is*

$$F^\circ(u;v) = \limsup_{w\to u,\, t\searrow 0} \frac{F(w+tv)-F(w)}{t}.$$

Moreover, Clarke's generalized gradient $\partial F\colon X \to 2^{X^*}$ of the locally Lipschitz function $F\colon X \to \mathbb{R}$ is given by

$$\partial F(u) = \{h \in X^*\colon F^\circ(u;v) \geq \langle h, v\rangle_{X^*\times X} \text{ for all } v \in X\} \quad \text{for all } u \in X.$$

The next proposition summarizes the main properties of generalized directional derivatives and generalized gradients in the sense of Clarke, see, for example Carl–Le [11].

Proposition 2.12. *Let $F\colon X \to \mathbb{R}$ be locally Lipschitz continuous at $u \in X$ with $L_u > 0$ being the Lipschitz constant. Then the following hold:*

(i) *the function $v \mapsto F^\circ(u;v)$ is positively homogeneous, subadditive, and satisfies*

$$|F^\circ(u;v)| \leq L_u \|v\|_X \quad \text{for all } v \in X;$$

(ii) *the function $(u,v) \mapsto F^\circ(u;v)$ is upper semicontinuous;*

(iii) *$\partial F(u)$ is a nonempty, convex, and weakly* compact subset of X^* with $\|\xi\|_{X^*} \leq L_u$ for all $\xi \in \partial F(u)$;*

(iv) *$F^\circ(u;v) = \max\{\langle \xi, v\rangle_{X^*\times X} : \xi \in \partial F(u)\}$ for all $v \in X$;*

(v) *the multivalued function $X \ni u \mapsto \partial F(u) \subset X^*$ is upper semicontinuous.*

The following lemma is taken from Ho–Winkert [33, Lemma 2.10] which is a necessary tool to show the boundedness of weak solutions of problem (P).

Lemma 2.13. *Let $\{Z_n\}, n = 0,1,2,\dots,$ be a sequence of positive numbers, satisfying the recursion inequality*

$$Z_{n+1} \leq M k^n \left(Z_n^{1+\lambda_1} + Z_n^{1+\lambda_2}\right), \quad n = 0,1,2,\dots,$$

for some $k > 1$, $M > 0$ and $\lambda_2 \geq \lambda_1 > 0$. If

$$Z_0 \leq \min\left(1, (2M)^{-\frac{1}{\lambda_1}} k^{-\frac{1}{\lambda_1^2}}\right)$$

or

$$Z_0 \leq \min\left((2M)^{-\frac{1}{\lambda_1}} k^{-\frac{1}{\lambda_1^2}}, (2M)^{-\frac{1}{\lambda_2}} k^{-\frac{1}{\lambda_1\lambda_2}-\frac{\lambda_2-\lambda_1}{\lambda_2^2}}\right),$$

then $Z_n \leq 1$ for some $n \in \mathbb{N}\cup\{0\}$. Moreover,

$$Z_n \leq \min\left(1, (2M)^{-\frac{1}{\lambda_1}} k^{-\frac{1}{\lambda_1^2}} k^{-\frac{n}{\lambda_1}}\right), \quad \text{for all } n \geq n_0,$$

where n_0 is the smallest $n \in \mathbb{N}\cup\{0\}$ satisfying $Z_n \leq 1$. In particular, $Z_n \to 0$ as $n \to \infty$.

3. New continuous and compact embedding results

In this section we are going to prove new continuous and compact embeddings related to our Musielak-Orlicz Sobolev spaces $W^{1,\mathcal{H}_L}(\Omega)$. We use ideas from the works of Cianchi [13, 14] and Cianchi–Diening [15].

We start with the definition of a Young function.

Definition 3.1. *A function $\varphi\colon [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, continuous, non-constant, $\varphi(0) = 0$ and $\varphi(t) = \int_0^t a(\tau)\,\mathrm{d}\tau$, where $a\colon [0,\infty) \to [0,\infty]$ is a non-decreasing function. In addition, $\varphi\colon \Omega\times[0,\infty) \to [0,\infty]$ is called a generalized Young function if $\varphi(x,\cdot)$ is a Young function for a.a. $x \in \Omega$ and $\varphi(\cdot,t)$ is measurable for all $t \geq 0$.*

Next, we introduce another function associated to the Young function φ (see also Cianchi [13]):

$$\overline{\varphi}(x,t) = \begin{cases} 2\varphi_0(x,t) - 1 & \text{if } t \geq 1,\\ \limsup\limits_{|x|\to\infty} \varphi_0(x,t) & \text{if } 0 \leq t < 1, \end{cases}$$

where $\varphi_0(x,t) = \max\{\varphi(x,\varphi^{-1}(x,1)t), 2t-1\}$ for all $x \in \Omega$ and for all $t \geq 0$. Furthermore, the Sobolev conjugate φ_N of φ is given by

$$\varphi_N(x,t) = \overline{\varphi}(x, T_N^{-1}(x,t))$$

for all $x \in \Omega$ and for all $t \geq 0$ with $T_N\colon \Omega \times [0,\infty) \to [0,\infty)$ defined as

$$T_N(x,t) = \left(\int_0^t \left(\frac{\tau}{\overline{\varphi}(x,\tau)} \right)^{\frac{1}{N-1}} \, \mathrm{d}\tau \right)^{\frac{1}{N'}}$$

for all $x \in \Omega$ and for all $t \geq 0$. Indeed, φ_N is a generalized Young function.

Coming back to our N-function $\mathcal{H}_L$ given in (1.2), we see that it is a generalized Young function and since it satisfies (A0) and the Δ_2-condition (see Cianchi–Diening [15] and Lu–Vetro–Zeng [43]), we deduce that $\overline{\mathcal{H}_L}$ is equivalent to $\mathcal{H}_L$ for all $x \in \Omega$ and for all $t \geq 0$ since Ω is a bounded domain in $\mathbb{R}^N$.

The following hypotheses are required to guarantee our main embedding results.

(H1) Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume $\mu\colon \overline{\Omega} \to [0,\infty)$ is a κ-Hölder continuous function and $p, q \in C^{0,1}(\overline{\Omega})$ such that
- (i) $1 < p(x) < N$ and $p(x) \leq q(x)$ with $\Omega_1 := \{x \in \Omega\colon p(x) < q(x)\} \nsubseteq \Omega_0 := \{x \in \Omega\colon \mu(x) = 0\}$;
- (ii) $\dfrac{q(x)}{p(x)} < 1 + \dfrac{\kappa}{N}$;

for all $x \in \overline{\Omega}$.

Next, we recall the conditions (A0) and (A1) for Young functions.

Definition 3.2. *A Young function φ is said to satisfy the condition:*

(A0) *if there exists $0 < \beta \leq 1$ satisfying*

$$\beta \leq \varphi^{-1}(x,1) \leq \frac{1}{\beta}$$

for a.a. $x \in \Omega$. This implies, in particular, that one can find a constant $\beta \in [0,1]$ satisfying $\varphi(x,\beta) \leq 1 \leq \varphi(x, \frac{1}{\beta})$ for a.a. $x \in \Omega$.

(A1) *if there exists $0 < \beta < 1$ satisfying*

$$\beta\varphi^{-1}(x,t) \leq \varphi^{-1}(y,t)$$

for all $t \in \left[1, \frac{1}{|B_R|}\right]$, for a.a. $x, y \in B_R \cap \Omega$ with $|B_R| \leq 1$.

According to Cianchi [13, Theorem 3.6, Theorem 3.7] along with Lu–Vetro–Zeng [43, Theorem 2.17] we obtain the following Proposition.

Proposition 3.3. *Let hypotheses* (H1) *be satisfied. Then the following hold:*
- (i) *$W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{(\mathcal{H}_L)_N}(\Omega)$;*
- (ii) *If $\xi\colon \Omega \times [0,\infty) \to [0,\infty)$ is a continuous function such that $\xi \in \mathrm{N}(\Omega)$ and $\xi \ll (\mathcal{H}_L)_N$, then $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow\hookrightarrow L^{\xi}(\Omega)$.*

Based on the above Proposition, we are ready to give the following embedding results for the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$ to suitable Musielak-Orlicz spaces.

Proposition 3.4. *Let hypotheses* (H1) *be satisfied. If*

$$\mathcal{B}(x,t) = t^{\tau(x)} \log^{\frac{\tau(x)}{p(x)}}(e + \omega t) + \mu(x)^{\frac{\pi(x)}{q(x)}} t^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e + \omega t)$$

for all $x \in \overline{\Omega}$, for all $t \in [0,\infty)$ with $\tau, \pi \in C(\overline{\Omega})$ such that $1 < \tau(x) \leq p^*(x)$ and $1 < \pi(x) \leq q^*(x)$ for all $x \in \overline{\Omega}$, then the embedding

$$W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega). \tag{3.1}$$

is continuous. Additionally, if $1 < \tau(x) < p^*(x)$ and $1 < \pi(x) < q^*(x)$ for all $x \in \overline{\Omega}$, the above embedding is compact.

Proof. According to [43, Theorem 2.17] we know that if (H1) hold, then $\mathcal{H}_L$ satisfies (A0) and (A1). First, let $\omega > 0$, since $\varphi(x,t) = t^{p(x)} \log(e + \omega t) \approx t^{p(x)} \log(1 + t)$ for $t > K > 1$. We see that there exists $K > 1$ large enough such that if $\varphi(x,t) = t^{p(x)} \log(e + \omega t)$ then according to Example 1.2 of Cianchi [13], we have

$$\varphi_N(x,t) = t^{p^*(x)} \log^{\frac{p^*(x)}{p(x)}}(e + \omega t)$$

for all $x \in \Omega$ and for all $t > K$. As for $\mathcal{H}_L(x,t) = [t^{p(x)} + \mu(x) t^{q(x)}] \log(e + \omega t)$, note that

$$\mathcal{H}_L(x,t) \approx \varphi(x,t) = t^{p(x)} \log(e + \omega t)$$

for $(x,t)\in(\{x\in\Omega:\mu(x)=0\}\times\{t\in\mathbb{R}\})\cup(\{x\in\Omega:\mu(x)\neq 0\}\times\{t\in\mathbb{R}:t<1\})$, and

$$\mathcal{H}_L(x,t)\approx t^{q(x)}\log(e+\omega t)\quad\text{for } (x,t)\in\{x\in\Omega:\mu(x)\neq 0\}\times\{t\in\mathbb{R}:t\geq 1\}.$$

Therefore

$$(\mathcal{H}_L)_N(x,t)\approx t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+\mu(x)^s t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t)=:\mathcal{H}^*(x,t)$$

with $s>0$. Thus, employing Theorem 3.6 by Cianchi–Diening [15], we obtain $W^{1,\mathcal{H}_L}(\Omega)\hookrightarrow L^{\mathcal{H}^*}(\Omega)$. Moreover, if $0\leq t\leq K$, then we see that $\mathcal{H}^*(x,t)\leq C(K,p,q)\leq C\mathcal{H}_L(x,t)$ for all $x\in\Omega$. It follows that $W^{1,\mathcal{H}_L}(\Omega)\hookrightarrow L^{\mathcal{H}_L}(\Omega)\hookrightarrow L^{\mathcal{H}^*}(\Omega)$. Also, it is not hard to verify that if $\omega=0$, the same embedding results hold true. Furthermore, since

$$\mathcal{B}(x,t)\leq t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+\mu(x)^s t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t)+2$$

for all $x\in\overline{\Omega}$ and all $t\in[0,\infty)$, which implies $\mathcal{B}(x,t)\prec\mathcal{H}^*(x,t)$, it follows the continuous embedding (3.1).

On the other hand, to verify the compact embedding we only need to show that $\mathcal{B}(x,t)\ll\mathcal{H}^*(x,t)$ under the assumption that $1<\tau(x)<p^*(x)$ and $1<\pi(x)<q^*(x)$ for all $x\in\overline{\Omega}$. In fact, for all $t\geq 0$ and $C\geq 1$, we have

$$\log(e+Ct)\leq C\log(e+t). \tag{3.2}$$

Thus, for any $k>0$, for all $x\in\Omega$, and for all $t\geq 0$, employing (3.2) we have

$$\begin{aligned}\frac{\mathcal{B}(x,t)}{\mathcal{H}^*(x,t)}&=\frac{(kt)^{\tau(x)}\log^{\frac{\tau(x)}{p(x)}}(e+\omega kt)+\mu(x)^{\frac{\pi(x)}{q(x)}}(kt)^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega kt)}{t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+\mu(x)^s t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t)}\\&\leq\left(1+k^{\tau^++\frac{\tau^+}{p^-}}+k^{\pi^++\frac{\pi^+}{q^-}}\right)\frac{t^{\tau(x)}\log^{\frac{\tau(x)}{p(x)}}(e+\omega t)+\mu(x)^{\frac{\pi(x)}{q(x)}}t^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega t)}{t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+\mu(x)^s t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t)}\end{aligned}$$

Applying Young's inequality, we get

$$t^{\tau(x)}\log^{\frac{\tau(x)}{p(x)}}(e+\omega t)\leq\varepsilon t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+C(\varepsilon),$$

analogously,

$$t^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega t)\leq\varepsilon t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t)+C(\varepsilon).$$

Due to $\mu\in L^\infty(\Omega)$, if $\varepsilon>0$ small enough, it follows that $\lim_{t\to\infty}\frac{\mathcal{B}(x,t)}{\mathcal{H}^*(x,t)}=0$ uniformly for a.e. $x\in\Omega$. □

When it comes to the trace embedding, note that for $\varphi(x,t)=t^{p(x)}\log(e+\omega t)$ we take

$$\varphi^-(x,t):=t^{p^-}\log(e+\omega t)\prec\varphi(x,t)=t^{p(x)}\log(e+\omega t).$$

And if $\omega>0$, invoking Example 3.3 and the embedding theorems given by Cianchi [14], we see that there exists $K_1>1$ such that $\log(t)\approx\log(e+\omega t)$ for all $t>K_1$, then $W^{1,\varphi}(\Omega)\hookrightarrow W^{1,\varphi^-}(\Omega)\hookrightarrow L^{\varphi_T}(\partial\Omega)$ with

$$\varphi_T(x,t):=t^{(p^-)_*}\log^{\frac{(p^-)_*}{p^-}}(e+\omega t).$$

As done above in the proof of Proposition 3.4, we get $W^{1,\mathcal{H}_L}(\Omega)\hookrightarrow L^{H_T}(\partial\Omega)$ where

$$H_T(x,t):=t^{(p^-)_*}\log^{\frac{(p^-)_*}{p^-}}(e+\omega t)+\mu(x)^s t^{(q^-)_*}\log^{\frac{(q^-)_*}{q^-}}(e+\omega t).$$

However, if $0\leq t\leq K_1$ then we see that $H_T(x,t)\leq C(K_1,p,q)\leq t^r$ with $r\leq p^-$. So, we have $W^{1,\mathcal{H}_L}(\Omega)\hookrightarrow L^r(\partial\Omega)\hookrightarrow L^{H_T}(\partial\Omega)$. Similarly, for $\omega=0$, we get the same results.

Next, we prove the trace embedding results related to the space $W^{1,\mathcal{H}_L}(\Omega)$.

Proposition 3.5. *Let hypotheses* (H1) *be satisfied. If*

$$\mathcal{B}_\Gamma(x,t)=t^{\theta(x)}\log^{\frac{\theta(x)}{p(x)}}(e+\omega t)+\mu(x)^{\frac{\vartheta(x)}{q(x)}}t^{\vartheta(x)}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega t)\quad\text{for all } x\in\overline{\Omega},\ \text{all } t\in[0,\infty),$$

with $\theta,\vartheta\in C(\overline{\Omega})$ such that $1<\theta(x)\leq(p_*)^-$ and $1<\vartheta(x)\leq(q_*)^-$ for all $x\in\overline{\Omega}$, then the following continuous embedding holds true

$$W^{1,\mathcal{H}_L}(\Omega)\hookrightarrow L^{\mathcal{B}_\Gamma}(\Gamma_1). \tag{3.3}$$

Moreover, if $1<\theta(x)<(p_*)^-$ and $1<\vartheta(x)<(q_*)^-$ for all $x\in\overline{\Omega}$, then the above embedding is compact.

Proof. First, one can observe that for $r \in C(\overline{\Omega})$ with $1 \leq r(x) < N$ for all $x \in \overline{\Omega}$, there hold $(r^-)^* = (r^*)^-$, $(r^-)_* = (r_*)^-$ and $\mathcal{B}_\Gamma(x,t) \leq H_T(x,t) + 2$, thus $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{H_T}(\partial\Omega) \hookrightarrow L^{\mathcal{B}_\Gamma}(\partial\Omega)$, which implies (3.3).

On the other hand, let $1 < \theta(x) < (p_*)^-$ and $1 < \vartheta(x) < (q_*)^-$ for all $x \in \overline{\Omega}$. Due to $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow W^{1,p^-}(\Omega) \hookrightarrow\hookrightarrow L^1(\Gamma_1)$, then $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_L}(\Omega)$ is a bounded sequence, we see that $u_n \to u$ in measure on Γ_1 (in the sense of subsequence). For fixed $\varepsilon > 0$ and $w_{j,k} := \frac{u_j(x)-u_k(x)}{\varepsilon}$ with $j,k \in \mathbb{N}$, we know that $\{w_{j,k}\}$ is bounded in $L^{\mathcal{B}_\Gamma}(\Gamma_1)$, i.e. $\|w_{j,k}\|_{\mathcal{B}_\Gamma,\Gamma_1} < K$, where $K > 0$. Similar to the proof of Proposition 3.4 we can show that

$$\lim_{t\to\infty} \frac{\mathcal{B}_\Gamma(x,t)}{H_T(x,t)} = 0 \quad \text{uniformly for a.e. } x \in \Gamma_1.$$

Hence there exists $t_1 > 0$ such that for all $t > t_1$ and $x \in \overline{\Omega}$, we get

$$\mathcal{B}_\Gamma(x,t) \leq \frac{1}{4} H_T\left(x, \frac{t}{K}\right).$$

Let $\delta > 0$ such that if $|E| < \delta$ we have

$$\int_E \mathcal{B}_\Gamma(x,t_1) \leq \frac{1}{4}.$$

Moreover, we choose $\Gamma_{j,k} = \{x \in \Gamma_1 : |w_{j,k}(x)| \geq \mathcal{B}_\Gamma^{-1}\left(\frac{1}{2|\Gamma_1|}\right)\}$. Note that $\{u_j\}$ converges in measure and according to the definition of $w_{j,k}$ we know that there exists $N \in \mathbb{N}^+$ such that if $j,k > N$, then $|\Gamma_{j,k}| \leq \delta$. We define

$$\Gamma'_{j,k} = \{x \in \Gamma_{j,k} : |w_{j,k}| > t_1\} \quad \text{and} \quad \Gamma''_{j,k} = \Gamma_{j,k} \backslash \Gamma'_{j,k}.$$

Based on the above setting, for $j,k \geq N$ we get

$$\begin{aligned}
\int_{\Gamma_1} \mathcal{B}_\Gamma(x,|w_{j,k}(x)|)d\varsigma &= \int_{\Gamma_1\backslash\Gamma_{j,k}} \mathcal{B}_\Gamma(x,|w_{j,k}(x)|)d\varsigma + \int_{\Gamma'_{j,k}} \mathcal{B}_\Gamma(x,|w_{j,k}(x)|)d\varsigma \\
&\quad + \int_{\Gamma''_{j,k}} \mathcal{B}_\Gamma(x,|w_{j,k}(x)|)d\varsigma \\
&\leq \frac{|\Gamma_1|}{2|\Gamma_1|} + \frac{1}{4}\int_{\Gamma'_{j,k}} H_T\left(x, \frac{|w_{j,k}(x)|}{K}\right)d\varsigma + \int_{\Gamma_{j,k}} \mathcal{B}_\Gamma(x,t_1)d\varsigma \leq 1.
\end{aligned}$$

This implies that $\|u_j - u_k\|_{\mathcal{B}_\Gamma,\Gamma_1} \leq \varepsilon$ and therefore by the completeness of $L^{\mathcal{B}_\Gamma}(\Gamma_1)$ we see that $u_n \to u$ in $L^{\mathcal{B}_\Gamma}(\Gamma_1)$. ☐

4. Boundedness of weak solutions to problem (P)

In this section, we prove the boundedness of weak solutions of problem (P) by employing De Giorgi's method along with localization arguments. The proofs of our main results are using ideas of the papers by Ho–Kim [31], Ho–Kim–Winkert–Zhang [32], Ho–Winkert [33], and Winkert–Zacher [57, 58].

For any $u \in M(\Omega)$ and $y \in [M(\Omega)]^N$ we define

$$\mathcal{F}(u) = \{\xi \in M(\Omega) \colon \xi(x) \in f(x,u(x),y(x)) \text{ for a.a. } x \text{ in } \Omega\},$$

as the measurable selections of $f(\cdot,u,y)$, due to the hypotheses (A) and (B), which will be introduced below, the above set is nonempty. Similarly, for any $u \in M(\Omega)$ we define

$$\mathcal{G}(u) = \{\zeta \in M(\Gamma_1) \colon \zeta(x) \in g(x,u(x)) \text{ for a.a. } x \text{ in } \Gamma_1\},$$

which is also nonempty.

4.1. **Subcritical growth.** We start with the subcritical case and suppose the following assumptions:

(A) Let $\mathcal{A} \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying

(i)

$$\begin{aligned}
&|\mathcal{A}(x,t,y)| \\
&\leq \alpha_1\Bigg[|t|^{\frac{p^*(x)}{p'(x)}} \log^{\frac{N-1}{N-p(x)}}(e+\omega|t|) + \mu(x)^{\frac{N-1}{N-q(x)}}|t|^{\frac{q^*(x)}{q'(x)}} \log^{\frac{N-1}{N-q(x)}}(e+\omega|t|) \\
&\qquad + |y|^{p(x)-1}\log(e+\omega|y|) + \mu(x)|y|^{q(x)-1}\log(e+\omega|y|) + 1\Bigg],
\end{aligned}$$

(ii)

$$\begin{aligned}&\mathcal{A}(x,t,y)\cdot y\\&\geq \alpha_2\Big[|y|^{p(x)}\log(e+\omega|y|)+\mu(x)|y|^{q(x)}\log(e+\omega|y|)\Big]\\&\quad-\alpha_3\left[|t|^{\iota(x)}\log^{\frac{\iota(x)}{p(x)}}(e+\omega|t|)+\mu(x)^{\frac{\pi(x)}{q(x)}}|t|^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega|t|)+1\right],\end{aligned}$$

for a.a. $x\in\Omega$, for all $t\in\mathbb{R}$ and for all $y\in\mathbb{R}^N$ with positive constants α_1,α_2 and α_3.

(B) (i) Let $f\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ and $g\colon\Gamma_1\times\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ be graph measurable functions. Moreover, $f(x,\cdot,\cdot)\colon\mathbb{R}\times\mathbb{R}^N\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ is upper semicontinuous for a.a. $x\in\Omega$ and $g(x,\cdot)\colon\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ is upper semicontinuous for a.a. $x\in\Gamma_1$.

(ii) Let $\iota,\pi\in C(\overline{\Omega})$ with $p(x)<\iota(x)<p^*(x)$ and $q(x)<\pi(x)<q^*(x)$ for all $x\in\overline{\Omega}$ and

$$\begin{aligned}&\sup\{|\xi|\colon\xi\in f(x,t,y)\}\\&\leq\beta\Bigg[|t|^{\iota(x)-1}\log^{\frac{\iota(x)}{p(x)}}(e+\omega|t|)+\mu(x)^{\frac{\pi(x)}{q(x)}}|t|^{\pi(x)-1}\log^{\frac{\pi(x)}{q(x)}}(e+\omega|t|)\\&\quad+|y|^{\frac{p(x)}{\iota'(x)}}\log^{\frac{1}{p(x)}+\frac{1}{\iota'(x)}}(e+\omega|y|)\\&\quad+\mu(x)^{\frac{1}{q(x)}+\frac{1}{\pi'(x)}}|y|^{\frac{q(x)}{\pi'(x)}}\log^{\frac{1}{q(x)}+\frac{1}{\pi'(x)}}(e+\omega|y|)+1\Bigg]\end{aligned}$$

for a.a. $x\in\Omega$, for all $t\in\mathbb{R}$ and for all $y\in\mathbb{R}^N$ with a positive constant β.

(iii) Let $\theta,\vartheta\in C(\overline{\Omega})$ with $p(x)<\theta(x)<(p_*)^-$ and $q(x)<\vartheta(x)<(q_*)^-$ for all $x\in\overline{\Omega}$ and

$$\begin{aligned}&\sup\{|\zeta|\colon\zeta\in g(x,t)\}\\&\leq\gamma\left[|t|^{\theta(x)-1}\log^{\frac{\theta(x)}{p(x)}}(e+\omega|t|)+\mu(x)^{\frac{\vartheta(x)}{q(x)}}|t|^{\vartheta(x)-1}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega|t|)+1\right]\end{aligned}$$

for a.a. $x\in\Gamma_1$ and for all $t\in\mathbb{R}$ with a positive constant γ.

According to the embedding results given by Propositions 3.4 and 3.5 we know that the following definition of weak solutions to problem (P) are well defined under hypotheses (A) and (B).

Definition 4.1. *A function $u\in W^{1,\mathcal{H}_L}(\Omega)$ is a weak solution to problem (P), if there exist $\xi(x)\in f(x,u(x),\nabla u(x))$ for a.a. $x\in\Omega$ and $\zeta(x)\in g(x,u(x))$ for a.a. $x\in\Gamma_1$ such that*

$$\int_\Omega\mathcal{A}(x,u,\nabla u)\cdot\nabla v\,\mathrm{d}x=\int_\Omega\xi v\,\mathrm{d}x+\int_{\Gamma_1}\zeta v\,\mathrm{d}\varsigma \tag{4.1}$$

is satisfied for all $v\in W^{1,\mathcal{H}_L}(\Omega)$.

Now, we are going to state and prove our main results concerning the boundedness of weak solutions.

Theorem 4.2. *Let hypotheses* (H1), (A) *and* (B) *be satisfied. Then every weak solution of problem* (P) *belongs to $L^\infty(\Omega)\cap L^\infty(\Gamma_1)$ and it holds*

$$\|u\|_{\infty,\Omega}+\|u\|_{\infty,\Gamma_1}\leq C\max\Big\{\left(\|u\|_{\mathcal{B},\Omega}+\|u\|_{\mathcal{B}_\Gamma,\Gamma_1}\right)^{r_1},\left(\|u\|_{\mathcal{B},\Omega}+\|u\|_{\mathcal{B}_\Gamma,\Gamma_1}\right)^{r_2}\Big\}, \tag{4.2}$$

with C,r_1,r_2 being positive constants independent of u.

Proof. Let $u\in W^{1,\mathcal{H}_L}(\Omega)$ be a weak solution of problem (P). We divided the proof into several steps.

Step 1. Construct the iteration sequence and make basic estimates.

First, for any $n\in\mathbb{N}_0$ we define

$$Z_n:=\int_{A_{\psi_n}}\left[(u-\psi_n)^{\iota(x)}\log^{\frac{\iota(x)}{p(x)}}(e+\omega(u-\psi_n))+\mu(x)^{\frac{\pi(x)}{q(x)}}(u-\psi_n)^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega(u-\psi_n))\right]\mathrm{d}x,$$

with

$$A_\psi:=\{x\in\Omega\colon u(x)>\psi\},\quad\psi\in\mathbb{R} \tag{4.3}$$

and

$$Y_n:=\int_{B_{\psi_n}}\left[(u-\psi_n)^{\theta(x)}\log^{\frac{\theta(x)}{p(x)}}(e+\omega(u-\psi_n))+\mu(x)^{\frac{\vartheta(x)}{q(x)}}(u-\psi_n)^{\vartheta(x)}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega(u-\psi_n))\right]\mathrm{d}\varsigma,$$

with

$$B_\psi:=\{x\in\Gamma_1\colon u(x)>\psi\},\quad\psi\in\mathbb{R}. \tag{4.4}$$

For $n \in \mathbb{N}_0$, we define

$$\psi_n := \psi_* \left(2 - \frac{1}{2^n}\right), \tag{4.5}$$

where $\psi_* > 0$ will be specified later. It is easy to see that for all $n \in \mathbb{N}_0$, there hold

$$\begin{aligned} &\psi_n \nearrow 2\psi_* \quad \text{and} \quad \psi_* \le \psi_n < 2\psi_*, \\ &A_{\psi_{n+1}} \subset A_{\psi_n} \quad \text{and} \quad Z_{n+1} \le Z_n, \\ &B_{\psi_{n+1}} \subset B_{\psi_n} \quad \text{and} \quad Y_{n+1} \le Y_n. \end{aligned} \tag{4.6}$$

Furthermore, we have the following estimates

$$u(x) - \psi_n \ge u(x)\left(1 - \frac{\psi_n}{\psi_{n+1}}\right) = \frac{u(x)}{2^{n+2}-1} \quad \text{for a.a. } x \in A_{\psi_{n+1}}$$

and

$$\begin{aligned} \left|A_{\psi_{n+1}}\right| &\le \int_{A_{\psi_{n+1}}} \left(\frac{u-\psi_n}{\psi_{n+1}-\psi_n}\right)^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e + \omega(u-\psi_n))\,\mathrm{d}x \\ &\le \int_{A_{\psi_n}} \frac{2^{\iota(x)(n+1)}}{\psi_*^{\iota(x)}} (u-\psi_n)^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e + \omega(u-\psi_n))\,\mathrm{d}x. \end{aligned}$$

It follows that

$$u(x) \le \left(2^{n+2}-1\right)(u(x)-\psi_n) \quad \text{for a.a. } x \in A_{\psi_{n+1}} \text{ and for all } n \in \mathbb{N}_0 \tag{4.7}$$

and

$$\left|A_{\psi_{n+1}}\right| \le \left(\psi_*^{-\iota^-} + \psi_*^{-\iota^+}\right) 2^{(n+1)\iota^+} Z_n \le 2\left(1 + \psi_*^{-\iota^+}\right) 2^{(n+1)\iota^+} Z_n \quad \text{for all } n \in \mathbb{N}_0. \tag{4.8}$$

Similarly, we get

$$u(x) \le \left(2^{n+2}-1\right)(u(x)-\psi_n) \quad \text{for a.a. } x \in B_{\psi_{n+1}} \text{ and for all } n \in \mathbb{N}_0 \tag{4.9}$$

and

$$\left|B_{\psi_{n+1}}\right|_\varsigma \le \left(\psi_*^{-\theta^-} + \psi_*^{-\theta^+}\right) 2^{(n+1)\theta^+} Y_n \le 2\left(1 + \psi_*^{-\theta^+}\right) 2^{(n+1)\theta^+} Y_n \quad \text{for all } n \in \mathbb{N}_0. \tag{4.10}$$

In the sequel, for any fixed $i \in \mathbb{N}$, we denote by C_i a positive constant independent of u, n, ψ_* and we set

$$X_n = Z_n + Y_n, \tag{4.11}$$

We claim that, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} &\int_{A_{\psi_{n+1}}} \left[|\nabla u|^{p(x)} \log(e + \omega|\nabla u|) + \mu(x)|\nabla u|^{q(x)} \log(e + \omega|\nabla u|)\right] \mathrm{d}x \\ &\le C_1 \left(1 + \psi_*^{-\alpha_0}\right) 2^{n(\beta_0 + \frac{\beta_0}{p^-})} X_n, \end{aligned} \tag{4.12}$$

where $\alpha_0 := \max\{\iota^+, \theta^+\}$ and $\beta_0 := \max\{\iota^+, \pi^+, \theta^+, \vartheta^+\}$.

First, we take $\varphi = (u - \psi_{n+1})_+ \in W^{1,\mathcal{H}_L}(\Omega)$ as test function in (4.1) and obtain

$$\int_{A_{\psi_{n+1}}} \mathcal{A}(x, u, \nabla u) \cdot \nabla u \,\mathrm{d}x = \int_{A_{\psi_{n+1}}} \xi\,(u - \psi_{n+1})\,\mathrm{d}x + \int_{B_{\psi_{n+1}}} \zeta\,(u - \psi_{n+1})\,\mathrm{d}\varsigma. \tag{4.13}$$

Note that $u \ge u - \psi_{n+1} > 0$ and $u \le u^{\iota(x)} + 1$ in $A_{\psi_{n+1}}$. Using this and applying (A) (ii), (B) (ii) and Young's inequality we get

$$\begin{aligned} &\int_{A_{\psi_{n+1}}} \mathcal{A}(x, u, \nabla u) \cdot \nabla u \,\mathrm{d}x \\ &\ge \alpha_2 \int_{A_{\psi_{n+1}}} \left[|\nabla u|^{p(x)} \log(e + \omega|\nabla u|) + \mu(x)|\nabla u|^{q(x)} \log(e + \omega|\nabla u|)\right] \mathrm{d}x \\ &\quad - \alpha_3 \int_{A_{\psi_{n+1}}} \left[u^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e + \omega u) + \mu(x)^{\frac{\pi(x)}{q(x)}} u^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e + \omega u) + 1\right] \mathrm{d}x \end{aligned}$$

and

$$\int_{A_{\psi_{n+1}}} \xi\,(u - \psi_{n+1})\,\mathrm{d}x$$

$$
\begin{aligned}
&\le \beta \int_{A_{\psi_{n+1}}} \Big[u^{\iota(x)-1} \log^{\frac{\iota(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\pi(x)}{q(x)}} u^{\pi(x)-1} \log^{\frac{\pi(x)}{q(x)}}(e+\omega u) \\
&\quad + |\nabla u|^{\frac{p(x)}{\iota'(x)}} \log^{\frac{1}{p(x)}+\frac{1}{\iota'(x)}}(e+|\nabla u|) + \mu(x)^{\frac{1}{q(x)}+\frac{1}{\pi'(x)}} |\nabla u|^{\frac{q(x)}{\pi'(x)}} \log^{\frac{1}{q(x)}+\frac{1}{\pi'(x)}}(e+|\nabla u|) + 1 \Big] u \, \mathrm{d}x \\
&\le \frac{\alpha_2}{2} \int_{A_{\psi_{n+1}}} \Big[|\nabla u|^{p(x)} \log(e+\omega|\nabla u|) + \mu(x)|\nabla u|^{q(x)} \log(e+\omega|\nabla u|) \Big] \mathrm{d}x \\
&\quad + C_2 \int_{A_{\psi_{n+1}}} \Big[u^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\pi(x)}{q(x)}} u^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e+\omega u) + 1 \Big] \mathrm{d}x.
\end{aligned}
$$

Similar, taking (B) (iii) into account, we get

$$
\begin{aligned}
&\int_{B_{\psi_{n+1}}} \zeta\,(u-\psi_{n+1})\, \mathrm{d}\varsigma \\
&\le \gamma \int_{B_{\psi_{n+1}}} \Big[u^{\theta(x)-1} \log^{\frac{\theta(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\vartheta(x)}{q(x)}} u^{\vartheta(x)-1} \log^{\frac{\vartheta(x)}{q(x)}}(e+\omega u) + 1 \Big] u \, \mathrm{d}\varsigma \\
&\le 2\gamma \int_{B_{\psi_{n+1}}} \Big[u^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\vartheta(x)}{q(x)}} u^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}}(e+\omega u) + 1 \Big] \mathrm{d}\varsigma.
\end{aligned}
$$

From the estimations above along with (4.7) , (4.9) as well as (4.13), we obtain

$$
\begin{aligned}
&\int_{A_{\psi_{n+1}}} \Big[|\nabla u|^{p(x)} \log(e+\omega|\nabla u|) + \mu(x)|\nabla u|^{q(x)} \log(e+\omega|\nabla u|) \Big] \mathrm{d}x \\
&\le C_3 \int_{A_{\psi_{n+1}}} \Big[u^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\pi(x)}{q(x)}} u^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e+\omega u) + 1 \Big] \mathrm{d}x \\
&\quad + C_4 \int_{B_{\psi_{n+1}}} \Big[u^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}}(e+\omega u) + \mu(x)^{\frac{\vartheta(x)}{q(x)}} u^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}}(e+\omega u) + 1 \Big] \mathrm{d}\varsigma \\
&\le C_3 \int_{A_{\psi_{n+1}}} \Big(\left[\left(2^{n+2}-1\right)(u-\psi_n) \right]^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}} \left[e + \omega \left(2^{n+2}-1\right)(u-\psi_n) \right] \\
&\quad + \mu(x)^{\frac{\pi(x)}{q(x)}} \left[\left(2^{n+2}-1\right)(u-\psi_n) \right]^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}} \left[e + \omega \left(2^{n+2}-1\right)(u-\psi_n) \right] \Big) \mathrm{d}x + C_3 \left| A_{\psi_{n+1}} \right| \\
&\quad + C_4 \int_{B_{\psi_{n+1}}} \Big(\left[\left(2^{n+2}-1\right)(u-\psi_n) \right]^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}} \left[e + \omega \left(2^{n+2}-1\right)(u-\psi_n) \right] \\
&\quad + \mu(x)^{\frac{\vartheta(x)}{q(x)}} \left[\left(2^{n+2}-1\right)(u-\psi_n) \right]^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}} \left[e + \omega \left(2^{n+2}-1\right)(u-\psi_n) \right] \Big) \mathrm{d}\varsigma + C_4 \left| B_{\psi_{n+1}} \right|_{\varsigma}.
\end{aligned}
$$

Hence, with (3.2) and (4.11), it follows, for $\alpha_0 := \max\{\iota^+, \theta^+\}$ and $\beta_0 := \max\{\iota^+, \pi^+, \theta^+, \vartheta^+\}$, that

$$
\begin{aligned}
&\int_{A_{\psi_{n+1}}} \Big[|\nabla u|^{p(x)} \log(e+\omega|\nabla u|) + \mu(x)|\nabla u|^{q(x)} \log(e+\omega|\nabla u|) \Big] \mathrm{d}x \\
&\le C_5 2^{n(\beta_0 + \frac{\beta_0}{p^-})} X_n + C_3 |A_{\psi_{n+1}}| + C_4 |B_{\psi_{n+1}}|_{\varsigma},
\end{aligned}
$$

From this, with (4.8) and (4.10), we see that (4.12) is satisfied.

Step 2. Establish the iteration inequalities between X_{n+1} and X_n.

For this purpose, we are going to estimate Z_{n+1} and Y_{n+1} by X_n for $n \in \mathbb{N}_0$. To this end, let $B_i \subset \mathbb{R}^N$ be open balls of radius R with $i \in \mathcal{I} := \{1, \cdots, m\}$ and assume that $\{B_i\}_{i=1}^m$ is a finite open covering of $\overline{\Omega}$ such that $\Omega_i := B_i \cap \Omega$ for $i \in \mathcal{I}$ are Lipschitz domains and also $\hat{\Gamma}_i := B_i \cap \Gamma_1 \neq \emptyset$. For $i \in \mathcal{I}$, we take R small enough fulfilling

$$
|\Omega_i| < 1, \quad \left| \hat{\Gamma}_i \right|_{\varsigma} < 1, \tag{4.14}
$$

$$
p_i^+ < \iota_i^- \le \iota_i^+ < (p^*)_i^- \quad \text{and} \quad q_i^+ < \pi_i^- \le \pi_i^+ < (q^*)_i^-, \tag{4.15}
$$

$$
p_i^+ < \theta_i^- \le \theta_i^+ < (p_*)_i^- \quad \text{and} \quad q_i^+ < \vartheta_i^- \le \vartheta_i^+ < (q_*)_i^-, \tag{4.16}
$$

where for a function $f \in C(\overline{\Omega})$ and $i \in \mathcal{I}$, we denote

$$
f_i^+ := \max_{x \in \overline{\Omega}_i} f(x) \quad \text{and} \quad f_i^- := \min_{x \in \overline{\Omega}_i} f(x).
$$

Next we take $v_n := (u-\psi_{n+1})_+$ for all $n \in \mathbb{N}_0$. Moreover, for any $i \in \mathcal{I}$, $\hat{\alpha}>0$, and $\hat{\beta}>0$, we define

$$T_{n,i}(\hat{\alpha},\hat{\beta}) := \int_{\Omega_i} \left[v_n^{\hat{\alpha}} \log^{\frac{\hat{\alpha}}{p(x)}}(e+\omega v_n) + \mu(x)^{\frac{\hat{\beta}}{q(x)}} v_n^{\hat{\beta}} \log^{\frac{\hat{\beta}}{q(x)}}(e+\omega v_n)\right] \mathrm{d}x.$$

Note that

$$\begin{aligned} Z_{n+1} &= \int_{\Omega} \left[v_n^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e+\omega v_n) + \mu(x)^{\frac{\pi(x)}{q(x)}} v_n^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e+\omega v_n)\right] \mathrm{d}x \\ &\le \sum_{i=1}^m \int_{\Omega_i} \left[v_n^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}}(e+\omega v_n) + \mu(x)^{\frac{\pi(x)}{q(x)}} v_n^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}}(e+\omega v_n)\right] \mathrm{d}x. \end{aligned}$$

Utilizing the above estimate along with the following interpolation inequality

$$t^{\tilde{\beta}} \le t^{\tilde{\alpha}} + t^{\tilde{\gamma}} \quad \text{for all } t \ge 0 \text{ and for all } \tilde{\alpha},\tilde{\beta},\tilde{\gamma} \text{ with } 0<\tilde{\alpha}\le\tilde{\beta}\le\tilde{\gamma}, \tag{4.17}$$

we have

$$Z_{n+1} \le \sum_{i=1}^m \left[T_{n,i}\left(\iota_i^-,\pi_i^-\right) + T_{n,i}\left(\iota_i^+,\pi_i^+\right)\right]. \tag{4.18}$$

With view to (4.15), we can fix $\varepsilon>0$ satisfying

$$\varepsilon < \min_{1\le i\le m} \min\left\{ (p^*)_i^- - \iota_i^+, (q^*)_i^- - \pi_i^+ \right\}. \tag{4.19}$$

Let $\star \in \{+,-\}$ for $i \in \mathcal{I}$. Applying Hölder's inequality and (4.14) we obtain

$$\begin{aligned} T_{n,i}(\iota_i^\star,\pi_i^\star) &= \int_{A_{\psi_{n+1}}\cap\Omega_i} \left[v_n^{\iota_i^\star} \log^{\frac{\iota_i^\star}{p(x)}}(e+\omega v_n) + \mu(x)^{\frac{\pi_i^\star}{q(x)}} v_n^{\pi_i^\star} \log^{\frac{\pi_i^\star}{q(x)}}(e+\omega v_n)\right] \mathrm{d}x \\ &\le \left(\int_{\Omega_i} v_n^{\iota_i^\star+\varepsilon} \log^{\frac{\iota_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\iota_i^\star}{\iota_i^\star+\varepsilon}} |A_{\psi_{n+1}}\cap\Omega_i|^{\frac{\varepsilon}{\iota_i^\star+\varepsilon}} \\ &\quad + \left(\int_{\Omega_i} \mu(x)^{\frac{\pi_i^\star+\varepsilon}{q(x)}} v_n^{\pi_i^\star+\varepsilon} \log^{\frac{\pi_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\pi_i^\star}{\pi_i^\star+\varepsilon}} |A_{\psi_{n+1}}\cap\Omega_i|^{\frac{\varepsilon}{\pi_i^\star+\varepsilon}} \\ &\le |A_{\psi_{n+1}}\cap\Omega_i|^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}} \left[\left(\int_{\Omega_1} v_n^{\iota_i^\star+\varepsilon} \log^{\frac{\iota_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\iota_i^\star}{\iota_i^\star+\varepsilon}} \right. \\ &\quad \left. + \left(\int_{\Omega_i} \mu(x)^{\frac{\pi_i^\star+\varepsilon}{q(x)}} v_n^{\pi_i^\star+\varepsilon} \log^{\frac{\pi_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\pi_i^\star}{\pi_i^\star+\varepsilon}} \right]. \end{aligned} \tag{4.20}$$

Next, we denote

$$\mathcal{B}_\star(x,t) := t^{\iota_i^\star+\varepsilon} \log^{\frac{\iota_i^\star+\varepsilon}{p(x)}}(e+\omega t) + \mu(x)^{\frac{\pi_i^\star+\varepsilon}{q(x)}} t^{\pi_i^\star+\varepsilon} \log^{\frac{\pi_i^\star+\varepsilon}{q(x)}}(e+\omega t),$$

and from (4.19) one can see that

$$\iota_i^\star+\varepsilon < (p^*)_i^- \quad \text{and} \quad \pi_i^\star+\varepsilon < (q^*)_i^- .$$

This along with $(p^*)_i^- = (p^-)_i^*$ and Propositions 2.7 and 3.4 indicates that

$$W^{1,p(\cdot)}(\Omega_i) \hookrightarrow W^{1,p_i^-}(\Omega_i) \hookrightarrow L^{\iota_i^\star+\varepsilon}(\Omega_i) \tag{4.21}$$

and

$$W^{1,\mathcal{H}_L}(\Omega_i) \hookrightarrow L^{\mathcal{B}_\star}(\Omega_i) \tag{4.22}$$

continuously. Taking the embeddings (4.21), (4.22), Proposition 2.6 (for the case $\mu\equiv 0$ and $\Omega=\Omega_i$) as well as (3.2) into account we see that there exist $\sigma>0$ such that

$$\sigma < \min\{\iota_i^- - q_i^+, \pi_i^- - q_i^+, \theta_i^- - q_i^+, \vartheta_i^- - q_i^+\} \quad \text{for } i \in \mathcal{I}$$

satisfying

$$\begin{aligned}&\left(\int_{\Omega_i} v_n^{\iota_i^\star+\varepsilon}\log^{\frac{\iota_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\iota_i^\star}{\iota_i^\star+\varepsilon}}\\&\le \|v_n\|_{\mathcal{B}_\star,\Omega_i}^{\tilde{\iota}_i^\star}\le C_6\,\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{\tilde{\iota}_i^\star}\le C_7\left(S_{n,i}^{\frac{\tilde{\iota}_i^\star}{p_i^-}}+S_{n,i}^{\frac{\tilde{\iota}_i^\star}{q_i^++\sigma}}\right),\end{aligned}\tag{4.23}$$

where

$$\tilde{\iota}_i^\star=\begin{cases}\iota_i^\star & \text{if } \|v_n\|_{\mathcal{B}_\star,\Omega_i}\le 1,\\ \iota_i^\star+\frac{\iota_i^\star}{p_i^-} & \text{if } \|v_n\|_{\mathcal{B}_\star,\Omega_i}>1,\end{cases}$$

and

$$\begin{aligned}S_{n,i}=&\int_{\Omega_i}\Big[|\nabla v_n|^{p(x)}\log(e+\omega|\nabla v_n|)+\mu(x)\,|\nabla v_n|^{q(x)}\log(e+\omega|\nabla v_n|)\Big]\,\mathrm{d}x\\&+\int_{\Omega_i}\Big[v_n^{p(x)}\log(e+\omega v_n)+\mu(x)v_n^{q(x)}\log(e+\omega v_n)\Big]\,\mathrm{d}x.\end{aligned}$$

Furthermore, by using the continuous embedding (4.22) and Proposition 2.6 we find that

$$\begin{aligned}&\left(\int_{\Omega_i}\mu(x)^{\frac{\pi_i^\star+\varepsilon}{q(x)}}v_n^{\pi_i^\star+\varepsilon}\log^{\frac{\pi_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}x\right)^{\frac{\pi_i^\star}{\pi_i^\star+\varepsilon}}\\&\le \|v_n\|_{\mathcal{B}_\star,\Omega_i}^{\tilde{\pi}_i^\star}\le C_8\,\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{\tilde{\pi}_i^\star}\le C_9\left(S_{n,i}^{\frac{\tilde{\pi}_i^\star}{p_i^-}}+S_{n,i}^{\frac{\tilde{\pi}_i^\star}{q_i^++\sigma}}\right),\end{aligned}\tag{4.24}$$

with

$$\tilde{\pi}_i^\star=\begin{cases}\pi_i^\star & \text{if } \|v_n\|_{\mathcal{B}_\star,\Omega_i}\le 1,\\ \pi_i^\star+\frac{\pi_i^\star}{q_i^-} & \text{if } \|v_n\|_{\mathcal{B}_\star,\Omega_i}>1.\end{cases}$$

From the inequalities (4.20), (4.23) and (4.24), we get

$$T_{n,i}(\iota_i^\star,\pi_i^\star)\le C_{10}\left|A_{\psi_{n+1}}\cap\Omega_i\right|^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}\left(S_{n,i}^{\frac{\tilde{\iota}_i^\star}{p_i^-}}+S_{n,i}^{\frac{\tilde{\iota}_i^\star}{q_i^++\sigma}}+S_{n,i}^{\frac{\tilde{\pi}_i^\star}{p_i^-}}+S_{n,i}^{\frac{\tilde{\pi}_i^\star}{q_i^++\sigma}}\right).$$

Combining this and (4.17) we infer

$$T_{n,i}(\iota_i^\star,\pi_i^\star)\le C_{11}\left|A_{\psi_{n+1}}\right|^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}\left(S_n^{1+\gamma_1}+S_n^{1+\gamma_2}\right),\tag{4.25}$$

with

$$\begin{aligned}S_n=&\int_{\Omega}\Big[|\nabla v_n|^{p(x)}\log(e+\omega|\nabla v_n|)+\mu(x)\,|\nabla v_n|^{q(x)}\log(e+\omega|\nabla v_n|)\Big]\,\mathrm{d}x\\&+\int_{\Omega}\Big[v_n^{p(x)}\log(e+\omega v_n)+\mu(x)v_n^{q(x)}\log(e+\omega v_n)\Big]\,\mathrm{d}x\end{aligned}\tag{4.26}$$

and

$$0<\gamma_1:=\min_{1\le i\le m}\min\left\{\frac{\iota_i^-}{q_i^++\sigma},\frac{\pi_i^-}{q_i^++\sigma}\right\}-1\le\gamma_2:=\max_{1\le i\le m}\max\left\{\frac{\iota_i^++\frac{\iota_i^+}{p_i^-}}{p_i^-},\frac{\pi_i^++\frac{\pi_i^+}{q_i^-}}{p_i^-}\right\}-1.$$

Invoking (4.18) and (4.25) we obtain

$$Z_{n+1}\le C_{12}\left|A_{\psi_{n+1}}\right|^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}\left(S_n^{1+\gamma_1}+S_n^{1+\gamma_2}\right).\tag{4.27}$$

In addition, since hypotheses (B) ensure that $p(x)<\iota(x)<p^*(x)$ and $q(x)<\pi(x)<q^*(x)$ for all $x\in\overline{\Omega}$, there hold

$$\begin{aligned}&\int_{A_{\psi_{n+1}}}\Big[(u-\psi_{n+1})^{p(x)}\log(e+\omega(u-\psi_{n+1}))+\mu(x)\,(u-\psi_{n+1})^{q(x)}\log(e+\omega(u-\psi_{n+1}))\Big]\,\mathrm{d}x\\&\le\int_{A_{\psi_{n+1}}}\Big[\,(u-\psi_{n+1})^{\iota(x)}\log^{\frac{\iota(x)}{p(x)}}(e+\omega(u-\psi_{n+1}))\end{aligned}$$

$$+\mu(x)^{\frac{\pi(x)}{q(x)}}\left(u-\psi_{n+1}\right)^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}\left(e+\omega(u-\psi_{n+1})\right)+2\Big]\,\mathrm{d}x.$$

This along with (4.6), (4.8) and (4.11) implies

$$\begin{aligned}&\int_{A_{\psi_{n+1}}}\left[v_n^{p(x)}\log(e+\omega v_n)+\mu(x)v_n^{q(x)}\log(e+\omega v_n)\right]\,\mathrm{d}x\\&\leq 5\left(1+\psi_*^{-\iota^+}\right)2^{(n+1)\iota^+}Z_n\leq 5\left(1+\psi_*^{-\iota^+}\right)2^{(n+1)\iota^+}X_n\end{aligned}$$

for all $n\in\mathbb{N}_0$. The above inequality along with (4.12) leads to

$$S_n\leq C_{13}\left(1+\psi_*^{-\alpha_0}\right)2^{n(\beta_0+\frac{\beta_0}{p^-})}X_n\quad\text{for all } n\in\mathbb{N}_0. \tag{4.28}$$

Therefore, we get

$$S_n^{1+\gamma_1}+S_n^{1+\gamma_2}\leq C_{14}\left(1+\psi_*^{-\alpha_0(1+\gamma_2)}\right)2^{n(\beta_0+\frac{\beta_0}{p^-})(1+\gamma_2)}\left(X_n^{1+\gamma_1}+X_n^{1+\gamma_2}\right). \tag{4.29}$$

In addition, (4.8) implies that

$$\begin{aligned}\left|A_{\psi_{n+1}}\right|^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}&\leq C_{15}\left(\psi_*^{-\frac{\varepsilon\iota^-}{\iota^++\pi^++\varepsilon}}+\psi_*^{-\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}}\right)2^{\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}n}Z_n^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}\\&\leq C_{15}\left(\psi_*^{-\frac{\varepsilon\iota^-}{\iota^++\pi^++\varepsilon}}+\psi_*^{-\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}}\right)2^{\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}n}X_n^{\frac{\varepsilon}{\iota^++\pi^++\varepsilon}}.\end{aligned} \tag{4.30}$$

Taking (4.17), (4.27), (4.29) and (4.30) into account, we arrive at

$$Z_{n+1}\leq C_{16}\left(\psi_*^{-\mu_1}+\psi_*^{-\mu_2}\right)k^n\left(X_n^{1+\varpi_1}+X_n^{1+\varpi_2}\right)\quad\text{for all } n\in\mathbb{N}_0, \tag{4.31}$$

where

$$\begin{aligned}&0<\mu_1:=\frac{\varepsilon r^-}{\iota^++\pi^++\varepsilon}<\mu_2:=\alpha_0\left(1+\gamma_2\right)+\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}\\&1<k:=2^{(\beta_0+\frac{\beta_0}{p^-})(1+\gamma_2)+\frac{\varepsilon\iota^+}{\iota^++\pi^++\varepsilon}},\\&0<\varpi_1:=\gamma_1+\frac{\varepsilon}{\iota^++\pi^++\varepsilon}\leq\varpi_2:=\gamma_2+\frac{\varepsilon}{\iota^++\pi^++\varepsilon}.\end{aligned}$$

Now, we estimate Y_{n+1} by X_n. For any $i\in\mathcal{I}=\{1,\cdots,m\}$, $\hat{\alpha}>0$, and $\hat{\beta}>0$, we define

$$H_{n,i}(\hat{\alpha},\hat{\beta}):=\int_{\hat{\Gamma}_i}\left[v_n^{\hat{\alpha}}\log^{\frac{\hat{\alpha}}{p(x)}}(e+\omega v_n)+\mu(x)^{\frac{\hat{\beta}}{q(x)}}v_n^{\hat{\beta}}\log^{\frac{\hat{\beta}}{q(x)}}(e+\omega v_n)\right]\mathrm{d}\varsigma.$$

We have

$$\begin{aligned}Y_{n+1}&=\int_{\Gamma_1}\left[v_n^{\theta(x)}\log^{\frac{\theta(x)}{p(x)}}(e+\omega v_n)+\mu(x)^{\frac{\vartheta(x)}{q(x)}}v_n^{\vartheta(x)}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega v_n)\right]\mathrm{d}\varsigma\\&\leq\sum_{i=1}^m\int_{\hat{\Gamma}_i}\left[v_n^{\theta(x)}\log^{\frac{\theta(x)}{p(x)}}(e+\omega v_n)+\mu(x)^{\frac{\vartheta(x)}{q(x)}}v_n^{\vartheta(x)}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega v_n)\right]\mathrm{d}\varsigma.\end{aligned}$$

Similarly, we obtain

$$Y_{n+1}\leq\sum_{i=1}^m\left[H_{n,i}\left(\theta_i^-,\vartheta_i^-\right)+H_{n,i}\left(\theta_i^+,\vartheta_i^+\right)\right]. \tag{4.32}$$

From (4.16), we can fix $\varepsilon>0$ such that

$$\varepsilon<\min_{1\leq i\leq m}\min\left\{(p_*)_i^--\theta_i^+,(q_*)_i^--\vartheta_i^+\right\}. \tag{4.33}$$

Let $\star \in \{+,-\}$ for $i \in \mathcal{I}$. We apply Hölder's inequality and (4.14) to obtain

$$\begin{aligned}
H_{n,i}(\theta_i^\star,\vartheta_i^\star) &= \int_{B_{\psi_{n+1}}\cap\hat{\Gamma}_i}\left[v_n^{\theta_i^\star}\log^{\frac{\theta_i^\star}{p(x)}}(e+\omega v_n)+\mu(x)^{\frac{\vartheta_i^\star}{q(x)}}v_n^{\vartheta_i^\star}\log^{\frac{\vartheta_i^\star}{q(x)}}(e+\omega v_n)\right]\mathrm{d}\varsigma\\
&\le\left(\int_{\hat{\Gamma}_i}v_n^{\theta_i^\star+\varepsilon}\log^{\frac{\theta_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\theta_i^\star}{\theta_i^\star+\varepsilon}}|B_{\psi_{n+1}}\cap\hat{\Gamma}_i|_\varsigma^{\frac{\varepsilon}{\theta_i^\star+\varepsilon}}\\
&\quad+\left(\int_{\hat{\Gamma}_i}\mu(x)^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}v_n^{\vartheta_i^\star+\varepsilon}\log^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\vartheta_i^\star}{\vartheta_i^\star+\varepsilon}}|B_{\psi_{n+1}}\cap\hat{\Gamma}_i|_\varsigma^{\frac{\varepsilon}{\vartheta_i^\star+\varepsilon}}\\
&\le|B_{\psi_{n+1}}\cap\hat{\Gamma}_i|_\varsigma^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}\left[\left(\int_{\hat{\Gamma}_i}v_n^{\theta_i^\star+\varepsilon}\log^{\frac{\theta_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\theta_i^\star}{\theta_i^\star+\varepsilon}}\right.\\
&\quad\left.+\left(\int_{\hat{\Gamma}_i}\mu(x)^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}v_n^{\vartheta_i^\star+\varepsilon}\log^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\vartheta_i^\star}{\vartheta_i^\star+\varepsilon}}\right].
\end{aligned}\tag{4.34}$$

We set

$$\mathcal{B}_{\star,\Gamma}(x,t):=t^{\theta_i^\star+\varepsilon}\log^{\frac{\theta_i^\star+\varepsilon}{p(x)}}(e+\omega t)+\mu(x)^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}t^{\vartheta_i^\star+\varepsilon}\log^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}(e+\omega t).$$

From (4.33) we conclude that

$$\theta_i^\star+\varepsilon<(p_*)_i^-\quad\text{and}\quad\vartheta_i^\star+\varepsilon<(q_*)_i^-.$$

Using this with $(p_*)_i^-=(p_i^-)_*$ and Propositions 2.7 and 3.5 gives us

$$W^{1,p(\cdot)}(\Omega_i)\hookrightarrow W^{1,p_i^-}(\Omega_i)\hookrightarrow L^{\theta_i^\star+\varepsilon}(\Gamma_1)\tag{4.35}$$

and

$$W^{1,\mathcal{H}_L}(\Omega_i)\hookrightarrow L^{\mathcal{B}_{\star,\Gamma}}(\Gamma_1)\tag{4.36}$$

Then, from (3.2), (4.35), Propositions 2.6 and 2.7, Remark 2.8 and Proposition 3.5 we have

$$\begin{aligned}
&\left(\int_{\hat{\Gamma}_i}v_n^{\theta_i^\star+\varepsilon}\,\mathrm{d}x\log^{\frac{\theta_i^\star+\varepsilon}{p(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\theta_i^\star}{\theta_i^\star+\varepsilon}}\\
&\le\|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}^{\tilde{\theta}_i^\star}\le C_{17}\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{\tilde{\theta}_i^\star}\le C_{18}\left(S_n^{\frac{\tilde{\theta}_i^\star}{p_i^-}}+S_n^{\frac{\tilde{\theta}_i^\star}{q_i^++\sigma}}\right),
\end{aligned}\tag{4.37}$$

where S_n is given by (4.26) and

$$\tilde{\theta}_i^\star=\begin{cases}\theta_i^\star & \text{if } \|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}\le 1,\\ \theta_i^\star+\frac{\theta_i^\star}{p_i^-} & \text{if } \|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}>1.\end{cases}$$

From the embedding (4.36) and Proposition 2.6 we derive that

$$\begin{aligned}
&\left(\int_{\hat{\Gamma}_i}\mu(x)^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}v_n^{\vartheta_i^\star+\varepsilon}\log^{\frac{\vartheta_i^\star+\varepsilon}{q(x)}}(e+\omega v_n)\,\mathrm{d}\varsigma\right)^{\frac{\vartheta_i^\star}{\vartheta_i^\star+\varepsilon}}\\
&\le\|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}^{\theta_i^\star}\le C_{19}\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{\tilde{\vartheta}_i^\star}\le C_{20}\left(S_n^{\frac{\tilde{\vartheta}_i^\star}{p_i^-}}+S_n^{\frac{\tilde{\vartheta}_i^\star}{q_i^++\sigma}}\right)
\end{aligned}\tag{4.38}$$

with

$$\tilde{\vartheta}_i^\star=\begin{cases}\vartheta_i^\star & \text{if } \|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}\le 1,\\ \vartheta_i^\star+\frac{\vartheta_i^\star}{q_i^-} & \text{if } \|v_n\|_{\mathcal{B}_{\star,\Gamma},\hat{\Gamma}_i}>1.\end{cases}$$

Combining (4.34), (4.37) and (4.38) yields

$$H_{n,i}(\theta_i^\star,\vartheta_i^\star)\le C_{21}\left|B_{\psi_{n+1}}\right|_\varsigma^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}\left(S_n^{\frac{\tilde{\theta}_i^\star}{p_i^-}}+S_n^{\frac{\tilde{\theta}_i^\star}{q_i^++\sigma}}+S_n^{\frac{\tilde{\vartheta}_i^\star}{p_i^-}}+S_n^{\frac{\tilde{\vartheta}_i^\star}{q_i^++\sigma}}\right).$$

From this and (4.17) it follows that

$$H_{n,i}\left(\theta_i^\star,\vartheta_i^\star\right)\le C_{21}\left|B_{\psi_{n+1}}\right|_\varsigma^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}\left(S_n^{1+\tilde\gamma_1}+S_n^{1+\tilde\gamma_2}\right) \tag{4.39}$$

with

$$0<\tilde\gamma_1:=\min_{1\le i\le m}\min\left\{\frac{\theta_i^-}{q_i^++\sigma},\frac{\vartheta_i^-}{q_i^++\sigma}\right\}-1\le\tilde\gamma_2:=\max_{1\le i\le m}\max\left\{\frac{\theta_i^++\frac{\theta_i^+}{p_i^-}}{p_i^-},\frac{\vartheta_i^++\frac{\vartheta_i^+}{q_i^-}}{p_i^-}\right\}-1.$$

Invoking (4.32) and (4.39) we obtain

$$Y_{n+1}\le C_{21}\left|B_{\psi_{n+1}}\right|_\varsigma^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}\left(S_n^{1+\tilde\gamma_1}+S_n^{1+\tilde\gamma_2}\right), \tag{4.40}$$

which implies, along with (4.28), that

$$S_n^{1+\tilde\gamma_1}+S_n^{1+\tilde\gamma_2}\le C_{23}\left(1+\psi_*^{-\alpha_0(1+\tilde\gamma_2)}\right)2^{n(\beta_0+\frac{\beta_0}{p^-})(1+\tilde\gamma_2)}\left(X_n^{1+\tilde\gamma_1}+X_n^{1+\tilde\gamma_2}\right). \tag{4.41}$$

Furthermore, from (4.10), we conclude that

$$\begin{aligned}\left|B_{\psi_{n+1}}\right|_\varsigma^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}&\le C_{24}\left(\psi_*^{-\frac{\varepsilon\theta^-}{\theta^++\vartheta^++\varepsilon}}+\psi_*^{-\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}}\right)2^{\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}n}Y_n^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}\\&\le C_{24}\left(\psi_*^{-\frac{\varepsilon\theta^-}{\theta^++\vartheta^++\varepsilon}}+\psi_*^{-\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}}\right)2^{\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}n}X_n^{\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}}.\end{aligned} \tag{4.42}$$

Combining (4.40), (4.41) and (4.42) leads to

$$Y_{n+1}\le C_{25}\left(\psi_*^{-\tilde\mu_1}+\psi_*^{-\tilde\mu_2}\right)\tilde k^n\left(X_n^{1+\tilde\varpi_1}+X_n^{1+\tilde\varpi_2}\right)\quad\text{for all } n\in\mathbb{N}_0, \tag{4.43}$$

where

$$\begin{aligned}&0<-\tilde\mu_1:=\frac{\varepsilon\theta^-}{\theta^++\vartheta^++\varepsilon}<-\tilde\mu_2:=\alpha_0\left(1+\tilde\gamma_2\right)+\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}\\&1<\tilde k:=2^{(\beta_0+\frac{\beta_0}{p^-})(1+\tilde\gamma_2)+\frac{\varepsilon\theta^+}{\theta^++\vartheta^++\varepsilon}},\\&0<\tilde\varpi_1:=\tilde\gamma_1+\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}\le\tilde\varpi_2:=\tilde\gamma_2+\frac{\varepsilon}{\theta^++\vartheta^++\varepsilon}.\end{aligned}$$

Then, employing (4.31) and (4.43) we have

$$X_{n+1}\le C_{26}\left(\psi_*^{-\varrho_1}+\psi_*^{-\varrho_2}\right)k_0^n\left(X_n^{1+\lambda_1}+X_n^{1+\lambda_2}\right)\quad\text{for all } n\in\mathbb{N}_0, \tag{4.44}$$

where

$$\begin{aligned}&0<\varrho_1:=\min\{\mu_1,\tilde\mu_1\}\le\varrho_2:=\max\{\mu_2,\tilde\mu_2\},\\&1<k_0:=\max\{k,\tilde k\},\\&0<\lambda_1:=\min\{\varpi_1,\tilde\varpi_1\}\le\lambda_2:=\max\{\varpi_2,\tilde\varpi_2\}.\end{aligned}$$

Step 3. Show the boundedness of solutions

Finally, we are going to verify (4.2). According to Lemma 2.13, the iteration inequalities (4.44) imply

$$X_n\to 0\quad\text{as } n\to\infty, \tag{4.45}$$

provided that

$$X_0\le\min\left\{\left(2C_{27}\left(\psi_*^{-\varrho_1}+\psi_*^{-\varrho_2}\right)\right)^{-\frac{1}{\lambda_1}}k_0^{-\frac{1}{\lambda_1^2}},\left(2C_{27}\left(\psi_*^{-\varrho_1}+\psi_*^{-\varrho_2}\right)\right)^{-\frac{1}{\lambda_2}}k_0^{-\frac{1}{\lambda_1\lambda_2}-\frac{\lambda_2-\lambda_1}{\lambda_2^2}}\right\}. \tag{4.46}$$

Next, we specify ψ_* in order to be satisfied (4.46). Note that

$$\begin{aligned}Z_0=&\int_\Omega\Big[(u-\psi_*)_+^{\iota(x)}\log^{\frac{\iota(x)}{p(x)}}(e+\omega(u-\psi_*)_+)\\&+\mu(x)^{\frac{\pi(x)}{q(x)}}(u-\psi_*)_+^{\pi(x)}\log^{\frac{\pi(x)}{q(x)}}(e+\omega(u-\psi_*)_+)\Big]\,\mathrm{d}x\\\le&\int_\Omega\mathcal{B}(x,|u|)\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned} Y_0 = & \int_{\Gamma_1} \Big[(u-\psi_*)_+^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}} (e+\omega(u-\psi_*)_+) \\ & +\mu(x)^{\frac{\vartheta(x)}{q(x)}} (u-\psi_*)_+^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}} (e+\omega(u-\psi_*)_+) \Big] \, \mathrm{d}\varsigma \\ \leq & \int_{\Gamma_1} \mathcal{B}_\Gamma(x,|u|) \, \mathrm{d}\varsigma. \end{aligned}$$

Hence

$$X_0 \leq \int_\Omega \mathcal{B}(x,|u|) \, \mathrm{d}x + \int_{\Gamma_1} \mathcal{B}_\Gamma(x,|u|) \, \mathrm{d}\varsigma = I(u). \tag{4.47}$$

We also see that

$$\begin{aligned} & I(u) \leq (2C_{27})^{-\frac{1}{\lambda_1}} \left(\psi_*^{-\varrho_1} + \psi_*^{-\varrho_2} \right)^{-\frac{1}{\lambda_1}} k_0^{-\frac{1}{\lambda_1^2}}, \\ & I(u) \leq (2C_{27})^{-\frac{1}{\lambda_2}} \left(\psi_*^{-\varrho_1} + \psi_*^{-\varrho_2} \right)^{-\frac{1}{\lambda_2}} k_0^{-\frac{1}{\lambda_1\lambda_2}-\frac{\lambda_2-\lambda_1}{\lambda_2^2}} \end{aligned} \tag{4.48}$$

is equivalent to

$$\begin{aligned} & \psi_*^{-\varrho_1} + \psi_*^{-\varrho_2} \leq (2C_{27})^{-1} k_0^{-\frac{1}{\lambda_1}} (I(u))^{-\lambda_1}, \\ & \psi_*^{-\varrho_1} + \psi_*^{-\varrho_2} \leq (2C_{27})^{-1} k_0^{-\frac{1}{\lambda_1}-\frac{\lambda_2-\lambda_1}{\lambda_2}} (I(u))^{-\lambda_2}. \end{aligned}$$

Moreover,

$$\begin{aligned} & 2\psi_*^{-\varrho_1} \leq (2C_{27})^{-1} k_0^{-\frac{1}{\lambda_1}-\frac{\lambda_2-\lambda_1}{\lambda_2}} \min\left\{ (I(u))^{-\lambda_1}, (I(u))^{-\lambda_2} \right\}, \\ & 2\psi_*^{-\varrho_2} \leq (2C_{27})^{-1} k_0^{-\frac{1}{\lambda_1}-\frac{\lambda_2-\lambda_1}{\lambda_2}} \min\left\{ (I(u))^{-\lambda_1}, (I(u))^{-\lambda_2} \right\}, \end{aligned}$$

is equivalent to

$$\begin{aligned} & \psi_* \geq (4C_{27})^{\frac{1}{\varrho_1}} k_0^{\frac{1}{\varrho_1}(\frac{1}{\lambda_1}+\frac{\lambda_2-\lambda_1}{\lambda_2})} \max\left\{ (I(u))^{\frac{\lambda_1}{\varrho_1}}, (I(u))^{\frac{\lambda_2}{\varrho_1}} \right\}, \\ & \psi_* \geq (4C_{27})^{\frac{1}{\varrho_2}} k_0^{\frac{1}{\varrho_2}(\frac{1}{\lambda_1}+\frac{\lambda_2-\lambda_1}{\lambda_2})} \max\left\{ (I(u))^{\frac{\lambda_1}{\varrho_2}}, (I(u))^{\frac{\lambda_2}{\varrho_2}} \right\}. \end{aligned} \tag{4.49}$$

Hence, if we take

$$\psi_* = \max\left\{ (4C_{27})^{\frac{1}{\varrho_1}}, (4C_{27})^{\frac{1}{\varrho_2}} \right\} k_0^{\frac{1}{\varrho_1}\left(\frac{1}{\lambda_1}+\frac{\lambda_2-\lambda_1}{\lambda_2}\right)} \cdot \max\left\{ (I(u))^{\frac{\lambda_1}{\varrho_2}}, (I(u))^{\frac{\lambda_2}{\varrho_1}} \right\},$$

it follows (4.49), which implies (4.48). Invoking (4.47) and (4.48), one can get (4.46). Thus we can employ (4.45) associating with Lebesgue's dominated convergence theorem to arrive at

$$\begin{aligned} X_n = & \int_\Omega \Big[(u-\psi_n)_+^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}} (e+\omega(u-\psi_n)_+) \\ & +\mu(x)^{\frac{\pi(x)}{q(x)}} (u-\psi_n)_+^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}} (e+\omega(u-\psi_n)_+) \Big] \, \mathrm{d}x \\ & + \int_{\Gamma_1} \Big[(u-\psi_n)_+^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}} (e+\omega(u-\psi_n)_+) \\ & +\mu(x)^{\frac{\vartheta(x)}{q(x)}} (u-\psi_n)_+^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}} (e+\omega(u-\psi_n)_+) \Big] \, \mathrm{d}\varsigma \\ \to & \int_\Omega \Big[(u-2\psi_*)_+^{\iota(x)} \log^{\frac{\iota(x)}{p(x)}} (e+\omega(u-2\psi_*)_+) \\ & +\mu(x)^{\frac{\pi(x)}{q(x)}} (u-2\psi_*)_+^{\pi(x)} \log^{\frac{\pi(x)}{q(x)}} (e+\omega(u-2\psi_*)_+) \Big] \, \mathrm{d}x \\ & + \int_{\Gamma_1} \Big[(u-2\psi_*)_+^{\theta(x)} \log^{\frac{\theta(x)}{p(x)}} (e+\omega(u-2\psi_*)_+) \\ & +\mu(x)^{\frac{\vartheta(x)}{q(x)}} (u-2\psi_*)_+^{\vartheta(x)} \log^{\frac{\vartheta(x)}{q(x)}} (e+\omega(u-2\psi_*)_+) \Big] \, \mathrm{d}\varsigma, \end{aligned}$$

as $n \to \infty$. This implies that the right-hand side of the above equals to zero, and then

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) + \operatorname*{ess\,sup}_{x\in\Gamma_1} u(x) \leq 4\psi_*.$$

Analogously, we get the following results by replacing u with $-u$

$$\operatorname*{ess\,sup}_{x\in\Omega}(-u)(x)+\operatorname*{ess\,sup}_{x\in\Gamma_1}(-u)(x)\leq 4\psi_*.$$

Therefore,

$$\|u\|_{\infty,\Omega}+\|u\|_{\infty,\Gamma_1}\leq C\max\left\{I(u)^{r_1},I(u)^{r_2}\right\}, \tag{4.50}$$

with C,r_1,r_2 being positive constants independent of u. Finally, through (4.50) applying Proposition 2.5, we obtain (4.2). ☐

4.2. **Critical growth.** In Subsection 4.1 we discussed the case that the exponents of the N-function possesses subcritical growth, namely, $p(x)<\iota(x)<p^*(x)$, $q(x)<\pi(x)<q^*(x)$ for all $x\in\overline{\Omega}$ and $p(x)<\theta(x)<p_*(x)$, $q(x)<\vartheta(x)<q_*(x)$ for all $x\in\Gamma_1$. However, the critical case has not been considered, since among the proof of Theorem 4.2, if we take $\iota(\cdot)=p^*(\cdot)$, $\pi(\cdot)=q^*(\cdot)$ and $\theta(\cdot)=(p_*)^-$, $\vartheta(\cdot)=(q_*)^-$, then we cannot apply Hölder's inequality in (4.20) and (4.34). Thus the method for showing the boundedness of weak solutions to problem (P) in Theorem 4.2 is not suitable for the critical case. Therefore, in this subsection, we use a different way to prove the boundedness result under critical growth assumptions. The proof is based on the ideas by Ho-Winkert [33].

Here and in the following, let hypotheses (A') and (B') to be (A) and (B) respectively, with $\iota(x)=p^*(x)$, $\pi(x)=q^*(x)$ for all $x\in\overline{\Omega}$ and $\theta(x)=(p_*)^-$, $\vartheta(x)=(q_*)^-$ for all $x\in\Gamma_1$.

Theorem 4.3. *Let hypotheses* (H1), (A') *and* (B)' *be satisfied. Then every weak solution of problem* (P) *belongs to* $L^\infty(\Omega)\cap L^\infty(\Gamma_1)$.

Proof. As before, let B_i be open balls with radius R for $i\in\mathcal{I}:=\{1,\dots,m\}$ and $\{B_i\}_{i=1}^m$ is a finite open covering of $\overline{\Omega}$ such that $\Omega_i:=B_i\cap\Omega$ are Lipschitz domains and $\hat{\Gamma}_i:=B_i\cap\Gamma_1\neq\emptyset$ for $i\in\mathcal{I}$. In the sequel, we choose R sufficiently small satisfying

$$q_i^+<(p^*)_i^- \quad \text{for all } i\in\mathcal{I}. \tag{4.51}$$

Recall that for a function $f\in C(\overline{\Omega})$ and $i\in\mathcal{I}$, we denote

$$f_i^+:=\max_{x\in\overline{\Omega}_i}f(x)\quad\text{and}\quad f_i^-:=\min_{x\in\overline{\Omega}_i}f(x).$$

Also, A_ψ and Γ_ψ are given by (4.3) and (4.4), respectively. Next, assume $u\in W^{1,\mathcal{H}_L}(\Omega)$ is a weak solution of problem (P) and choose $\psi_*\geq 1$ large enough such that

$$\int_{A_{\psi_*}}\mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x+\int_{A_{\psi_*}}\mathcal{H}^*(x,|u|)\,\mathrm{d}x+\int_{B_{\psi_*}}H_T(x,|u|)\,\mathrm{d}\varsigma<1. \tag{4.52}$$

Recall the definitions

$$\begin{aligned}
\mathcal{H}_L(x,t)&:=t^{p(x)}\log(e+\omega t)+\mu(x)t^{q(x)}\log(e+\omega t),\\
\mathcal{H}^*(x,t)&:=t^{p^*(x)}\log^{\frac{p^*(x)}{p(x)}}(e+\omega t)+\mu(x)^{\frac{q^*(x)}{q(x)}}t^{q^*(x)}\log^{\frac{q^*(x)}{q(x)}}(e+\omega t),\\
H_T(x,t)&:=t^{(p_*)^-}\log^{\frac{(p_*)^-}{p^-}}(e+\omega t)+\mu(x)^{\frac{(q_*)^-}{q^-}}t^{(q_*)^-}\log^{\frac{(q_*)^-}{q^-}}(e+\omega t),
\end{aligned}$$

for all $x\in\overline{\Omega}$ and for all $t\in[0,\infty)$ and for each $n\in\mathbb{N}_0$, $\{\psi_n\}_{n\in\mathbb{N}_0}$ are defined by (4.5). Moreover, we take $v_n:=(u-\psi_{n+1})_+$ and set

$$L_n:=\int_{A_{\psi_n}}\mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x+\int_{A_{\psi_n}}\mathcal{H}^*\left(x,u-\psi_n\right)\,\mathrm{d}x+\int_{B_{\psi_n}}H_T\left(x,u-\psi_n\right)\,\mathrm{d}\varsigma. \tag{4.53}$$

Similar to the proof of Theorem 4.2, it follows that

$$L_{n+1}\leq L_n, \tag{4.54}$$

$$u(x)\leq\left(2^{n+2}-1\right)(u(x)-\psi_n)\ \text{ for a.a. } x\in A_{\psi_{n+1}}, \tag{4.55}$$

$$u(x)\leq\left(2^{n+2}-1\right)(u(x)-\psi_n)\ \text{ for a.a. } x\in B_{\psi_{n+1}}, \tag{4.56}$$

$$\left|A_{\psi_{n+1}}\right|\leq\frac{2^{(n+1)(p^*)^+}}{\psi_*^{(p^*)^-}}L_n\leq 2^{(n+1)(p^*)^+}L_n, \tag{4.57}$$

for all $n \in \mathbb{N}_0$. In the following, we assume again that C_i for $i \in \mathbb{N}$ are positive constants independent of u, n and ψ_*. We prove the main result in several steps.

Step 1 We are going to prove that

$$\int_{A_{\psi_{n+1}}} \mathcal{H}^*(x, v_n)\,\mathrm{d}x \le C_1 \cdot 2^{\frac{n\left((p^*)^+\right)^2}{q^-}} \left(L_n^{1+\eta_1} + L_n^{1+\eta_2}\right) \quad \text{for all } n \in \mathbb{N}_0 \tag{4.58}$$

with

$$0 < \eta_1 := \min_{1\le i\le m} \frac{(p^*)_i^-}{q_i^+ + \sigma} - 1 \le \eta_2 := \max_{1\le i\le m} \frac{(p^*)_i^-}{q_i^+ + \sigma} - 1,$$

where $\sigma > 0$ satisfies

$$\sigma < \min\{(p^*)_i^- - q_i^+, (p_*)^- - q_i^-\} \quad \text{for } i \in \mathcal{I}.$$

For $i \in \mathcal{I}$, it holds that

$$\int_{A_{\psi_{n+1}}} \mathcal{H}^*(x, v_n)\,\mathrm{d}x = \int_\Omega \mathcal{H}^*(x, v_n)\,\mathrm{d}x \le \sum_{i=1}^m \int_{\Omega_i} \mathcal{H}^*(x, v_n)\,\mathrm{d}x. \tag{4.59}$$

Furthermore, from (4.52) and Proposition 2.5 we see that

$$\int_{\Omega_i} \mathcal{H}^*(x, v_n)\,\mathrm{d}x \le \|v_n\|_{\mathcal{H}^*,\Omega_i}^{(p^*)_i^-}.$$

This together with Proposition 3.4 implies

$$\int_{\Omega_i} \mathcal{H}^*(x, v_n)\,\mathrm{d}x \le C_2\left[\|\nabla v_n\|_{\mathcal{H}_L,\Omega_i} + \|v_n\|_{\mathcal{H}_L,\Omega_i}\right]^{(p^*)_i^-}.$$

Based on this and employing the equivalent norm given in (2.1), Proposition 2.6 as well as (4.52) we get

$$\begin{aligned}\int_{\Omega_i} \mathcal{H}_L^*(x, v_n)\,\mathrm{d}x &\le C_3\left(\int_{\Omega_i} \mathcal{H}_L(x, |\nabla v_n|)\,\mathrm{d}x + \int_{\Omega_i} \mathcal{H}_L(x, v_n)\,\mathrm{d}x\right)^{\frac{(p^*)_i^-}{q_i^+ + \sigma}}\\ &\le C_4\left(\int_{A_{\psi_{n+1}}} \mathcal{H}_L(x, |\nabla v_n|)\,\mathrm{d}x + \int_{A_{\psi_{n+1}}} \mathcal{H}^*(x, v_n)\,\mathrm{d}x + |A_{\psi_{n+1}}|\right)^{\frac{(p^*)_i^-}{q_i^+ + \sigma}}\end{aligned}$$

Then, from (4.53), (4.54) and (4.57) we obtain

$$\int_{\Omega_i} \mathcal{H}^*(x, v_n)\,\mathrm{d}x \le C_5 2^{\frac{n(p^*)^+(p^*)_i^-}{q_i^+ + \sigma}} L_n^{\frac{(p^*)_i^-}{q_i^+ + \sigma}}.$$

Taking this and (4.59), (4.17) as well as (4.51) we obtain (4.58). Thus, Step 1 is completed.

Step 2 There exist $\eta_3, \eta_4 > 0$ such that

$$\int_{B_{\psi_{n+1}}} H_T(x, v_n)\,\mathrm{d}\varsigma \le C_6 2^{\frac{n(p^*)^+(q_*)^+}{q^-}} \left(L_n^{1+\eta_3} + L_n^{1+\eta_4}\right) \quad \text{for all } n \in \mathbb{N}_0, \tag{4.60}$$

with η_3 and η_4 to be specified later.

For $i \in \mathcal{I}$ it holds that

$$\int_{B_{\psi_{n+1}}} H_T(x, v_n)\,\mathrm{d}\varsigma = \int_{\Gamma_1} H_T(x, v_n)\,\mathrm{d}\varsigma \le \sum_{1\le i\le m} \int_{\hat{\Gamma}_i} H_T(x, v_n)\,\mathrm{d}\varsigma. \tag{4.61}$$

Furthermore, invoking Propositions 2.5, 2.6 and 2.7, Remark 2.8, and Proposition 3.5 along with (4.52), we see that

$$\int_{\hat{\Gamma}_i} v_n^{(p_*)^-} \log^{\frac{(p_*)^-}{p^-}}(e + \alpha v_n)\,\mathrm{d}\varsigma \le \|v_n\|_{\mathcal{H}_T,\hat{\Gamma}_i}^{(p_*)_i^-} \le C_7\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{(p_*)_i^-} \le C_8 S_n^{\frac{(p_*)_i^-}{q_i^- + \sigma}}, \tag{4.62}$$

and

$$\int_{\hat{\Gamma}_i} \mu(x)^{\frac{(q_*)^-}{q^-}} v_n^{(q_*)^-} \log^{\frac{(q_*)^-}{q^-}}(e + \alpha v_n)\,\mathrm{d}\varsigma \le \|v_n\|_{\mathcal{H}_T,\hat{\Gamma}_i}^{(q_*)_i^-} \le C_9\|v_n\|_{1,\mathcal{H}_L,\Omega_i}^{(q_*)_i^-} \le C_{10} S_n^{\frac{(q_*)_i^-}{q_i^- + \sigma}}, \tag{4.63}$$

where

$$S_n := \int_\Omega \mathcal{H}_L(x, |\nabla v_n|)\,\mathrm{d}x + \int_\Omega \mathcal{H}_L(x, v_n)\,\mathrm{d}x.$$

From (4.62) and (4.63), we get

$$\begin{aligned}
&\int_{\hat{\Gamma}_i} H_T(x, v_n)\,\mathrm{d}\varsigma \leq C_8 S_n^{\frac{(p_*)_i^-}{q_i^-+\sigma}} + C_{10} S_n^{\frac{(q_*)_i^-}{q_i^-+\sigma}}\\
&\leq C_{11}\left(\int_{A_{\psi_n+1}} \mathcal{H}_L(x, |\nabla v_n|)\,\mathrm{d}x + \int_{A_{\psi_n+1}} \mathcal{H}_T(x, v_n)\,\mathrm{d}x + |A_{\psi_n+1}|\right)^{\frac{(p_*)_i^-}{q_i^-+\sigma}}\\
&\quad + C_{11}\left(\int_{A_{\psi_n+1}} \mathcal{H}_L(x, |\nabla v_n|)\,\mathrm{d}x + \int_{A_{\psi_n+1}} \mathcal{H}_T(x, v_n)\,\mathrm{d}x + |A_{\psi_n+1}|\right)^{\frac{(q_*)_i^-}{q_i^-+\sigma}}.
\end{aligned}$$

Now, by using (4.53), (4.54) and (4.57), we conclude that

$$\int_{\hat{\Gamma}_i} H_T(x, v_n)\,\mathrm{d}\sigma \leq C_{12} 2^{\frac{n(p^*)^+(p_*)_i^-}{q_i^-+\sigma}} L_n^{\frac{(p_*)_i^-}{q_i^-+\sigma}} + C_{13} 2^{\frac{n(p^*)^+(q_*)_i^-}{q_i^-+\sigma}} L_n^{\frac{(q_*)_i^-}{q_i^-+\sigma}}.$$

Finally, taking this along with (4.61), (4.17) as well as (4.51), we infer that

$$\int_\Gamma H_T(x, v_n)\,\mathrm{d}\sigma \leq C_{14} 2^{\frac{n(p^*)^+(q_*)^+}{q^-}} \left(L_n^{1+\eta_3} + L_n^{1+\eta_4}\right),$$

where

$$0 < \eta_3 := \min_{1\leq i\leq m} \min\left\{\frac{(p_*)_i^-}{q_i^-+\sigma}, \frac{(q_*)_i^-}{q_i^-+\sigma}\right\} - 1 \leq \eta_4 := \max_{1\leq i\leq m} \max\left\{\frac{(p_*)_i^-}{q_i^-+\sigma}, \frac{(q_*)_i^-}{q_i^-+\sigma}\right\} - 1.$$

Hence, we have finished Step 2.

Step 3 We show that

$$\begin{aligned}
&\int_{A_{\psi_n+1}} \mathcal{H}_L(x, |\nabla u|)\,\mathrm{d}x\\
&\leq C_{15} 2^{n\left[\frac{((p^*)^+)^2}{q^-} + \frac{(p^*)^+(q_*)^+}{q^-} + (q^*)^+ + \frac{(q^*)^+}{q^-}\right]} \left(L_{n-1}^{1+\gamma_1} + L_{n-1}^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N},
\end{aligned} \tag{4.64}$$

where $0 < \gamma_1 := \min_{1\leq i\leq 4} \eta_i \leq \gamma_2 := \max_{1\leq i\leq 4} \eta_i$.

Taking $\varphi = v_n \in W^{1,\mathcal{H}_L}(\Omega)$ as test function in (4.1) yields

$$\int_{A_{\psi_{n+1}}} \mathcal{A}(x, u, \nabla u)\cdot\nabla u\,\mathrm{d}x = \int_{A_{\psi_{n+1}}} \xi\left(u - \psi_{n+1}\right)\,\mathrm{d}x + \int_{B_{\psi_{n+1}}} \zeta\left(u - \psi_{n+1}\right)\mathrm{d}\varsigma.$$

Note that $u \geq u - \psi_{n+1} > 0$ and $u > \psi_{n+1} \geq 1$ on $A_{\psi_{n+1}}$. Applying the assumptions on (A')(ii) and (B'), we obtain

$$\begin{aligned}
\int_{A_{\psi_{n+1}}} \mathcal{A}(x, u, \nabla u)\nabla u\,\mathrm{d}x &\geq \alpha_2 \int_{A_{\psi_{n+1}}} \mathcal{H}_L(x, |\nabla u|)\,\mathrm{d}x - \alpha_3 \int_{A_{\psi_{n+1}}} \left[\mathcal{H}^*(x, u) + 1\right]\,\mathrm{d}x\\
&\geq \alpha_2 \int_{A_{\psi_{n+1}}} \mathcal{H}_L(x, |\nabla u|)\,\mathrm{d}x - 2\alpha_3 \int_{A_{\psi_{n+1}}} \mathcal{H}^*(x, u)\,\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_{\psi_{n+1}}} \xi\left(u - \psi_{n+1}\right)\,\mathrm{d}x\\
&\leq \beta \int_{A_{\psi_{n+1}}} \Big[u^{p^*(x)-1} \log^{\frac{p^*(x)}{p(x)}}(e + \omega u) + \mu(x)^{\frac{q^*(x)}{q(x)}} u^{q^*(x)-1} \log^{\frac{q^*(x)}{q(x)}}(e + \omega u)\\
&\quad + |\nabla u|^{\frac{p(x)}{(p^*)'(x)}} \log^{\frac{N+1}{N}}(e + \omega|\nabla u|) + \mu(x)^{\frac{N+1}{N}} |\nabla u|^{\frac{q(x)}{(q^*)'(x)}} \log^{\frac{N+1}{N}}(e + \omega|\nabla u|) + 1\Big]\,u\,\mathrm{d}x\\
&\leq \frac{\alpha_2}{2} \int_{A_{\psi_{n+1}}} \mathcal{H}_L(x, |\nabla u|)\,\mathrm{d}x + C_{16} \int_{A_{\psi_{n+1}}} \mathcal{H}^*(x, u)\,\mathrm{d}x,
\end{aligned}$$

as well as

$$\begin{aligned}&\int_{B_{\psi_{n+1}}} \zeta\left(u-\psi_{n+1}\right) \mathrm{d}\varsigma\\ &\leq \gamma \int_{B_{\psi_{n+1}}}\left[u^{p_*(x)-1} \log^{\frac{p_*(x)}{p(x)}}(e+\omega u)+\mu(x)^{\frac{q_*(x)}{q(x)}} u^{q_*(x)-1} \log^{\frac{q_*(x)}{q(x)}}(e+\omega u)+1\right] u \,\mathrm{d}\varsigma\\ &\leq 2\gamma \int_{B_{\psi_{n+1}}} H_T(x,u)\,\mathrm{d}\varsigma.\end{aligned}$$

Combining the three estimates above, we get

$$\int_{A_{\psi_{n+1}}} \mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x \leq C_{17}\int_{A_{\psi_{n+1}}} \mathcal{H}^*(x,u)\,\mathrm{d}x + C_{18}\int_{B_{\psi_{n+1}}} H_T(x,u)\,\mathrm{d}\varsigma.$$

From this, (4.55) and (4.56) we arrive at

$$\int_{A_{\psi_{n+1}}} \mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x \leq C_{19} 2^{n\left[(q^*)^+ + \frac{(q^*)^+}{q^-}\right]}\left[\int_{A_{\psi_n}} \mathcal{H}^*\left(x, v_{n-1}\right)\mathrm{d}x + \int_{B_{\psi_n}} H_T\left(x, v_{n-1}\right)\mathrm{d}\varsigma\right].$$

Combining this and the results given by Step 1 and Step 2 we show (4.64). Therefore, (4.54), (4.58) and (4.60) indicate

$$L_{n+1} \leq C_{20} k^n\left(L_{n-1}^{1+\gamma_1}+L_{n-1}^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}$$

where

$$k := 2^{n\left[\frac{\left((p^*)^+\right)^2}{q^-}+\frac{(p^*)^+(q_*)^+}{q^-}+(q^*)^+ + \frac{(q^*)^+}{q^-}\right]} > 1.$$

It follows that

$$L_{2(n+1)} \leq C_{21} k^{2n+1}\left(L_{2n}^{1+\gamma_1}+L_{2n}^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}_0.$$

Now, we set $\widetilde{L}_n := L_{2n}$ and $\tilde{k} := k^2$ and obtain

$$\widetilde{L}_{n+1} \leq k C_{21} \tilde{k}^n\left(\widetilde{L}_n^{1+\gamma_1}+\widetilde{L}_n^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}_0. \tag{4.65}$$

Invoking Lemma 2.13 we see that if

$$\widetilde{L}_0 \leq \min\left\{(2kC_{21})^{-\frac{1}{\gamma_1}}\, \tilde{k}^{-\frac{1}{\gamma_1^2}}, (2kC_{21})^{-\frac{1}{\gamma_2}}\, \tilde{k}^{-\frac{1}{\gamma_1\gamma_2}-\frac{\gamma_2-\gamma_1}{\gamma_2^2}}\right\}, \tag{4.66}$$

then

$$L_{2n} = \widetilde{L}_n \to 0 \quad \text{as } n \to \infty. \tag{4.67}$$

Note that (4.65) implies

$$L_{2(n+1)+1} \leq C_{21} k^{2(n+1)}\left(L_{2n+1}^{1+\gamma_1}+L_{2n+1}^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}_0$$

and if we take $\overline{L}_n := L_{2n+1}$ as well as $\tilde{k} := k^2$ the former inequality equals to

$$\overline{L}_{n+1} \leq \tilde{k} C_{21} \tilde{k}^n\left(\overline{L}_n^{1+\gamma_1}+\overline{L}_n^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}_0.$$

This combined with Lemma 2.13 means that if

$$\overline{L}_0 \leq \min\left\{\left(2\tilde{k}C_{21}\right)^{-\frac{1}{\gamma_1}} \tilde{k}^{-\frac{1}{\gamma_1^2}}, \left(2\tilde{k}C_{21}\right)^{-\frac{1}{\gamma_2}} \tilde{k}^{-\frac{1}{\gamma_1\gamma_2}-\frac{\gamma_2-\gamma_1}{\gamma_2^2}}\right\}, \tag{4.68}$$

then

$$L_{2n+1} = \overline{L}_n \to 0 \quad \text{as } n \to \infty. \tag{4.69}$$

Note that

$$\overline{L}_0 = L_1 \leq L_0 = \widetilde{L}_0 \leq \int_{A_{\psi_*}} \mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x + \int_{A_{\psi_*}} \mathcal{H}^*(x,u)\,\mathrm{d}x + \int_{B_{\psi_*}} H_T(x,u)\,\mathrm{d}\varsigma.$$

Then, if we take $\psi_* > 1$ large enough, it holds that

$$\int_{A_{\psi_*}} \mathcal{H}_L(x,|\nabla u|)\,\mathrm{d}x + \int_{A_{\psi_*}} \mathcal{H}^*(x,u)\,\mathrm{d}x + \int_{B_{\psi_*}} H_T(x,u)\,\mathrm{d}\varsigma$$

$$\leq \min\left\{1, \left(2\tilde{k}C_{21}\right)^{\frac{1}{\gamma_1}} \tilde{k}^{\frac{1}{\gamma_1^2}}, \left(2\tilde{k}C_{21}\right)^{\frac{1}{\gamma_2}} \tilde{k}^{\frac{1}{\gamma_1\gamma_2}-\frac{\gamma_2-\gamma_1}{\gamma_2^2}}\right\},$$

where $\tilde{k} := k^2$. Therefore, (4.52), (4.66) and (4.68) hold and thus (4.67) and (4.69) hold true as well. Hence

$$L_n = \int_{A_{\psi_n}} \mathcal{H}_L(x, |\nabla u|)\,\mathrm{d}x + \int_{A_{\psi_n}} \mathcal{H}^*\left(x, u-\psi_n\right)\,\mathrm{d}x + \int_{B_{\psi_n}} H_T\left(x, u-\psi_n\right)\mathrm{d}\varsigma \to 0 \quad \text{as } n\to\infty.$$

We deduce that

$$\int_\Omega (u-2\psi_*)_+^{p^*(x)}\,\mathrm{d}x + \int_{\Gamma_1} (u-2\psi_*)_+^{(p_*)^-}\,\mathrm{d}\varsigma = 0.$$

So, $(u-2\psi_*)_+ = 0$ a.e. in Ω and $(u-2\psi_*)_+ = 0$ a.e. on Γ_1. This means

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) + \operatorname*{ess\,sup}_{x\in\Gamma_1} u(x) \leq 4\psi_*.$$

Replacing u by $-u$ in the above arguments we obtain

$$\operatorname*{ess\,sup}_{x\in\Omega} (-u)(x) + \operatorname*{ess\,sup}_{x\in\Gamma_1} (-u)(x) \leq 4\psi_*.$$

Hence

$$\|u\|_{\infty,\Omega} + \|u\|_{\infty,\Gamma_1} \leq 4\psi_*,$$

where $\psi_* \in \mathbb{R}$, which completes the proof. □

5. Special cases

In this section, by applying Theorem 4.2 and Theorem 4.3, we establish the boundedness of weak solutions to the problems mentioned in Remark 1.1. As we discussed in Remark 1.1, problem (Pi) $(1 \leq i \leq 9, i \in \mathbb{N})$ are special cases to (P). Thus, under suitable assumptions, we obtain the following corollaries directly. We focus on Dirichlet boundary value problem (P5), Neumann boundary value problem (P6), and generalized hemivariational inequality (P9). In this section, we denote by C, τ_1, τ_2 positive constants independent of u.

Definition 5.1. *A function $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ is a weak solution to problem* (P5)*, if there exist $\xi(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$ such that*

$$\int_\Omega \mathcal{A}(x, u, \nabla u)\cdot\nabla v\,\mathrm{d}x = \int_\Omega \xi v\,\mathrm{d}x$$

is satisfied for all $v \in W_0^{1,\mathcal{H}_L}(\Omega)$.

Corollary 5.2. *Let hypotheses* (H1), (A) *and* (B) *be satisfied. Then every weak solution of problem* (P5) *in the sense of Definition 5.1 belongs to $L^\infty(\Omega)$ and it holds*

$$\|u\|_{\infty,\Omega} \leq C\max\{\|u\|_{\mathcal{B},\Omega}^{\tau_1}, \|u\|_{\mathcal{B},\Omega}^{\tau_2}\}.$$

Moreover, if hypotheses (H1), (A') *and* (B') *hold, then any weak solution of problem* (P5) *belongs to $L^\infty(\Omega)$.*

Definition 5.3. *A function $u \in W^{1,\mathcal{H}_L}(\Omega)$ is a weak solution to problem* (P6)*, if there exist $\xi(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$ and $\zeta(x) \in g(x, u(x))$ for a.a. $x \in \Gamma_1$ such that*

$$\int_\Omega \mathcal{A}(x, u, \nabla u)\cdot\nabla v\,\mathrm{d}x = \int_\Omega \xi v\,\mathrm{d}x + \int_{\partial\Omega} \zeta v\,\mathrm{d}\varsigma$$

is satisfied for all $v \in W^{1,\mathcal{H}_L}(\Omega)$.

Corollary 5.4. *Let hypotheses* (H1), (A) *and* (B) *be satisfied. Then every weak solution of problem* (P6) *in the sense of Definition 5.3 belongs to $L^\infty(\Omega)\cap L^\infty(\partial\Omega)$ and it holds*

$$\|u\|_{\infty,\Omega} + \|u\|_{\infty,\partial\Omega} \leq C\max\left\{\left(\|u\|_{\mathcal{B},\Omega} + \|u\|_{\mathcal{B}_\Gamma,\partial\Omega}\right)^{\tau_1}, \left(\|u\|_{\mathcal{B},\Omega} + \|u\|_{\mathcal{B}_\Gamma,\partial\Omega}\right)^{\tau_2}\right\}.$$

Moreover, if hypotheses (H1), (A') *and* (B') *hold, then any weak solution of problem* (P6) *belongs to $L^\infty(\Omega)\cap L^\infty(\partial\Omega)$.*

For problem (P9), we consider $\mathcal{F}$ and $\mathcal{G}$ as Clarke's generalized gradients of two locally Lipschitz functions

$$j\colon \Omega\times\mathbb{R}\to\mathbb{R} \quad \text{with} \quad (x,s)\mapsto j(x,s),$$
$$j_\Gamma\colon \Gamma\times\mathbb{R}\to\mathbb{R} \quad \text{with} \quad (x,s)\mapsto j_\Gamma(x,s),$$

namely, $f(x,u) := \partial j(x,u)$ and $g(x,u) := \partial j_\Gamma(x,u)$.

Definition 5.5. *A function $u \in W^{1,\mathcal{H}_L}(\Omega)$ is a weak solution to problem* (P9)*, if there exist $\xi(x) \in \partial j(x,u(x))$ for a.a. $x \in \Omega$ and $\zeta(x) \in \partial j_\Gamma(x,u(x))$ for a.a. $x \in \Gamma_1$ such that*

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v\,\mathrm{d}x = \int_\Omega \xi v\,\mathrm{d}x + \int_{\Gamma_1} \zeta v\,\mathrm{d}\varsigma \tag{5.1}$$

is satisfied for all $v \in W^{1,\mathcal{H}_L}(\Omega)$.

Now, we suppose the following assumptions on j and j_Γ as follows:

(J) (i) For all $s \in \mathbb{R}$, the mappings $x \mapsto j(x,s)$ and $x \mapsto j_\Gamma(x,s)$ are measurable in Ω and on Γ_1, respectively. In addition, for a.a. $x \in \Omega$ and for a.a. $x \in \Gamma_1$ the functions $s \mapsto j(x,s)$ and $s \mapsto j_\Gamma(x,s)$ are locally Lipschitz on $\mathbb{R}$.

(ii) Let $\iota, \pi \in C(\overline{\Omega})$ with $p(x) < \iota(x) < p^*(x)$ and $q(x) < \pi(x) < q^*(x)$ for all $x \in \overline{\Omega}$ such that

$$\begin{aligned}&\sup\{|y| \colon y \in f(x,t)\}\\ &\leq \beta\left[|t|^{\iota(x)-1}\log^{\frac{\iota(x)}{p(x)}}(e+\omega|t|) + \mu(x)^{\frac{\pi(x)}{q(x)}}|t|^{\pi(x)-1}\log^{\frac{\pi(x)}{q(x)}}(e+\omega|t|)+1\right]\end{aligned}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$ with a positive constants β.

(iii) Let $\theta, \vartheta \in C(\overline{\Omega})$ with $p(x) < \theta(x) < (p_*)^-$ and $q(x) < \vartheta(x) < (q_*)^-$ for all $x \in \overline{\Omega}$ such that

$$\begin{aligned}&\sup\{|y| \colon y \in f(x,t)\}\\ &\leq \gamma\left[|t|^{\theta(x)-1}\log^{\frac{\theta(x)}{p(x)}}(e+\omega|t|) + \mu(x)^{\frac{\vartheta(x)}{q(x)}}|t|^{\vartheta(x)-1}\log^{\frac{\vartheta(x)}{q(x)}}(e+\omega|t|)+1\right].\end{aligned}$$

for a.a. $x \in \Gamma_1$ and for all $t \in \mathbb{R}$ with a positive constant γ.

Moreover, assume hypotheses (J') to be (J) with $\iota(x) = p^*(x), \pi(x) = q^*(x)$ for all $x \in \overline{\Omega}$ and $\theta(x) = (p_*)^-, \vartheta(x) = (q_*)^-$ for all $x \in \Gamma_1$. As done by Carl [10, Proof of Lemma 2.5], we deduce that hypotheses (J) implies (B) (i), also, hypotheses (J') implies (B') (i).

Next, we are going to verify that $u \in W^{1,\mathcal{H}_L}(\Omega)$ is a solution to problem (P9) in the sense of Definition 5.5 if and only if u is a solution to the following generalized hemivariational inequality

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla(v-u)\,\mathrm{d}x + \int_\Omega j^\circ(x,u;v-u)\,\mathrm{d}x + \int_{\Gamma_1} j_\Gamma^\circ(x,u;v-u)\,\mathrm{d}\varsigma \geq 0 \tag{5.2}$$

for all $v \in W^{1,\mathcal{H}_L}(\Omega)$.

First, we give the notion of sub- and supersolution to problem (P9).

Definition 5.6. *A function $\underline{u} \in W^{1,\mathcal{H}_L}(\Omega)$ is a subsolution to problem* (P9) *provided the following conditions are satisfied:*

(i) *$\underline{u} \leq 0$ on Γ_1;*

(ii) *$\underline{\xi}(x) \in \partial j(x,\underline{u}(x))$ for a.a. $x \in \Omega$ and $\underline{\zeta}(x) \in \partial j_\Gamma(x,\underline{u}(x))$ for a.a. $x \in \Gamma_1$;*

(iii)

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v\,\mathrm{d}x + \int_\Omega \underline{\xi} v\,\mathrm{d}x + \int_{\Gamma_1} \underline{\zeta} v\,\mathrm{d}\varsigma \leq 0$$

holds for all $v \in W^{1,\mathcal{H}_L}(\Omega)$ with $v(x) \geq 0$ for all $x \in \overline{\Omega}$.

Definition 5.7. *A function $\overline{u} \in W^{1,\mathcal{H}_L}(\Omega)$ is a supersolution to problem* (P9) *provided the following conditions are satisfied:*

(i) *$\overline{u} \geq 0$ on Γ_1;*

(ii) *$\overline{\xi}(x) \in \partial j(x,\overline{u}(x))$ for a.a. $x \in \Omega$ and $\overline{\zeta}(x) \in \partial j_\Gamma(x,\overline{u}(x))$ for a.a. $x \in \Gamma_1$;*

(iii)

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v\,\mathrm{d}x + \int_\Omega \overline{\xi} v\,\mathrm{d}x + \int_{\Gamma_1} \overline{\zeta} v\,\mathrm{d}\varsigma \geq 0$$

holds for all $v \in W^{1,\mathcal{H}_L}(\Omega)$ with $v(x) \geq 0$ for all $x \in \overline{\Omega}$.

Lemma 5.8. *Let hypotheses* (H1), (A) *and* (J) *be satisfied. Then u is solution to the generalized hemivariational inequality* (5.2) *if and only if it is a solution to problem* (P9).

Proof. Let u be a solution of problem (P9). Then, we can find $\xi(x) \in f(x,u(x)) = \partial j(x,u(x))$ for a.a. $x \in \Omega$ and $\zeta(x) \in g(x,u(x)) = \partial j_\Gamma(x,u(x))$ for a.a. $x \in \Gamma_1$ satisfying (5.1). Recalling the definitions of ∂j and ∂j_Γ, we have

$$\begin{aligned} j^\circ(x,u;v-u) &\geq \xi(x)(v-u) \quad \text{in } \Omega,\\ j_\Gamma^\circ(x,u;v-u) &\geq \zeta(x)(v-u) \quad \text{on } \Gamma_1, \end{aligned} \tag{5.3}$$

for all $v \in W^{1,\mathcal{H}_L}(\Omega)$. Since ∂j and ∂j_Γ fulfill hypotheses (J), we infer that $j^\circ(x,u;v-u)$ belongs to $L^1(\Omega)$ and $j_\Gamma^\circ(x,u;v-u)$ belongs to $L^1(\Gamma_1)$. Taking (5.1) and (5.3) into account, we see that (5.2) holds true.

On the other hand, assume u is a solution of (5.2). As done by Carl-Le [11, Proof of Theorem 3.2], we only need to show that u is a subsolution and also a supersolution of problem (P9), since it indicates that u is solution of problem (P9). So, we first prove that u is a subsolution to problem (P9). Taking the test function $v = u \wedge \phi = u - (u-\phi)^+$ for any $\phi \in W^{1,\mathcal{H}_L}(\Omega)$ we have

$$-\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla(u-\phi)^+\,\mathrm{d}x + \int_\Omega j^\circ(x,u;-(u-\phi)^+)\,\mathrm{d}x + \int_{\Gamma_1} j_\Gamma^\circ(x,u;-(u-\phi)^+)\,\mathrm{d}\varsigma \geq 0,$$

for any $\phi \in W^{1,\mathcal{H}_L}(\Omega)$. Applying the positively homogeneity (see Proposition 2.12) of $\varrho \mapsto j^\circ(x,t;\varrho)$ (resp. $\varrho \mapsto j_\Gamma^\circ(x,t;\varrho)$), we get from the above inequality

$$\begin{aligned} &-\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla(u-\phi)^+\,\mathrm{d}x + \int_\Omega j^\circ(x,u;-1)(u-\phi)^+\,\mathrm{d}x\\ &+\int_{\Gamma_1} j_\Gamma^\circ(x,u;-1)(u-\phi)^+\,\mathrm{d}\varsigma \geq 0, \end{aligned} \tag{5.4}$$

for all $\phi \in W^{1,\mathcal{H}_L}(\Omega)$. Invoking Proposition 2.12 (iv) we see that

$$\begin{aligned} j^\circ(x,u(x);-1) &= \max\{-\eta(x) : \eta(x) \in \partial j(x,u(x))\}\\ &= -\min\{\eta(x) : \eta(x) \in \partial j(x,u(x))\}\\ &= -\underline{\xi}(x), \end{aligned} \tag{5.5}$$

for a.a. $x \in \Omega$ and

$$\begin{aligned} j_\Gamma^\circ(x,u(x);-1) &= \max\{-\eta_\Gamma(x) : \eta_\Gamma(x) \in \partial j_\Gamma(x,u(x))\}\\ &= -\min\{\eta_\Gamma(x) : \eta_\Gamma(x) \in \partial j_\Gamma(x,u(x))\}\\ &= -\underline{\zeta}(x), \end{aligned} \tag{5.6}$$

for a.a. $x \in \Gamma_1$. Taking (5.4), (5.5) and (5.6) into account we obtain

$$-\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla(u-\phi)^+\,\mathrm{d}x + \int_\Omega -\underline{\xi}(u-\phi)^+\,\mathrm{d}x + \int_{\Gamma_1} -\underline{\zeta}(u-\phi)^+\,\mathrm{d}\varsigma \geq 0$$

for all $\phi \in W^{1,\mathcal{H}_L}(\Omega)$. The above inequality equals to

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v\,\mathrm{d}x + \int_\Omega \underline{\xi} v\,\mathrm{d}x + \int_{\Gamma_1} \underline{\zeta} v\,\mathrm{d}\varsigma \leq 0$$

for all $v \in W^{1,\mathcal{H}_L}(\Omega)$ with $v(x) \geq 0$ for all $x \in \overline{\Omega}$. Thus, u is a subsolution of problem (P9). In a similar way, we can show the u is also a supersolution of problem (P9), hence a weak solution of (P9). □

Then, utilizing Lemma 5.8, as done in the proof of Theorems 4.2 and 4.3, we have the following corollary.

Corollary 5.9. *Let hypotheses* (H1)*,* (A) *and* (J) *be satisfied. Then every weak solution of the generalized hemivariational inequality* (5.2) *belongs to* $L^\infty(\Omega)\cap L^\infty(\Gamma_1)$ *and it holds*

$$\|u\|_{\infty,\Omega} + \|u\|_{\infty,\Gamma_1} \leq C\max\left\{(\|u\|_{\mathcal{B},\Omega} + \|u\|_{\mathcal{B}_\Gamma,\Gamma_1})^{\tau_1}, (\|u\|_{\mathcal{B},\Omega} + \|u\|_{\mathcal{B}_\Gamma,\Gamma_1})^{\tau_2}\right\}$$

with positive constants C,τ_1,τ_2 *of* u*. Moreover, if hypotheses* (H1)*,* (A') *and* (J') *hold, then any weak solution of* (5.2) *belongs to* $L^\infty(\Omega)\cap L^\infty(\Gamma_1)$*.*

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