

ON SINGULAR PROBLEMS INVOLVING THE $(1, q)$ -LAPLACIAN

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ABSTRACT. In this paper we study the asymptotic behavior of the singular (p, q) -equation

$$-\Delta_p u - \Delta_q u = u^{-\gamma} + \lambda u^{r-1} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

as $p \rightarrow 1^+$, where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\lambda > 0$, $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Based on a purely singular auxiliary problem along with the Nehari manifold and the fibering map we prove the existence of at least two weak solutions belonging to the usual Sobolev space $W_0^{1,q}(\Omega)$ of the $(1, q)$ -Laplace problem

$$-\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) - \Delta_q u = u^{-\gamma} + \lambda u^{r-1} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

provided the parameter λ is sufficiently small. As far as we know this is the first result for singular $(1, q)$ -problems. Moreover, in contrast to other works dealing with the 1-Laplacian, we get much more regularity of the solutions in the sense that the solutions of the problem above belong to the Sobolev space $W_0^{1,q}(\Omega)$ instead of the space $BV(\Omega)$ of functions of bounded variation.

1. INTRODUCTION AND NOTATION

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, the main objective of this work is to study the asymptotic behavior of solutions of the singular (p, q) -problem

$$\begin{cases} -\Delta_p u - \Delta_q u = u^{-\gamma} + \lambda u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

as $p \rightarrow 1^+$, where $\lambda > 0$ is a parameter to be specified, $0 < \gamma < 1 < p < q < N$, $q < r < q^*$ and the operator Δ_s for $s \in (1, \infty)$ denotes the usual s -Laplacian. As $p \rightarrow 1^+$, solutions of problem (1.1) are expected to converge to a solution of the $(1, q)$ -problem

$$\begin{cases} -\Delta_1 u - \Delta_q u = u^{-\gamma} + \lambda u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where Δ_1 denotes the 1-Laplace operator, which is formally defined by

$$\Delta_1 u = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right).$$

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Over the past three decades, research interest in problems involving the 1-Laplacian has grown considerably, with numerous studies employing various techniques. We refer to the pioneering works by Andreu–Ballester–Caselles–Mazón [1, 2, 3], Demengel [12], and Kawohl [19], see also the monograph by Andreu–Vaillo–Caselles–Mazón [4]. This operator also appears in several applications in image restoration, see for example the papers by Chen–Levine–Rao [9] and Rudin–Osher–Fatemi [26].

Our main result reads as follows.

Theorem 1.1. *Let $0 < \gamma < 1 < q < N$ and $q < r < q^*$. Then there exists $\Lambda > 0$ such that, for all $\lambda \in (0, \Lambda)$, problem (1.2) has at least two nontrivial weak solutions.*

The idea in the proof of Theorem 1.1 is first to study the solvability of the purely singular problem

$$\begin{cases} -\Delta_1 u - \Delta_q u = u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

As we shall see in Lemma 4.8, the existence of a solution of problem (1.3) ensures the nontriviality of the solutions of (1.2). In our approach we use an existence result for problem (1.1) obtained by Liu–Dai–Papageorgiou–Winkert [22] which gives us two nontrivial solutions provided the parameter λ is sufficiently small. Then we let $p \rightarrow 1^+$ and show the existence of solutions for our original problem (1.2).

To the best of our knowledge, there does not exist any result for singular $(1, q)$ -problems. Recently, Ortiz Chata–Pimenta–Segura de León [24] studied singular 1-Laplacian problems and obtained two solutions of the problem

$$\begin{cases} -\Delta_1 u = \lambda u^{-\gamma} + u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1 < r < 1^*$ and $\lambda > 0$ is small enough. In [20], Latorre–Oliva–Petitta–Segura de León studied a general problem involving a nonlinearity which can blow up at the origin and the 1-Laplacian operator. We also mention the work by Giachetti–Oliva–Petitta [17] who proved existence, nonexistence and regularity results for nonnegative solutions of the Dirichlet problem involving the 1-Laplacian operator

$$\begin{cases} -\Delta_1 u = g(u)|Du| + h(u)f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^N(\Omega)$, and g as well as h are continuous functions which may be singular at zero. As far as the regularity is concerned, in all of these papers, it is known that the solutions belong to the space $BV(\Omega)$. We point out that in our case, we get much more regularity of the solutions. Indeed, instead of having solutions in the space $BV(\Omega)$, we get solutions in the usual Sobolev space $W_0^{1,q}(\Omega)$. Here, we also mention the problem

$$\begin{cases} -\Delta_1 u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

which has been studied in the paper by Mercaldo–Segura de León–Trombetti [23] with $f \in L^1(\Omega)$. In particular, the convergence as $p \rightarrow 1^+$ of the problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

has been analyzed by Cicalese–Trombetti [10] and Mercaldo–Segura de León–Trombetti [23]. In [10] the authors showed that if $f \in L^N(\Omega)$ has small norm, then solutions of (1.5) converge to zero, but if the norm of f is large, then they converge to $+\infty$ as $p \rightarrow 1^+$. In [23], the authors proved that a sequence of renormalized solutions of (1.5) converge to a solution of (1.4) in a very weak sense. In general, singular problems involving the 1-Laplacian operator are of special interest. Here we also mention the work by De Cicco–Giachetti–Segura de León [14] who studied the problem

$$\begin{cases} -\Delta_1 u = g(u)f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $g(u) = u^{-\gamma}$ and $f \in L^N(\Omega)$. De Cicco–Giachetti–Oliva–Petitta [15] considered a general singular function g and obtained solutions provided $\|f\|_{L^{N,\infty}(\Omega)} < 1/\tilde{S}_1 g(\infty)$, where $g(\infty) = \lim_{s \rightarrow \infty} g(s)$ and $\tilde{S}_1$ is the best constant of the immersion $W_0^{1,1}(\Omega) \hookrightarrow L^{1^*,1}(\Omega)$. In addition, problems involving so-called convection terms have been studied. For example, Balducci–Oliva–Petitta [5] considered the problem

$$\begin{cases} -\Delta_1 u + h(u)|Du| = g(u)f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where g was again allowed to be singular at the origin. We used some ideas from [5] in our work. For more problems involving the p -Laplacian operator for singular and non-singular problems, we refer to the seminal works Bensoussan–Boccardo–Murat [7], Boccardo–Murat–Puel [8], Crandall–Rabinowitz–Tartar [11], Lazer–McKenna [21], Stuart [30] and De Cave [13].

This paper is organized as follows. In Section 2 we recall some basic facts about the space $BV(\Omega)$ and the usual Sobolev space $W_0^{1,q}(\Omega)$. In Section 3, we obtain a sequence u_p of nontrivial solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x)u^{-\gamma} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided f is nonnegative and in $L^\tau(\Omega)$ with $\tau > \frac{N}{q}$. In the next step we prove that this sequence u_p converges to a nontrivial solution of the problem

$$\begin{cases} -\Delta_1 u - \Delta_q u = f(x)u^{-\gamma} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, in Section 4 we use the Nehari manifold method (in a similar fashion as it was done by Ortiz Chata–Pimenta–Segura de León [24], Figueiredo–Pimenta–Winkert [16] and Liu–Dai–Papageorgiou–Winkert [22]) to obtain sequences u_p and v_p of solutions of problem (1.1) for small values of the parameter $\lambda > 0$ while the sequence u_p has negative energy and the sequence v_p has positive energy. Finally we

show that these sequences converge to nontrivial and distinct solutions of problem (1.2).

2. PRELIMINARIES

In this section we state the main preliminaries that will be needed in the sequel. To this end, let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For $1 \leq r \leq \infty$, we denote by $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ and $W_0^{1,r}(\Omega)$ is the related Sobolev space with zero traces equipped with the equivalent norm $\|\nabla \cdot\|_r$.

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,q}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. Clearly, one has

$$u^\pm \in W_0^{1,q}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

For a number $r \in (1, \infty)$, we write $r' = r/(r-1)$.

Next we introduce the space $BV(\Omega)$ of functions of bounded variation. We say that $u \in BV(\Omega)$, if $u \in L^1(\Omega)$, and its distributional derivative Du is a vectorial Radon measure, i.e.,

$$BV(\Omega) = \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega, \mathbb{R}^N)\}.$$

It can be proved that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_{\Omega} |Du| := \sup \left\{ \int_{\Omega} u \operatorname{div} \phi \, dx : \phi \in C_c^1(\Omega, \mathbb{R}^N), \|\phi\|_{\infty} \leq 1 \right\} < +\infty.$$

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} := \int_{\Omega} |Du| + \int_{\Omega} |u| \, dx.$$

The Sobolev embeddings for this space also hold, in such a way that it is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*]$, where $1^* = N/(N-1)$. If Ω is bounded, these embeddings are compact into $L^r(\Omega)$ for all $r \in [1, 1^*)$.

3. PURELY SINGULAR PROBLEMS

In this section we first study the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x)u^{-\gamma} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

with $0 < \gamma < 1 < p < q < N$. Let $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be given by

$$a(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi.$$

A function $u \in W_0^{1,q}(\Omega)$ is a weak solution of problem (3.1), if $u > 0$ a.e. in Ω , $fu^{-\gamma} \in L_{\text{loc}}^1(\Omega)$ and

$$\int_{\Omega} a(\nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} fu^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in C_c^1(\Omega). \quad (3.2)$$

We assume that

$$f \text{ is nonnegative and } f \in L^{\tau}(\Omega) \text{ for some } \tau > \frac{N}{q}. \quad (3.3)$$

The idea is to find a solution of problem (3.1) via approximation. For this purpose, we consider for $n \in \mathbb{N}$ the problem

$$\begin{cases} -\Delta_p u_n - \Delta_q u_n = f_n(x) \left(|u_n| + \frac{1}{n} \right)^{-\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

with $f_n = T_n(f) = \min\{f, n\}$. We say that $u_n \in W_0^{1,q}(\Omega)$ is a weak solution of problem (3.4) if

$$\int_{\Omega} a(\nabla u_n) \cdot \nabla \varphi \, dx = \int_{\Omega} f_n(x) \left(|u_n| + \frac{1}{n} \right)^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in W_0^{1,q}(\Omega). \quad (3.5)$$

We have the following result.

Theorem 3.1. *Let $0 < \gamma < 1 < p < q < N$ and assume that f satisfies (3.3). Then there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ of nonnegative solutions to (3.4). Moreover, there exists a constant $C > 0$ independent of p such that*

$$\|u_n\|_\infty \leq C \quad \text{for all } n \in \mathbb{N}. \quad (3.6)$$

Proof. We use ideas of Balducci–Oliva–Petitta [5, Theorem 3.3]. Let $v \in L^q(\Omega)$. We know from Theorem 1 by Boccardo–Murat–Puel [8] that there exists a solution $w \in W_0^{1,q}(\Omega)$ of

$$\begin{cases} -\Delta_p u - \Delta_q u = f_n(x) \left(|v| + \frac{1}{n} \right)^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

Moreover, w is indeed unique, see B enilan–Boccardo–Gallou et–Gari epy–Pierre–V azquez [6] and Segura de Le on [27]. Consequently, we may define the operator $S: L^q(\Omega) \rightarrow L^q(\Omega)$ by $w = S(v)$, so that

$$\begin{aligned} & \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla w|^{q-2} \nabla w \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} f_n(x) \left(|v| + \frac{1}{n} \right)^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in W_0^{1,q}(\Omega). \end{aligned} \quad (3.8)$$

Taking $\varphi = w^-$ as test function in (3.8), we conclude that $w \geq 0$. Moreover, if we choose $\varphi = w$ in (3.8), we get

$$\int_{\Omega} |\nabla w|^q \, dx \leq n^{1+\gamma} \int_{\Omega} w \, dx. \quad (3.9)$$

From this along with Poincar e’s and H older’s inequality, we obtain

$$\left(\int_{\Omega} |w|^q \, dx \right) \leq C n^{1+\gamma} \left(\int_{\Omega} |w|^q \, dx \right)^{\frac{1}{q}}$$

for some $C > 0$. Thus,

$$\|w\|_{L^q(\Omega)} \leq (C n^{1+\gamma})^{\frac{1}{q-1}}.$$

Since $w = S(v)$, we conclude that a ball is invariant under S . Moreover, (3.9) and the Sobolev embedding imply that $S(L^q(\Omega))$ is relatively compact in $L^q(\Omega)$.

We also show that $S: L^q(\Omega) \rightarrow L^q(\Omega)$ is continuous. Indeed, assume that $v_k \rightarrow v$ in $L^q(\Omega)$ and let $w_k = S(v_k)$. Then,

$$\|S(v_k)\|_{L^q(\Omega)} \leq (Cn^{1+\gamma})^{\frac{1}{q-1}} \quad \text{for all } k \in \mathbb{N}.$$

Thus, from (3.9), we deduce that $S(v_k)$ is bounded in $W_0^{1,q}(\Omega)$. Therefore, up to a subsequence,

$$S(v_k) \rightharpoonup w \quad \text{weakly in } W_0^{1,q}(\Omega).$$

We claim that $w = S(v)$. Choosing $u_k = S(v_k) - w$ as a test function in (3.7), we get

$$\begin{aligned} & \int_{\Omega} |\nabla w_k|^{p-2} \nabla w_k \cdot \nabla u_k \, dx + \int_{\Omega} |\nabla w_k|^{q-2} \nabla w_k \cdot \nabla u_k \, dx \\ &= \int_{\Omega} f_n(x) \left(|v_k| + \frac{1}{n} \right)^{-\gamma} u_k \, dx \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Consequently,

$$\left| \int_{\Omega} a(\nabla w_k) \cdot \nabla u_k \, dx \right| \leq \omega_k,$$

where $\omega_k \rightarrow 0$ as $k \rightarrow \infty$. Thus,

$$\left| \int_{\Omega} (a(\nabla w_k) - a(\nabla w)) \cdot \nabla (w_k - w) \, dx \right| \leq \omega_k + \left| \int_{\Omega} a(\nabla w) \cdot \nabla u_k \, dx \right| \rightarrow 0,$$

as $k \rightarrow \infty$, since $u_k \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$. As a consequence, from Boccardo–Murat–Puel [8, Lemma 5], we know that

$$w_k \rightarrow w \quad \text{strongly in } W_0^{1,q}(\Omega).$$

This in turn implies that $w = S(v)$ and thus $S: L^q(\Omega) \rightarrow L^q(\Omega)$ is continuous. Schauder's fixed point theorem then shows that S has a nonnegative fixed point $S(u_n) = u_n$. We have thus proved that for all $n \in \mathbb{N}$ there exists $u_n \in W_0^{1,q}(\Omega)$ that solves (3.4). Next, we need to show that u_n is uniformly bounded in $L^\infty(\Omega)$ independently of p . This is a consequence of Stampacchia's argument. Indeed, take $G_\kappa(u_n) = (u_n - \kappa)^+$ with $\kappa > 0$ as a test function in (3.5). We get

$$\int_{\Omega} |\nabla G_\kappa(u_n)|^q \, dx \leq \int_{\Omega} f(x) \left(u_n + \frac{1}{n} \right)^{-\gamma} G_\kappa(u_n) \, dx \leq \kappa^{-\gamma} \int_{\Omega} f(x) G_\kappa(u_n) \, dx.$$

The Sobolev embedding and Hölder's inequality imply that

$$\mathcal{S}_q^{-q} \left(\int_{\Omega} |G_\kappa(u_n)|^{q^*} \, dx \right)^{\frac{q}{q^*}} \leq \kappa^{-\gamma} \left(\int_{A_\kappa} f(x)^{(q^*)'} \, dx \right)^{\frac{1}{(q^*)'}} \left(\int_{\Omega} |G_\kappa(u_n)|^{q^*} \, dx \right)^{\frac{1}{q^*}},$$

where $\mathcal{S}_q$ denotes the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$, $\frac{1}{(q^*)'} + \frac{1}{q^*} = 1$, and $A_\kappa = \{x \in \Omega: u_n(x) > \kappa\}$. Thus,

$$\left(\int_{\Omega} |G_\kappa(u_n)|^{q^*} \, dx \right)^{\frac{q-1}{q^*}} \leq \mathcal{S}_q^q \kappa^{-\gamma} \left(\int_{A_\kappa} f(x)^{(q^*)'} \, dx \right)^{\frac{1}{(q^*)'}}.$$

Since $f \in L^\tau(\Omega)$ with $\tau > \frac{N}{q}$ and taking into account that $\frac{N}{q} > (q^*)'$, we have

$$\left(\int_{\Omega} |G_{\kappa}(u_n)|^{q^*} dx \right)^{\frac{q-1}{q^*}} \leq \mathcal{S}_q^q \kappa^{-\gamma} \left(\left(\int_{\Omega} f(x)^\tau dx \right)^{\frac{(q^*)'}{\tau}} |A_{\kappa}|^{1-\frac{(q^*)'}{\tau}} \right)^{\frac{1}{(q^*)'}}.$$

Consequently,

$$\left(\int_{\Omega} |G_{\kappa}(u_n)|^{q^*} dx \right)^{\frac{q-1}{q^*}} \leq \mathcal{S}_q^q \|f\|_{L^\tau(\Omega)} \kappa^{-\gamma} |A_{\kappa}|^{\frac{1}{(q^*)'} - \frac{1}{\tau}}.$$

We conclude that

$$\int_{\Omega} |G_{\kappa}(u_n)|^{q^*} dx \leq (\mathcal{S}_q^q \|f\|_{L^\tau(\Omega)} \kappa^{-\gamma})^{\frac{q^*}{q-1}} |A_{\kappa}|^{\left(\frac{1}{(q^*)'} - \frac{1}{\tau}\right) \left(\frac{q^*}{q-1}\right)},$$

and thus

$$|A_h| \leq \frac{1}{(h - \kappa)^{q^*}} (\mathcal{S}_q^q \|f\|_{L^\tau(\Omega)} \kappa^{-\gamma})^{\frac{q^*}{q-1}} |A_{\kappa}|^{\left(\frac{1}{(q^*)'} - \frac{1}{\tau}\right) \left(\frac{q^*}{q-1}\right)} \quad \text{for all } h > \kappa > 0.$$

Here we have used that $A_h \subset A_{\kappa}$ and $G_{\kappa}(u_n) \geq h - \kappa$ in A_h for all $h > \kappa$. Finally, since $\tau > \frac{N}{q}$, we have $\left(\frac{1}{(q^*)'} - \frac{1}{\tau}\right) \left(\frac{q^*}{q-1}\right) > 1$. Hence, from Stampacchia's Theorem, see Stampacchia [29, Lemma 4.1], we conclude that there exists $d > 0$ depending only on $q, \tau, \|f\|_{L^\tau(\Omega)}$ and Ω such that $|A_d| = 0$. This proves the result. $\square$

Let us study now the convergence properties of u_n . First, by taking u_n as test in (3.5), we get

$$\int_{\Omega} |\nabla u_n|^q dx \leq \int_{\Omega} u_n^{1-\gamma} f_n(x) dx \leq C^{1-\gamma} \int_{\Omega} f(x) dx,$$

with $C > 0$ given by (3.6). We conclude that u_n is bounded in $W_0^{1,q}(\Omega)$. The Sobolev embedding then implies that there exists $u \in W_0^{1,q}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,q}(\Omega), \quad (3.10)$$

$$u_n \rightarrow u \quad \text{strongly in } L^r(\Omega) \text{ for all } 1 \leq r < q^*. \quad (3.11)$$

Also observe that $a(\nabla u_n)$ is bounded in $L^{\frac{q}{q-1}}(\Omega)^N$. Thus, there exists $C_2 > 0$ such that

$$\int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi dx = \int_{\Omega} a(\nabla u_n) \cdot \nabla \varphi dx \leq C_2 \quad \text{for all } n \in \mathbb{N}.$$

Since φ is arbitrary, it follows that $f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma}$ is bounded in $L_{\text{loc}}^1(\Omega)$. As a consequence of Fatou's Lemma, we conclude that

$$u^{-\gamma} f \in L_{\text{loc}}^1(\Omega).$$

The function u is a candidate solution to the original problem (3.1). Before we can prove this, we need the following auxiliary result.

Proposition 3.2. *Let $0 < \gamma < 1 < p < q < N$ and assume that f satisfies (3.3). Then, $u_n \rightarrow u$ strongly in $W_0^{1,q}(\Omega)$.*

Proof. Let $w_n = u_n - u$ and take w_n as a test function in (3.5). We get

$$\int_{\Omega} a(\nabla u_n) \cdot \nabla w_n \, dx = \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx.$$

On the other hand, the weak convergence of u_n implies that

$$\int_{\Omega} a(\nabla u) \cdot \nabla w_n \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We conclude that

$$\int_{\Omega} (a(\nabla u_n) - a(\nabla u)) \cdot \nabla w_n \, dx \leq \omega_n + \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx,$$

where $\lim_{n \rightarrow \infty} \omega_n = 0$. We will now show that the second term on the right-hand side also goes to zero. Indeed,

$$\begin{aligned} & \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx \\ &= \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx + \int_{\{u_n > \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx \\ &=: J_{\delta,1} + J_{\delta,2}. \end{aligned}$$

It is clear that $J_{\delta,2}$ goes to zero as $n \rightarrow \infty$ for all δ . For $J_{\delta,1}$ we obtain

$$J_{\delta,1} \leq \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} u_n \, dx \leq \delta^{1-\gamma} \int_{\{u_n \leq \delta\}} f(x) \, dx.$$

Consequently,

$$\lim_{n \rightarrow \infty} \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} w_n \, dx \leq \delta^{1-\gamma} \int_{\Omega} f(x) \, dx \quad \text{for all } \delta > 0.$$

Since

$$\lim_{\delta \rightarrow 0} \delta^{1-\gamma} \int_{\Omega} f(x) \, dx = 0,$$

we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a(\nabla u_n) - a(\nabla u)) \cdot \nabla w_n \, dx = 0.$$

Thus, from Lemma 5 by Boccardo–Murat–Puel [8], we conclude that $u_n \rightarrow u$ strongly in $W_0^{1,q}(\Omega)$. This proves the result. $\square$

Theorem 3.3. *Let $0 < \gamma < 1 < p < q < N$ and assume that f satisfies (3.3). Then, the function $u \in W_0^{1,q}(\Omega)$ given by (3.10) and (3.11) is a solution of problem (3.1).*

Proof. Indeed, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is such that

$$\begin{aligned} & \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in C_c^1(\Omega) \text{ and for all } n \in \mathbb{N}. \end{aligned}$$

Proposition 3.2 ensures that the left-hand side converges to

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx,$$

for all $\varphi \in C_c^1(\Omega)$. Thus, we only need to show the convergence of the right-hand side. By linearity, it is enough to consider nonnegative $\varphi \in C_c^1(\Omega)$. We argue as follows: For all $\delta > 0$, it holds

$$\begin{aligned} \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx &= \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx \\ &\quad + \int_{\{u_n > \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx. \end{aligned} \quad (3.12)$$

Next, we are going to estimate the two terms in the right-hand side of (3.12) separately. We begin with the first one. To this end, let

$$V_{\delta}(s) = \begin{cases} 1 & \text{for } |s| \leq \delta, \\ \frac{2\delta - s}{\delta} & \text{for } \delta < s < 2\delta \\ 0 & \text{for } s > 2\delta. \end{cases} \quad (3.13)$$

Choosing $V_{\delta}(u_n)\varphi$ as a test function in (3.5), we get

$$\begin{aligned} &\int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx \\ &\leq \int_{\Omega} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} V_{\delta}(u_n) \varphi \, dx \\ &\leq \int_{\Omega} V_{\delta}(u_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, dx + \int_{\Omega} V_{\delta}(u_n) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla \varphi \, dx. \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx \\ &\leq \int_{\Omega} V_{\delta}(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} V_{\delta}(u) |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx. \end{aligned}$$

Consequently,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\{u_n \leq \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx = 0.$$

On the other hand, for almost all $\delta > 0$ the set $\{u = \delta\}$ is Lebesgue negligible, and thus

$$\limsup_{n \rightarrow \infty} \int_{\{u_n > \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx = \int_{\{u > \delta\}} f(x) u^{-\gamma} \varphi \, dx,$$

and thus

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\{u_n > \delta\}} f_n(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, dx \\ &= \int_{\{u > 0\}} f(x) u^{-\gamma} \varphi \, dx = \int_{\Omega} f(x) u^{-\gamma} \varphi \, dx. \end{aligned}$$

Here we have used the fact that $\{u = 0\} \subset \{f = 0\}$. From the above arguments, we conclude that (3.2) holds. This proves the result. $\square$

Next, we show that the space $W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ is suitable as the space of the test functions.

Proposition 3.4. *The space of test functions can be extended up to $W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let $v \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ be nonnegative. Then there exists $\varphi_n \in C_c^\infty(\Omega)$ such that $\varphi_n \rightarrow v$ strongly in $W_0^{1,q}(\Omega)$. Moreover, let ρ_η be the standard mollifier, see Papageorgiou–Winkert [25, Definition 4.1.27 (c)]. We note that $\psi_{n,\eta} = \rho_\eta * \min\{v, \varphi_n\} \in C_c^1(\Omega)$ for $\eta > 0$. Hence it is an admissible test function and we can write

$$\int_{\Omega} a(\nabla u) \cdot \nabla \psi_{n,\eta} \, dx = \int_{\Omega} f(x) \psi_{n,\eta} u^{-\gamma} \, dx.$$

Observe that $\psi_{n,\eta} \rightarrow \min\{v, \varphi_n\} := \psi_n$ as $\eta \rightarrow 0$ strongly in $W_0^{1,q}(\Omega)$, weakly-* in $L^\infty(\Omega)$ and it has compact support in Ω . Consequently,

$$\int_{\Omega} a(\nabla u) \cdot \nabla \psi_n \, dx = \int_{\Omega} f(x) \psi_n u^{-\gamma} \, dx,$$

where we have used the fact that $v \in L^\infty(\Omega)$. The strong convergence in $W_0^{1,q}(\Omega)$ implies that the left-hand side of the last equality goes to $\int_{\Omega} a(\nabla u) \cdot \nabla v \, dx$ as $n \rightarrow \infty$. As for the right-hand side, using Fatou's Lemma, gives

$$\int_{\Omega} f(x) v u^{-\gamma} \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f(x) \psi_n u^{-\gamma} \, dx = \int_{\Omega} a(\nabla u) \cdot \nabla v \, dx.$$

Consequently, $f(x) v u^{-\gamma} \in L^1(\Omega)$. Since $\psi_n \leq v \in L^1(\Omega)$, Lebesgue's dominated convergence theorem yields

$$\int_{\Omega} a(\nabla u) \cdot \nabla v \, dx = \int_{\Omega} f(x) v u^{-\gamma} \, dx \quad \text{for all } v \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega), v \geq 0.$$

A simple argument shows that this equality can be extended for sign-changing $v \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$. This proves the result. $\square$

In the second part of this section we are going to show that the solutions obtained in Theorem 3.3 converge to a solution of the associated $(1, q)$ -Laplacian problem. Assuming $0 < \gamma < 1 < p < q < N$ and (3.3), Theorem 3.3 yields a sequence u_p of solutions of the problem

$$\begin{cases} -\Delta_p u_p - \Delta_q u_p = f(x) u_p^{-\gamma} & \text{in } \Omega, \\ u_p \geq 0 & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.14)$$

in the sense that $u_p \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $f u_p^{-\gamma} \in L_{\text{loc}}^1(\Omega)$ and

$$\int_{\Omega} a(\nabla u_p) \cdot \nabla \varphi \, dx = \int_{\Omega} f(x) u_p^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega) \quad (3.15)$$

and for all $p > 1$. Due to (3.6), it is clear that the sequence u_p is uniformly bounded in $L^\infty(\Omega)$. We now show that this sequence is also bounded in $W_0^{1,q}(\Omega)$. Indeed,

since $u_p \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, we may take it as test function in (3.15), thus getting

$$\int_{\Omega} |\nabla u_p|^q dx \leq \int_{\Omega} u_p^{1-\gamma} f(x) dx \leq C^{1-\gamma} \int_{\Omega} f(x) dx =: \bar{C} < \infty. \quad (3.16)$$

This proves the claim. In addition, since $1 < p < q$, this implies that $(\int_{\Omega} |\nabla u_p|^p dx)_{p>1}$ is bounded. Indeed,

$$\begin{aligned} \int_{\Omega} |\nabla u_p|^p dx &\leq \int_{\Omega} (\max\{1, |\nabla u_p|\})^p dx \\ &\leq \int_{\Omega} (\max\{1, |\nabla u_p|\})^q dx \\ &\leq \int_{\Omega} (1 + |\nabla u_p|^q) dx \\ &\leq |\Omega| + \bar{C} = \tilde{C} \end{aligned} \quad (3.17)$$

for all $p > 1$.

It immediately follows that there exists $u \in W_0^{1,q}(\Omega)$ such that, up to a subsequence,

$$u_p \rightharpoonup u \quad \text{weakly in } W_0^{1,q}(\Omega), \quad (3.18)$$

$$u_p \rightarrow u \quad \text{strongly in } L^\eta(\Omega) \text{ for all } 1 \leq \eta < q^*. \quad (3.19)$$

We are going to prove that u is a solution of the problem

$$\begin{cases} -\Delta_1 u - \Delta_q u = f(x)u^{-\gamma} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.20)$$

Definition 3.5. We say that $u \in W_0^{1,q}(\Omega)$ is a weak solution of problem (3.20) if

$$fu^{-\gamma} \in L_{\text{loc}}^1(\Omega), \quad (3.21)$$

and if there exists a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$ such that

$$-\operatorname{div} z - \Delta_q u = f(x)u^{-\gamma} \quad \text{in } \mathcal{D}'(\Omega), \quad (3.22)$$

and

$$z \cdot \nabla u = |\nabla u| \quad \text{a.e. in } \Omega. \quad (3.23)$$

We have the following existence result for problem (3.20).

Theorem 3.6. Let $0 < \gamma < 1 < q < N$ and assume that f satisfies (3.3). Then, the function $u \in W_0^{1,q}(\Omega)$ given by (3.18) and (3.19) is a solution of problem (3.20) in the sense of Definition 3.5.

Proof. First we show (3.21). For $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} f(x)u_p^{-\gamma}\varphi dx &= \int_{\Omega} |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla \varphi dx + \int_{\Omega} |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla \varphi dx \\ &\leq \int_{\Omega} |\nabla u_p|^{p-1}|\nabla \varphi| dx + \int_{\Omega} |\nabla u_p|^{q-1}|\nabla \varphi| dx \\ &\leq \int_{\Omega} |\nabla \varphi| dx + 2 \int_{\Omega} |\nabla u_p|^{q-1}|\nabla \varphi| dx \\ &\leq \int_{\Omega} |\nabla \varphi| dx + 2 \frac{(q-1)}{q} \int_{\Omega} |\nabla u_p|^q dx + \frac{2}{q} \int_{\Omega} |\nabla \varphi|^q dx. \end{aligned}$$

Since the right-hand side is uniformly bounded with respect to p , one has that $f u_p^{-\gamma}$ is locally bounded in $L^1(\Omega)$ with respect to p . An application of Fatou's Lemma as $p \rightarrow 1^+$ ensures that $f u^{-\gamma} \in L^1_{\text{loc}}(\Omega)$. This proves (3.21).

To proceed further, let

$$z_p = |\nabla u_p|^{p-2} \nabla u_p.$$

We now prove the existence of a vector field $z \in L^\infty(\Omega; \mathbb{R}^N)$ such that $z_p \rightharpoonup z$ weakly in $L^s(\Omega; \mathbb{R}^N)$ for all $1 \leq s < \infty$. Let us fix $1 < s < \infty$. Since $p \rightarrow 1^+$, we may assume that $1 < p < s'$. By Hölder's inequality and (3.17), we obtain

$$\begin{aligned} \int_{\Omega} ||\nabla u_p|^{p-2} \nabla u_p|^s dx &= \int_{\Omega} |\nabla u_p|^{s(p-1)} dx \\ &\leq \left(\int_{\Omega} |\nabla u_p|^p dx \right)^{\frac{s}{p'}} |\Omega|^{1-\frac{s}{p'}} \\ &\leq \tilde{C}^{\frac{s}{p'}} |\Omega|^{1-\frac{s}{p'}}. \end{aligned}$$

Therefore,

$$\| |\nabla u_p|^{p-2} \nabla u_p \|_{L^s(\Omega)} \leq \tilde{C}^{\frac{1}{p'}} |\Omega|^{\frac{1}{s} - \frac{1}{p'}}. \quad (3.24)$$

Hence, for every $1 < s < \infty$, the family $\{ |\nabla u_p|^{p-2} \nabla u_p \}_p$ is bounded in $L^s(\Omega; \mathbb{R}^N)$ for $p > 1$ sufficiently close to 1. By a diagonal argument, there exist a subsequence and a vector field z , independent of s , such that

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z \quad \text{weakly in } L^s(\Omega; \mathbb{R}^N)$$

for every $1 < s < \infty$ as $p \rightarrow 1^+$. Since Ω has finite measure, the same convergence also holds weakly in $L^1(\Omega; \mathbb{R}^N)$. Consequently,

$$z_p = |\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z \quad \text{weakly in } L^s(\Omega; \mathbb{R}^N) \quad \text{for every } 1 \leq s < \infty \quad (3.25)$$

as $p \rightarrow 1^+$. For every $1 < s < \infty$, the lower semicontinuity of the norm and (3.24) yield

$$\|z\|_{L^s(\Omega)} \leq |\Omega|^{\frac{1}{s}}. \quad (3.26)$$

Finally, letting $s \rightarrow \infty$ in (3.26), we conclude that $z \in L^\infty(\Omega; \mathbb{R}^N)$ and $\|z\|_{L^\infty(\Omega)} \leq 1$.

Thus, letting $p \rightarrow 1^+$ in (3.14), we get

$$\int_{\Omega} z \cdot \nabla \varphi dx + \lim_{p \rightarrow 1^+} \int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx = \lim_{p \rightarrow 1^+} \int_{\Omega} f(x) u_p^{-\gamma} \varphi dx.$$

Arguing as in the proof of Theorem 4.9, we conclude that

$$\nabla u_p \rightarrow \nabla u \quad \text{a.e. in } \Omega.$$

Moreover, from (3.16),

$$\int_{\Omega} ||\nabla u_p|^{q-2} \nabla u_p|^{\frac{q}{q-1}} dx = \int_{\Omega} |\nabla u_p|^q dx \leq C,$$

where C does not depend on p . Hence, $(|\nabla u_p|^{q-2} \nabla u_p)_p$ is bounded in $L^{\frac{q}{q-1}}(\Omega)^N$. From Lemma 4.8 by Kaviani [18], we have

$$|\nabla u_p|^{q-2} \nabla u_p \rightharpoonup |\nabla u|^{q-2} \nabla u \quad \text{weakly in } L^{\frac{q}{q-1}}(\Omega)^N \text{ as } p \rightarrow 1^+.$$

Hence, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi \, dx \rightarrow \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx.$$

The term on the right-hand side converges to $\int_{\Omega} f(x) u^{-\gamma} \varphi \, dx$ as in the proof of Theorem 3.3. Therefore, it holds

$$\int_{\Omega} z \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f(x) u^{-\gamma} \varphi \, dx, \quad (3.27)$$

for all $\varphi \in C_c^\infty(\Omega)$. We have thus shown that

$$-\operatorname{div} z - \Delta_q u = f(x) u^{-\gamma} \quad \text{in } \mathcal{D}'(\Omega).$$

This proves (3.22). Finally, we have to show that $z \cdot \nabla u = |\nabla u|$ a.e. in Ω . For this purpose, let $\varphi \in C_c^1(\Omega)$ be nonnegative and take $u\varphi * \rho_\varepsilon$ as a test function in (3.27). Then we have

$$\begin{aligned} & \int_{\Omega} z \cdot (\nabla u \varphi * \rho_\varepsilon) \, dx \\ &= \int_{\Omega} z \cdot \nabla (u \varphi * \rho_\varepsilon) \, dx - \int_{\Omega} z \cdot (\nabla \varphi u * \rho_\varepsilon) \, dx \\ &= \int_{\Omega} f(x) u \varphi * \rho_\varepsilon u^{-\gamma} \, dx - \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla (u \varphi * \rho_\varepsilon) \, dx \\ & \quad - \int_{\Omega} z \cdot (\nabla \varphi u * \rho_\varepsilon) \, dx. \end{aligned} \quad (3.28)$$

Next, we will estimate the terms in the right-hand side of (3.28). First of all, we know that

$$f(x) (u \varphi * \rho_\varepsilon) u^{-\gamma} \rightarrow f(x) u^{1-\gamma} \varphi \quad \text{a.e. in } \Omega \text{ as } \varepsilon \rightarrow 0.$$

Furthermore,

$$f(x) ((u \varphi) * \rho_\varepsilon) u^{-\gamma} \leq f(x) ((u \varphi) * \rho_\varepsilon) + \|u\|_\infty f(x) (\varphi * \rho_\varepsilon) u^{-\gamma} \quad \text{in } \Omega.$$

Indeed, note that

$$f(x) (u \varphi * \rho_\varepsilon) u^{-\gamma} \leq f(x) (u \varphi * \rho_\varepsilon) \quad \text{in } \{u > 1\}.$$

On the other hand, in $\{u \leq 1\}$, since $u \in L^\infty(\Omega)$,

$$\begin{aligned} f(x) ((u \varphi) * \rho_\varepsilon) u^{-\gamma} &\leq \|u\|_\infty f(x) \int_{\Omega} \rho_\varepsilon(y) \varphi(x-y) \, dy u^{-\gamma} \\ &= \|u\|_\infty f(x) (\varphi * \rho_\varepsilon) u^{-\gamma}. \end{aligned}$$

Putting together the last two inequalities yields the estimate above. Since the right-hand side converges in $L^1(\Omega)$, the generalized Lebesgue's dominated convergence theorem implies that the first term on the right-hand side of (3.28) converges to $\int_{\Omega} f u^{1-\gamma} \varphi \, dx$. The estimates for the other terms of (3.28) are trivial. Thus, we get

$$\begin{aligned} & \int_{\Omega} z \cdot \nabla u \varphi \, dx \\ &= \int_{\Omega} f u^{1-\gamma} \varphi \, dx - \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla (u \varphi) \, dx - \int_{\Omega} z \cdot \nabla \varphi u \, dx \end{aligned} \quad (3.29)$$

for all nonnegative $\varphi \in C_c^1(\Omega)$. On the other hand, taking $u_p \varphi$ in (3.14), we have

$$\begin{aligned} & \int_{\Omega} \varphi |\nabla u_p|^p dx + \int_{\Omega} u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx + \int_{\Omega} \varphi |\nabla u_p|^q dx \\ & + \int_{\Omega} u_p |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx = \int_{\Omega} f u_p^{1-\gamma} \varphi dx. \end{aligned} \quad (3.30)$$

To estimate the first term on the left-hand side, observe that the weak lower semicontinuity of the functional $u \mapsto \int_{\Omega} |\nabla u| \varphi dx$ with respect to the L^1 -convergence yields

$$\begin{aligned} \int_{\Omega} \varphi |\nabla u| dx & \leq \liminf_{p \rightarrow 1^+} \int_{\Omega} \varphi |\nabla u_p| dx \\ & \leq \liminf_{p \rightarrow 1^+} \frac{1}{p} \int_{\Omega} \varphi |\nabla u_p|^p dx + \liminf_{p \rightarrow 1^+} \frac{p-1}{p} \int_{\Omega} \varphi dx \\ & = \liminf_{p \rightarrow 1^+} \int_{\Omega} \varphi |\nabla u_p|^p dx. \end{aligned}$$

In the same way, the functional $u \mapsto \int_{\Omega} \varphi |\nabla u|^q dx$ is lower semicontinuous with respect to the L^q -convergence. Indeed, this is a consequence of the convexity of the integrand and Serrin's Lower Semicontinuity Theorem (see Serrin [28]). Consequently, letting $p \rightarrow 1^+$ in (3.30) and using (3.6), (3.18), (3.19), (3.25), we get

$$\begin{aligned} & \int_{\Omega} \varphi |\nabla u| dx + \int_{\Omega} u z \cdot \nabla \varphi dx + \int_{\Omega} \varphi |\nabla u|^q dx \\ & + \int_{\Omega} u |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi dx \leq \int_{\Omega} f u^{1-\gamma} \varphi dx. \end{aligned}$$

From (3.29), we conclude that

$$\int_{\Omega} |\nabla u| \varphi dx \leq \int_{\Omega} (z \cdot \nabla u) \varphi dx \quad \text{for all } \varphi \in C_c^1(\Omega) \text{ with } \varphi \geq 0.$$

This proves (3.23) and the assertion of the theorem. $\square$

4. CONCAVE-CONVEX PROBLEMS

In this section we first study the solvability of the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \frac{1}{u^\gamma} + \lambda u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

for small values of λ , where $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. We are looking for weak solutions $u \in W_0^{1,q}(\Omega)$ of problem (4.1) in the sense that

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi dx \\ & = \int_{\Omega} \varphi u^{-\gamma} dx + \lambda \int_{\Omega} u^{r-1} \varphi dx \end{aligned} \quad (4.2)$$

is satisfied for all $\varphi \in W_0^{1,q}(\Omega)$. To this end, we define the energy functional $I_\lambda: W_0^{1,q}(\Omega) \rightarrow \mathbb{R}$ of (4.1) by

$$I_\lambda(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - \frac{1}{1-\gamma} \int_\Omega |u|^{1-\gamma} dx - \frac{\lambda}{r} \|u\|_r^r.$$

Since the functional I_λ is not differentiable, we will make use of the Nehari manifold method. For $u \in W_0^{1,q}(\Omega)$, we define the fibering function $\omega_u: [0, +\infty) \rightarrow \mathbb{R}$ by

$$\omega_u(t) = I_\lambda(tu).$$

Then, the Nehari manifold related to (4.1) is defined by

$$\begin{aligned} \mathcal{N}_\lambda &= \left\{ u \in W_0^{1,q}(\Omega) \setminus \{0\} : \|\nabla u\|_p^p + \|\nabla u\|_q^q = \int_\Omega |u|^{1-\gamma} dx + \lambda \|u\|_r^r \right\} \\ &= \{u \in W_0^{1,q}(\Omega) \setminus \{0\} : \omega'_u(1) = 0\}. \end{aligned}$$

As shown in the paper by Liu–Dai–Papageorgiou–Winkert [22, Proposition 3.1], the restriction to $\mathcal{N}_\lambda$ of the functional I_λ is coercive. We split the Nehari manifold into three disjoint parts given by

$$\begin{aligned} \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : (p + \gamma - 1)\|\nabla u\|_p^p + (q + \gamma - 1)\|\nabla u\|_q^q = \lambda(r + \gamma - 1)\|u\|_r^r\} \\ &= \{u \in \mathcal{N}_\lambda : \omega''_u(1) = 0\}, \\ \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : (p + \gamma - 1)\|\nabla u\|_p^p + (q + \gamma - 1)\|\nabla u\|_q^q - \lambda(r + \gamma - 1)\|u\|_r^r > 0\} \\ &= \{u \in \mathcal{N}_\lambda : \omega''_u(1) > 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : (p + \gamma - 1)\|\nabla u\|_p^p + (q + \gamma - 1)\|\nabla u\|_q^q - \lambda(r + \gamma - 1)\|u\|_r^r < 0\} \\ &= \{u \in \mathcal{N}_\lambda : \omega''_u(1) < 0\}. \end{aligned}$$

Our objective is to obtain a solution u_p of problem (4.1) which belongs to $\mathcal{N}_\lambda^+$ and a second solution v_p which belongs to $\mathcal{N}_\lambda^-$. Our first result states that for small enough λ the component $\mathcal{N}_\lambda^0$ of the Nehari manifold $\mathcal{N}_\lambda$ is empty.

Proposition 4.1. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, there exists $\Lambda_1 > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$. The number Λ_1 does not depend on p .*

Proof. Let us suppose that $\lambda > 0$ is such that there exists $u \in \mathcal{N}_\lambda^0$. Since $u \in \mathcal{N}_\lambda$, we get

$$\begin{aligned} &(r + \gamma - 1)\|\nabla u\|_p^p + (r + \gamma - 1)\|\nabla u\|_q^q \\ &= (r + \gamma - 1) \int_\Omega |u|^{1-\gamma} dx + \lambda(r + \gamma - 1)\|u\|_r^r. \end{aligned}$$

Substituting this in the expression $\omega''_u(1) = 0$, we get

$$(r - p)\|\nabla u\|_p^p + (r - q)\|\nabla u\|_q^q = (r + \gamma - 1) \int_\Omega |u|^{1-\gamma} dx.$$

Since $r > p$, we may ignore the first term in the left-hand side. The right-hand side can be estimated via Hölder's inequality and the Sobolev embedding theorem. From this, we get a constant $c_1 > 0$ such that

$$(r - q)\|\nabla u\|_q^q \leq c_1(r + \gamma - 1)\|\nabla u\|_q^{1-\gamma}.$$

Thus, we can find $c_2 > 0$, which does not depend on λ nor on p such that

$$\|\nabla u\|_q \leq c_2. \quad (4.3)$$

On the other hand, since $u \in \mathcal{N}_\lambda^0$, we have

$$(q + \gamma - 1) \|\nabla u\|_q^q \leq \lambda(r + \gamma - 1) \|u\|_r^r \leq c_3 \lambda(r + \gamma - 1) \|\nabla u\|_q^r,$$

for some $c_3 > 0$. Consequently,

$$\lambda \|\nabla u\|_q^{r-q} \geq c_4.$$

Here, c_4 does not depend on λ nor on p . Using (4.3), it follows that

$$c_4 \leq \lambda c_2^{r-q}.$$

Therefore, $\lambda \geq \frac{c_4}{c_2^{r-q}}$. This proves the result. $\square$

Proposition 4.2. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, there exists $\Lambda_2 > 0$ such that $\mathcal{N}_\lambda^\pm \neq \emptyset$ for all $0 < \lambda < \Lambda_2$. This number does not depend on p . Furthermore, for each $u \in W_0^{1,q}(\Omega) \setminus \{0\}$, there exist unique numbers $t_1, t_2 > 0$ such that $t_1 u \in \mathcal{N}_\lambda^+$ and $t_2 u \in \mathcal{N}_\lambda^-$.*

Proof. We follow arguments similar to those in the paper by Liu–Dai–Papageorgiou–Winkert [22]. Let $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ and consider the function $\hat{\psi}_u: (0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\hat{\psi}_u(t) = t^{q-r} \|\nabla u\|_q^q - t^{1-r-\gamma} \int_\Omega |u|^{1-\gamma} dx.$$

Since $0 > q - r > 1 - r - \gamma$, there exists $\hat{t}_0 > 0$ such that $\hat{\psi}_u(\hat{t}_0) = \max_{t>0} \hat{\psi}_u(t)$. Moreover, $\hat{\psi}'_u(\hat{t}_0) = 0$, and consequently,

$$\hat{t}_0 = \left(\frac{(r + \gamma - 1) \int_\Omega |u|^{1-\gamma} dx}{(r - q) \|\nabla u\|_q^q} \right)^{\frac{1}{q+\gamma-1}}.$$

Moreover,

$$\hat{\psi}_u(\hat{t}_0) = \frac{q + \gamma - 1}{r - q} \left(\frac{r - q}{r + \gamma - 1} \right)^{\frac{r+\gamma-1}{q+\gamma-1}} \frac{\|\nabla u\|_q^{\frac{q(r+\gamma-1)}{q+\gamma-1}}}{\left(\int_\Omega |u|^{1-\gamma} dx \right)^{\frac{r-q}{q+\gamma-1}}}.$$

On the other hand,

$$\int_\Omega |u|^{1-\gamma} dx \leq \left(\int_\Omega |u|^{q^*} dx \right)^{\frac{1-\gamma}{q^*}} |\Omega|^{1-\frac{1-\gamma}{q^*}} = \|u\|_{q^*}^{1-\gamma} |\Omega|^{1-\frac{1-\gamma}{q^*}},$$

and

$$\mathcal{S}_q^{-1} \|u\|_{q^*}^q \leq \|\nabla u\|_q^q,$$

where $\mathcal{S}_q$ is the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$. Combining the preceding estimates, we obtain

$$\hat{\psi}_u(\hat{t}_0) \geq \frac{q + \gamma - 1}{r - q} \left(\frac{r - q}{r + \gamma - 1} \right)^{\frac{r+\gamma-1}{q+\gamma-1}} \frac{\mathcal{S}_q^{\frac{-r-\gamma+1}{q+\gamma-1}}}{|\Omega|^{\frac{(r-q)(q^*+\gamma-1)}{(q+\gamma-1)q^*}}} \|u\|_{q^*}^r.$$

Hence, $\hat{\psi}_u(\hat{t}_0) \geq c_{q,r,\gamma} \|u\|_{q^*}^r$ for some constant $c_{q,r,\gamma}$ that does not depend on p . Thus,

$$\hat{\psi}_u(\hat{t}_0) - \lambda \|u\|_r^r \geq c_{q,r,\gamma} \|u\|_{q^*}^r - \lambda |\Omega|^{1-\frac{r}{q^*}} \|u\|_{q^*}^r.$$

We conclude that if

$$\lambda < \frac{c_{q,r,\gamma}}{|\Omega|^{1-\frac{r}{q^*}}},$$

then

$$\widehat{\psi}_u(\widehat{t}_0) - \lambda \|u\|_r^r > 0.$$

Now let

$$\psi_u(t) = t^{p-r} \|\nabla u\|_p^p + t^{q-r} \|\nabla u\|_q^q - t^{1-r-\gamma} \int_{\Omega} |u|^{1-\gamma} dx.$$

Since $0 > q - r > p - r > 1 - r - \gamma$, there exists $t_0 > 0$ such that

$$\psi_u(t_0) = \max_{t>0} \psi_u(t).$$

Moreover,

$$\psi_u(t_0) - \lambda \|u\|_r^r > 0 \quad \text{for all } 0 < \lambda < \frac{c_{q,r,\gamma}}{|\Omega|^{1-\frac{r}{q^*}}}.$$

Thus, for each λ in this range, there exist $t_1 < t_0 < t_2$ such that

$$\psi_u(t_1) = \lambda \|u\|_r^r = \psi_u(t_2) \quad \text{and} \quad \psi'_u(t_2) < 0 < \psi'_u(t_1).$$

Arguing as in [22], one can conclude that $t_1 u \in \mathcal{N}_{\lambda}^+$ and that $t_2 u \in \mathcal{N}_{\lambda}^-$ and that t_1 and t_2 are unique. This shows the result. $\square$

Theorem 4.3. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, there exists $\Lambda_0 > 0$ which does not depend on p such that for all $0 < \lambda < \Lambda_0$, problem (4.1) has at least one weak solution $u^* \in W_0^{1,q}(\Omega)$ such that $u^* \in \mathcal{N}_{\lambda}^+$ and $I_{\lambda}(u^*) = \min_{\mathcal{N}_{\lambda}^+} I_{\lambda} < 0$.*

Proof. The proof of this result follows the same steps as in Propositions 3.6 and 3.7 by Liu–Dai–Papageorgiou–Winkert [22]. Here we only give a sketch. We first choose $\Lambda_0 = \min\{\Lambda_1, \Lambda_2\}$, so that Propositions 4.1 and 4.2 are applicable for $0 < \lambda < \Lambda_0$. For any such λ , we define

$$m_{\lambda}^+ = \inf_{\mathcal{N}_{\lambda}^+} I_{\lambda}.$$

Proposition 4.2 implies that m_{λ}^+ is well defined. Furthermore, one can check that (see [22, Proposition 3.2])

$$m_{\lambda}^+ < 0 \quad \text{for all } 0 < \lambda < \Lambda_0.$$

Thus, one can consider a minimizing sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{N}_{\lambda}^+$. The coercivity of the restriction of I_{λ} to $\mathcal{N}_{\lambda}$ implies that $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $W_0^{1,q}(\Omega)$ and thus there exists $u^* \in W_0^{1,q}(\Omega)$ so that, up to a subsequence,

$$u_n \rightharpoonup u^* \quad \text{weakly in } W_0^{1,q}(\Omega).$$

Next, it can be shown that this convergence is actually strong, so that $I_{\lambda}(u^*) = m_{\lambda}^+ < 0$. Finally, one can show that $u^* > 0$ a.e. in Ω and that u^* is a weak solution of problem (4.1) which belongs to $\mathcal{N}_{\lambda}^+$. This finishes the proof. $\square$

Our goal is now to obtain a second solution of problem (4.1) which belongs to $\mathcal{N}_{\lambda}^-$.

Lemma 4.4. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, there exists $\Lambda_3 > 0$ such that the restriction of I_λ to $\mathcal{N}_\lambda^-$ is positive for all $0 < \lambda < \Lambda_3$. The number Λ_3 does not depend on p .*

Proof. First, by the definition of $\mathcal{N}_\lambda^-$ together with the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$, one has

$$\lambda(r + \gamma - 1)\|u\|_r^r \geq (q + \gamma - 1)\|\nabla u\|_q^q \geq c_1^q(q + \gamma - 1)\|u\|_r^q \quad \text{for all } u \in \mathcal{N}_\lambda^-,$$

for some $c_1 > 0$. Thus,

$$\|u\|_r \geq \left(\frac{c_1^q(q + \gamma - 1)}{\lambda(r + \gamma - 1)} \right)^{\frac{1}{r-q}} := \frac{c_{q,\gamma,r}}{\lambda^{\frac{1}{r-q}}} \quad \text{for all } u \in \mathcal{N}_\lambda^-. \quad (4.4)$$

On the other hand, since

$$\frac{1}{q}\|\nabla u\|_q^q = \frac{1}{q} \int_\Omega |u|^{1-\gamma} dx + \frac{\lambda}{q}\|u\|_r^r - \frac{1}{q}\|\nabla u\|_p^p \quad \text{for all } u \in \mathcal{N}_\lambda,$$

we get

$$I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{q} \right) \|\nabla u\|_p^p + \left(\frac{1}{q} - \frac{1}{1-\gamma} \right) \int_\Omega |u|^{1-\gamma} dx + \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \|u\|_r^r$$

for all $u \in \mathcal{N}_\lambda^-$. We may ignore the first term on the right-hand side while the second one can be estimated via Hölder's inequality with exponents $\frac{r}{1-\gamma}$ and $\frac{r}{r+\gamma-1}$. This gives

$$I_\lambda(u) \geq \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \|u\|_r^r - \frac{c_{\gamma,r}}{1-\gamma} \|u\|_r^{1-\gamma} \quad \text{for all } u \in \mathcal{N}_\lambda^-,$$

with $c_{\gamma,r} > 0$ not depending on λ nor on p . Using (4.4) it follows that

$$\begin{aligned} I_\lambda(u) &\geq \|u\|_r^{1-\gamma} \left(\lambda \left(\frac{1}{q} - \frac{1}{r} \right) \|u\|_r^{r+\gamma-1} - \frac{c_{\gamma,r}}{1-\gamma} \right) \\ &\geq \|u\|_r^{1-\gamma} \left(\left(\frac{1}{q} - \frac{1}{r} \right) \frac{c_{q,\gamma,r}^{r+\gamma-1}}{\lambda^{\frac{r+\gamma-1}{r-q}}} - \frac{c_{\gamma,r}}{1-\gamma} \right). \end{aligned} \quad (4.5)$$

Therefore, $I_\lambda(u) > 0$ for all $u \in \mathcal{N}_\lambda^-$ provided

$$0 < \lambda < \left(\frac{(1-\gamma) \left(\frac{1}{q} - \frac{1}{r} \right) c_{q,\gamma,r}^{r+\gamma-1}}{c_{\gamma,r}} \right)^{\frac{r-q}{q+\gamma-1}} := \Lambda_3.$$

□

Next we obtain the existence of a second solution.

Theorem 4.5. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, there exists $\Lambda_0 > 0$ which does not depend on p such that for all $0 < \lambda < \Lambda_0$, problem (4.1) has a weak solution $v^* \in W_0^{1,q}(\Omega)$ such that $v^* \in \mathcal{N}_\lambda^-$ and $I_\lambda(v^*) = \min_{\mathcal{N}_\lambda^-} I_\lambda > 0$.*

Proof. We choose $\overline{\Lambda_0} = \min\{\Lambda_0, \Lambda_3\}$. The rest of the proof works in a very similar way as in [22, Propositions 3.8, 3.9 and 3.10]. □

Now we are interested in solving the problem

$$\begin{cases} -\Delta_1 u - \Delta_q u = \frac{1}{u^\gamma} + \lambda u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.6)$$

In order to do this, we will show that the solutions of problem (4.1) converge to solutions of problem (4.6) as $p \rightarrow 1^+$. First, we establish uniform estimates.

Lemma 4.6. *Let $0 < \gamma < 1 < p < q < N$, $q < r < q^*$ and $\{u_p\}$ be the sequence of weak solutions of problem (4.1) obtained in Theorem 4.3. Then, $\{u_p\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$.*

Proof. Indeed, since $I_\lambda(u_p) < 0$ for all p , we know that

$$\frac{1}{p} \|\nabla u_p\|_p^p + \frac{1}{q} \|\nabla u_p\|_q^q < \frac{1}{1-\gamma} \int_\Omega |u_p|^{1-\gamma} dx + \frac{\lambda}{r} \|u_p\|_r^r.$$

On the other hand, $u_p \in \mathcal{N}_\lambda$, and thus

$$\lambda \|u_p\|_r^r = \|\nabla u_p\|_p^p + \|\nabla u_p\|_q^q - \int_\Omega |u_p|^{1-\gamma} dx.$$

This leads to

$$\left(\frac{1}{p} - \frac{1}{r}\right) \|\nabla u_p\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_p\|_q^q < \left(\frac{1}{1-\gamma} - \frac{1}{r}\right) \int_\Omega |u_p|^{1-\gamma} dx.$$

Since $0 < \gamma < 1 < p < q < r$ and $|u_p|^{1-\gamma} \leq 1 + |u_p|$, we get

$$\left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_p\|_q^q < \left(\frac{1}{1-\gamma} - \frac{1}{r}\right) |\Omega| + \left(\frac{1}{1-\gamma} - \frac{1}{r}\right) S_1 \|\nabla u_p\|_q,$$

where we have also used the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^1(\Omega)$. Thus, $\{u_p\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$. $\square$

Next, we show the same result for the solutions v_p which have positive energy.

Lemma 4.7. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$ and $\{v_p\}$ be the sequence of weak solutions of problem (4.1) obtained in Theorem 4.5. Then, $\{v_p\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$.*

Proof. It is clear from the previous proof that it is enough to find a constant $a_1 > 0$ independent of p such that

$$I_\lambda(v_p) < a_1 \quad \text{for all } p > 1. \quad (4.7)$$

Recall that $I_\lambda(v_p) = \min_{\mathcal{N}_\lambda^-} I_\lambda$, and that for all $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ there exists a unique $t > 0$ such that $tu \in \mathcal{N}_\lambda^-$. Let $\omega \in C_c^\infty(\Omega)$ be a smooth function with compact support. Then $t_0\omega \in \mathcal{N}_\lambda^-$ for some $t_0 > 0$, and thus

$$I_\lambda(v_p) \leq \sup_{t>0} I_\lambda(t\omega).$$

On the other hand, we have

$$\begin{aligned} I_\lambda(t\omega) &= \frac{t^p}{p} \|\nabla \omega\|_p^p + \frac{t^q}{q} \|\nabla \omega\|_q^q - \frac{t^{1-\gamma}}{1-\gamma} \int_\Omega |\omega|^{1-\gamma} dx - \frac{\lambda t^r}{r} \|\omega\|_r^r \\ &\leq (1+t^q)(1 + \sup_\Omega |\nabla \omega|^q) |\Omega| + \frac{t^q}{q} \|\nabla \omega\|_q^q - \frac{\lambda t^r}{r} \|\omega\|_r^r \rightarrow -\infty \end{aligned}$$

as $t \rightarrow \infty$. Thus, there exists $a_1 > 0$ such that

$$\sup_{t>0} I_\lambda(t\omega) < a_1 \quad \text{for all } p > 1.$$

Observe that a_1 depends on λ but does not depend on p . This proves (4.7). $\square$

From Lemmas 4.6 and 4.7, we know that there exist $u_0, v_0 \in W_0^{1,q}(\Omega)$ such that $u_0, v_0 \geq 0$ in Ω and, up to subsequences,

$$\begin{aligned} u_p &\rightharpoonup u_0 \quad \text{and} \quad v_p \rightharpoonup v_0 \quad \text{weakly in } W_0^{1,q}(\Omega), \\ u_p &\rightarrow u_0 \quad \text{and} \quad v_p \rightarrow v_0 \quad \text{in } L^\eta(\Omega) \text{ for all } 1 \leq \eta < q^*. \end{aligned}$$

Lemma 4.8. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, the functions u_0, v_0 are nontrivial and $u_0^{-\gamma}, v_0^{-\gamma} \in L_{\text{loc}}^1(\Omega)$. In particular, $u_0, v_0 > 0$ a.e. in Ω .*

Proof. Let $w_p \in W_0^{1,q}(\Omega)$ satisfy

$$\begin{aligned} &\int_{\Omega} |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla w_p|^{q-2} \nabla w_p \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} \varphi w_p^{-\gamma} \, dx \quad \text{for all } \varphi \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega). \end{aligned}$$

Such a sequence of functions exists due to Theorem 3.3. Choosing $\varphi = (w_p - u_p)^+$ as a test function (this function belongs to $L^\infty(\Omega)$ because $w_p \in L^\infty(\Omega)$), we get

$$\begin{aligned} &\int_{\Omega} |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla (w_p - u_p)^+ \, dx + \int_{\Omega} |\nabla w_p|^{q-2} \nabla w_p \cdot \nabla (w_p - u_p)^+ \, dx \\ &= \int_{\Omega} (w_p - u_p)^+ w_p^{-\gamma} \, dx \\ &\leq \int_{\Omega} (w_p - u_p)^+ u_p^{-\gamma} \, dx + \lambda \int_{\Omega} u_p^{r-1} (w_p - u_p)^+ \, dx \\ &= \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla (w_p - u_p)^+ \, dx + \int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla (w_p - u_p)^+ \, dx. \end{aligned}$$

Thus, $(w_p - u_p)^+ = 0$ and so $u_p \geq w_p$. Since $w_p \rightarrow w$ a.e. in Ω and $w^{-\gamma} \in L_{\text{loc}}^1(\Omega)$ (see Theorem 3.6), we conclude that $u_0 \geq w$. Therefore, u_0 is nontrivial and $u_0^{-\gamma} \in L_{\text{loc}}^1(\Omega)$. A similar argument can be made for v_0 . The result is proved. $\square$

Theorem 4.9. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then, the sequences u_p and v_p converge strongly to u_0 and v_0 , respectively, in $W_0^{1,q}(\Omega)$.*

Proof. We prove the result only for u_0 , the proof for v_0 is analogous. Taking $w_p = u_p - u_0$ as test function in (4.2) yields

$$\begin{aligned} &\int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w_p \, dx + \int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla w_p \, dx \\ &= \int_{\Omega} (u_p - u_0) u_p^{-\gamma} \, dx + \lambda \int_{\Omega} u_p^{r-1} (u_p - u_0) \, dx. \end{aligned}$$

Let us estimate the two terms on the right-hand side. First note that for any $\frac{q^*}{q^* - (r-1)} < \eta < q^*$, it holds that $\frac{(r-1)\eta}{\eta-1} < q^*$ and thus u_p^{r-1} is bounded in $L^{\frac{\eta}{\eta-1}}(\Omega)$. Moreover, $u_p - u_0 \rightarrow 0$ strongly in $L^\eta(\Omega)$ for any such η . Therefore, we have

$$\int_{\Omega} \left| u_p^{r-1} (u_p - u_0) \right| \, dx \leq \|u_p - u_0\|_{L^\eta(\Omega)} \left(\int_{\Omega} u_p^{\frac{(r-1)\eta}{\eta-1}} \, dx \right)^{\frac{\eta-1}{\eta}} \rightarrow 0,$$

as $p \rightarrow 1^+$. To estimate the singular term, observe that, for all $\delta > 0$,

$$\begin{aligned} & \int_{\Omega} (u_p - u_0) u_p^{-\gamma} dx \\ &= \int_{\{u_p \leq \delta\}} (u_p - u_0) u_p^{-\gamma} dx + \int_{\{u_p > \delta\}} (u_p - u_0) u_p^{-\gamma} dx \\ &\leq \delta^{1-\gamma} |\Omega| + \int_{\{u_p > \delta\}} (u_p - u_0) u_p^{-\gamma} dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w_p dx + \int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla w_p dx \\ &\leq \delta^{1-\gamma} |\Omega| + \int_{\{u_p > \delta\}} (u_p - u_0) u_p^{-\gamma} dx + \omega_p, \end{aligned}$$

with $\lim_{p \rightarrow 1^+} \omega_p = 0$. Since

$$\lim_{p \rightarrow 1^+} \int_{\{u_p > \delta\}} (u_p - u_0) u_p^{-\gamma} dx = 0 \quad \text{for all } \delta > 0,$$

we conclude that

$$\begin{aligned} & \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w_p dx + \int_{\Omega} |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla w_p dx \\ &\leq \omega_{p,\delta}, \end{aligned}$$

where $\lim_{\delta \rightarrow 0} \lim_{p \rightarrow 1^+} \omega_{p,\delta} = 0$.

On the other hand, the weak convergence to zero of w_p implies that

$$\int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla w_p dx \rightarrow 0 \quad \text{as } p \rightarrow 1^+,$$

and thus

$$\begin{aligned} & \int_{\Omega} (|\nabla u_p|^{q-2} \nabla u_p - |\nabla u_0|^{q-2} \nabla u_0) \cdot \nabla w_p dx \\ &\leq \tilde{\omega}_{p,\delta} - \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w_p dx, \end{aligned}$$

where again $\lim_{\delta \rightarrow 0} \lim_{p \rightarrow 1^+} \tilde{\omega}_{p,\delta} = 0$. Rewriting this inequality gives

$$\begin{aligned} & \int_{\Omega} (|\nabla u_p|^{q-2} \nabla u_p - |\nabla u_0|^{q-2} \nabla u_0) \cdot \nabla w_p dx \\ &\leq \tilde{\omega}_{p,\delta} - \int_{\Omega} |\nabla u_p|^p dx + \int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla u_0 dx. \end{aligned}$$

Now we set

$$z_p = |\nabla u_p|^{p-2} \nabla u_p.$$

Since

$$\int_{\Omega} |\nabla u_p|^p dx \leq C,$$

where $C > 0$ is independent of p , we can argue as in the proof of Theorem 3.6. Thus, up to a subsequence, there exists $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|z\|_{L^\infty(\Omega)} \leq 1$ such that

$$z_p \rightharpoonup z \quad \text{weakly in } L^s(\Omega; \mathbb{R}^N)$$

for every $1 \leq s < \infty$ as $p \rightarrow 1^+$. Letting $p \rightarrow 1^+$ and $\delta \rightarrow 0$ we obtain

$$\begin{aligned} & \limsup_{p \rightarrow 1^+} \int_{\Omega} (|\nabla u_p|^{q-2} \nabla u_p - |\nabla u_0|^{q-2} \nabla u_0) \cdot \nabla w_p \, dx \\ & \leq -\liminf_{p \rightarrow 1^+} \int_{\Omega} |\nabla u_p|^p \, dx + \int_{\Omega} z \cdot \nabla u_0 \, dx \\ & \leq -\liminf_{p \rightarrow 1^+} \int_{\Omega} |\nabla u_p|^p \, dx + \int_{\Omega} |\nabla u_0| \, dx, \end{aligned}$$

where we have used that $z \in L^\infty(\Omega)$ with $\|z\|_\infty \leq 1$. Observe that the weak lower-semicontinuity of the norm implies that

$$\begin{aligned} \int_{\Omega} |\nabla u_0| \, dx & \leq \liminf_{p \rightarrow 1^+} \int_{\Omega} |\nabla u_p| \, dx \\ & \leq \liminf_{p \rightarrow 1^+} \left(\frac{1}{p} \int_{\Omega} |\nabla u_p|^p \, dx + \frac{p-1}{p} |\Omega| \right) \\ & = \liminf_{p \rightarrow 1^+} \int_{\Omega} |\nabla u_p|^p \, dx. \end{aligned}$$

We finally conclude that

$$\lim_{p \rightarrow 1^+} \int_{\Omega} (|\nabla u_p|^{q-2} \nabla u_p - |\nabla u_0|^{q-2} \nabla u_0) \cdot \nabla w_p \, dx = 0.$$

This proves the result due to Lemma 5 by Boccardo–Murat–Puel [8]. $\square$

Lemma 4.10. *Let $0 < \gamma < 1 < p < q < N$ and $q < r < q^*$. Then it holds $u_0 \not\equiv v_0$.*

Proof. Since $v_p \in \mathcal{N}_\lambda^-$ for all $p > 1$, we may use (4.5) to get

$$I_\lambda(v_p) \geq c \|v_p\|_r^{1-\gamma}, \quad (4.8)$$

where the constant $c > 0$ does not depend on p . Using Theorem 4.9, it holds that $\nabla v_p \rightarrow \nabla v_0$ a.e. in Ω . This leads to

$$\lim_{p \rightarrow 1^+} I_\lambda(v_p) = \|\nabla v_0\|_1 + \frac{1}{q} \lim_{p \rightarrow 1^+} \|\nabla v_p\|_q^q - \frac{1}{1-\gamma} \int_{\Omega} v_0^{1-\gamma} \, dx - \frac{\lambda}{r} \|v_0\|_r^r.$$

On the other hand, since $v_p \in \mathcal{N}_\lambda$, we know that

$$\|\nabla v_p\|_q^q = \int_{\Omega} |v_p|^{1-\gamma} \, dx + \lambda \|v_p\|_r^r - \|\nabla v_p\|_p^p \quad \text{for all } p > 1,$$

and thus,

$$\lim_{p \rightarrow 1^+} I_\lambda(v_p) = \left(1 - \frac{1}{q}\right) \|\nabla v_0\|_1 + \left(\frac{1}{q} - \frac{1}{1-\gamma}\right) \int_{\Omega} v_0^{1-\gamma} \, dx + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \|v_0\|_r^r.$$

Letting $p \rightarrow 1^+$ in (4.8) yields

$$0 < c \|v_0\|_r^{1-\gamma} \leq \lim_{p \rightarrow 1^+} I_\lambda(v_p) = \left(1 - \frac{1}{q}\right) \|\nabla v_0\|_1 + \left(\frac{1}{q} - \frac{1}{1-\gamma}\right) \int_{\Omega} v_0^{1-\gamma} \, dx$$

$$+ \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \|v_0\|_r^r,$$

where we used the fact that v_0 is nontrivial, see Lemma 4.8. We now estimate $I_\lambda(u_p)$. From Theorem 4.3 we know that $I_\lambda(u_p) < 0$ for all $p > 1$. Thus, since $\nabla u_p \rightarrow \nabla u_0$ a.e. in Ω ,

$$\begin{aligned} & \left(1 - \frac{1}{q}\right) \|\nabla u_0\|_1 + \left(\frac{1}{q} - \frac{1}{1-\gamma}\right) \int_\Omega u_0^{1-\gamma} dx + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \|u_0\|_r^r \\ &= \lim_{p \rightarrow 1^+} I_\lambda(u_p) \leq 0 < \lim_{p \rightarrow 1^+} I_\lambda(v_p) \\ &\leq \left(1 - \frac{1}{q}\right) \|\nabla v_0\|_1 + \left(\frac{1}{q} - \frac{1}{1-\gamma}\right) \int_\Omega v_0^{1-\gamma} dx + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \|v_0\|_r^r. \end{aligned}$$

Consequently, u_0 and v_0 must be different. This proves the result. $\square$

We will now show that u_0 and v_0 are solutions of problem (4.6) in the following sense.

Definition 4.11. We say that $u \in W_0^{1,q}(\Omega)$ is a weak solution of problem (4.6) if

$$u^{-\gamma} \in L_{\text{loc}}^1(\Omega), \quad (4.9)$$

and if there exists a vector field $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$ such that

$$-\operatorname{div} z - \Delta_q u = u^{-\gamma} + \lambda u^{r-1} \quad \text{in } \mathcal{D}'(\Omega), \quad (4.10)$$

and

$$z \cdot \nabla u = |\nabla u| \quad \text{a.e. in } \Omega. \quad (4.11)$$

Proof of Theorem 1.1. We will prove the result only for u_0 , the one for v_0 is analogous. From Lemma 4.8, we obtain (4.9). Let's show (4.10). Recall that u_0 is the weak limit of u_p in $W_0^{1,q}(\Omega)$ satisfying

$$\begin{aligned} & \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx + \int_\Omega |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx \\ &= \int_\Omega \varphi u_p^{-\gamma} dx + \lambda \int_\Omega u_p^{r-1} \varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega). \end{aligned} \quad (4.12)$$

We estimate the left-hand side of (4.12). Let $z_p = |\nabla u_p|^{p-2} \nabla u_p$. From Lemma 4.6 and the arguments in the proof of Theorem 3.6, it follows that there exists a vector field $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|z\|_{L^\infty(\Omega)} \leq 1$ such that

$$z_p \rightharpoonup z \quad \text{weakly in } L^s(\Omega; \mathbb{R}^N)$$

for every $1 \leq s < \infty$ as $p \rightarrow 1^+$. Thus,

$$\lim_{p \rightarrow 1^+} \int_\Omega z_p \cdot \nabla \varphi dx = \int_\Omega z \cdot \nabla \varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Furthermore, Theorem 4.9 implies that $\nabla u_p \rightarrow \nabla u_0$ a.e. in Ω . From Lemma 4.6, we know that $\{\nabla u_p\}$ is uniformly bounded in $L^q(\Omega)$. Thus,

$$\lim_{p \rightarrow 1^+} \int_\Omega |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx = \int_\Omega |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla \varphi dx.$$

We now estimate the right-hand side of (4.12). For the non-singular term, we have

$$\lim_{p \rightarrow 1^+} \int_\Omega u_p^{r-1} \varphi dx = \int_\Omega u_0^{r-1} \varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

because u_p^{r-1} is bounded in $L^{\frac{q^*}{r-1}}(\Omega)$ and $u_p^{r-1} \rightarrow u_0^{r-1}$ a.e. in Ω . For the singular term, by linearity it is enough to consider nonnegative $\varphi \in C_c^\infty(\Omega)$. Let $\delta > 0$. Then,

$$\int_{\Omega} \varphi u_p^{-\gamma} dx = \int_{\{u_p \leq \delta\}} \varphi u_p^{-\gamma} dx + \int_{\{u_p > \delta\}} \varphi u_p^{-\gamma} dx.$$

Let $V_\delta(s)$ be given by (3.13). Choosing $V_\delta(u_p)\varphi$ as a test function in (4.12), we get

$$\begin{aligned} & \int_{\{u_p \leq \delta\}} u_p^{-\gamma} \varphi dx \\ & \leq \int_{\Omega} u_p^{-\gamma} V_\delta(u_p) \varphi dx \\ & \leq \int_{\Omega} V_\delta(u_p) |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx + \int_{\Omega} V_\delta(u_p) |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx. \end{aligned}$$

Letting $p \rightarrow 1^+$, we get

$$\limsup_{p \rightarrow 1^+} \int_{\{u_p \leq \delta\}} u_p^{-\gamma} \varphi dx \leq \int_{\Omega} V_\delta(u_0) z \cdot \nabla \varphi dx + \int_{\Omega} V_\delta(u_0) |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla \varphi dx.$$

Consequently,

$$\lim_{\delta \rightarrow 0} \limsup_{p \rightarrow 1^+} \int_{\{u_p \leq \delta\}} u_p^{-\gamma} \varphi dx = 0.$$

On the other hand, for almost every $\delta > 0$,

$$\limsup_{p \rightarrow 1^+} \int_{\{u_p > \delta\}} u_p^{-\gamma} \varphi dx = \int_{\{u_0 > \delta\}} u_0^{-\gamma} \varphi dx,$$

and thus

$$\lim_{\delta \rightarrow 0} \lim_{p \rightarrow 1^+} \int_{\{u_p > \delta\}} u_p^{-\gamma} \varphi dx = \int_{\{u_0 > 0\}} u_0^{-\gamma} \varphi dx = \int_{\Omega} u_0^{-\gamma} \varphi dx.$$

Here we used the fact that the set $\{u_0 = 0\}$ is negligible with respect to Lebesgue's measure. This finishes the estimate for the singular term in (4.12). Taking everything together and letting $p \rightarrow 1^+$ in (4.12), we deduce that

$$\int_{\Omega} z \cdot \nabla \varphi dx + \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla \varphi dx = \int_{\Omega} \varphi u_0^{-\gamma} dx + \lambda \int_{\Omega} u_0^{r-1} \varphi dx \quad (4.13)$$

for all $\varphi \in C_c^1(\Omega)$, which proves (4.10). It only remains to show that (4.11) holds. To this aim, let $\varphi \in C_c^1(\Omega)$ be nonnegative. Taking $(T_k(u_0)\varphi) * \rho_\varepsilon$ as a test function in (4.13), we have

$$\begin{aligned} & \int_{\Omega} z \cdot ((\varphi \nabla T_k(u_0)) * \rho_\varepsilon) dx \\ & = \int_{\Omega} z \cdot \nabla ((T_k(u_0)\varphi) * \rho_\varepsilon) dx - \int_{\Omega} z \cdot ((T_k(u_0)\nabla \varphi) * \rho_\varepsilon) dx \\ & = \int_{\Omega} ((T_k(u_0)\varphi) * \rho_\varepsilon) u_0^{-\gamma} dx + \lambda \int_{\Omega} u_0^{r-1} ((T_k(u_0)\varphi) * \rho_\varepsilon) dx \\ & \quad - \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla ((T_k(u_0)\varphi) * \rho_\varepsilon) dx - \int_{\Omega} z \cdot ((T_k(u_0)\nabla \varphi) * \rho_\varepsilon) dx. \end{aligned}$$

We will estimate all the terms in this equation. We start with the singular one. Observe that

$$((T_k(u_0)\varphi) * \rho_\varepsilon)u_0^{-\gamma}(x) \rightarrow T_k(u_0)u_0^{-\gamma}\varphi(x) \quad \text{a.e. in } \Omega \text{ as } \varepsilon \rightarrow 0.$$

Furthermore,

$$((T_k(u_0)\varphi) * \rho_\varepsilon)u_0^{-\gamma} \leq (\varphi * \rho_\varepsilon)ku_0^{-\gamma}.$$

Since $\varphi * \rho_\varepsilon \rightarrow \varphi$ uniformly in the norm $\|\cdot\|_\infty$, we know that the right-hand side converges in $L^1(\Omega)$. Thus, the generalized Lebesgue's dominated convergence theorem implies that

$$\int_{\Omega} ((T_k(u_0)\varphi) * \rho_\varepsilon)u_0^{-\gamma} dx \rightarrow \int_{\Omega} T_k(u_0)u_0^{-\gamma}\varphi(x) dx \quad \text{as } \varepsilon \rightarrow 0.$$

For the term involving the parameter λ , we know that $(T_k(u_0)\varphi) * \rho_\varepsilon \rightarrow T_k(u_0)\varphi$ in $L^{q^*}(\Omega)$. Thus

$$\begin{aligned} & \int_{\Omega} |u_0^{r-1}((T_k(u_0)\varphi) * \rho_\varepsilon) - u_0^{r-1}T_k(u_0)\varphi| dx \\ & \leq \int_{\Omega} u_0^{r-1} |(T_k(u_0)\varphi) * \rho_\varepsilon - T_k(u_0)\varphi| dx \\ & \leq \left(\int_{\Omega} u_0^{q^*} dx \right)^{\frac{r-1}{q^*}} \left(\int_{\Omega} |(T_k(u_0)\varphi) * \rho_\varepsilon - T_k(u_0)\varphi|^{\frac{q^*}{q^*+1-r}} dx \right)^{\frac{q^*+1-r}{q^*}} \\ & \leq C \left(\int_{\Omega} |(T_k(u_0)\varphi) * \rho_\varepsilon - T_k(u_0)\varphi|^{\frac{q^*}{q^*+1-r}} dx \right)^{\frac{q^*+1-r}{q^*}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

It is also clear that

$$\begin{aligned} & \int_{\Omega} z \cdot ((T_k(u_0)\nabla\varphi) * \rho_\varepsilon) dx \rightarrow \int_{\Omega} T_k(u_0)z \cdot \nabla\varphi dx, \\ & \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla((T_k(u_0)\varphi) * \rho_\varepsilon) dx \rightarrow \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla(T_k(u_0)\varphi) dx, \\ & \int_{\Omega} z \cdot ((\varphi \nabla T_k(u_0)) * \rho_\varepsilon) dx \rightarrow \int_{\Omega} z \cdot (\varphi \nabla T_k(u_0)) dx, \end{aligned}$$

as $\varepsilon \rightarrow 0$. All together, we get

$$\begin{aligned} \int_{\Omega} z \cdot \varphi \nabla T_k(u_0) dx &= \int_{\Omega} T_k(u_0)u_0^{-\gamma}\varphi dx + \lambda \int_{\Omega} u_0^{r-1}T_k(u_0)\varphi dx \\ &\quad - \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla(T_k(u_0)\varphi) dx - \int_{\Omega} z \cdot T_k(u_0)\nabla\varphi dx. \end{aligned}$$

Thus, using $T_k(u_0) \rightarrow u_0$ strongly in $W_0^{1,q}(\Omega)$ and the monotone convergence theorem for the singular term, and letting $k \rightarrow \infty$, we conclude that

$$\begin{aligned} \int_{\Omega} z \cdot \varphi \nabla u_0 dx &= \int_{\Omega} u_0^{1-\gamma}\varphi dx + \lambda \int_{\Omega} u_0^r \varphi dx \\ &\quad - \int_{\Omega} |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla(u_0\varphi) dx - \int_{\Omega} z \cdot u_0 \nabla\varphi dx. \end{aligned} \tag{4.14}$$

On the other hand, taking $u_p\varphi$ in (4.12), we have

$$\int_{\Omega} \varphi |\nabla u_p|^p dx + \int_{\Omega} u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla\varphi dx$$

$$\begin{aligned}
& + \int_{\Omega} \varphi |\nabla u_p|^q dx + \int_{\Omega} u_p |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi dx \\
& = \int_{\Omega} u_p^{1-\gamma} \varphi dx + \lambda \int_{\Omega} u_p^r \varphi dx \quad \text{for all } p > 1.
\end{aligned}$$

Letting $p \rightarrow 1^+$ and using weak lower semicontinuity properties, we arrive at

$$\begin{aligned}
& \int_{\Omega} \varphi |\nabla u_0| dx + \int_{\Omega} u_0 z \cdot \nabla \varphi dx + \int_{\Omega} \varphi |\nabla u_0|^q dx + \int_{\Omega} u_0 |\nabla u_0|^{q-2} \nabla u_0 \cdot \nabla \varphi dx \\
& \leq \int_{\Omega} u_0^{1-\gamma} \varphi dx + \lambda \int_{\Omega} u_0^r \varphi dx.
\end{aligned}$$

Using (4.14) yields

$$\int_{\Omega} |\nabla u_0| \varphi dx \leq \int_{\Omega} (z \cdot \nabla u_0) \varphi dx \quad \text{for all } \varphi \in C_c^1(\Omega) \text{ with } \varphi \geq 0.$$

This proves (4.11) and the result. $\square$

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REFERENCES

- [1] F. Andreu, C. Ballester, V. Caselles, J.M. Mazón, *Minimizing total variation flow*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 11, 867–872.
- [2] F. Andreu, C. Ballester, V. Caselles, J.M. Mazón, *Minimizing total variation flow*, Differential Integral Equations **14** (2001), no. 3, 321–360.
- [3] F. Andreu, C. Ballester, V. Caselles, J.M. Mazón, *The Dirichlet problem for the total variation flow*, J. Funct. Anal. **180** (2001), no. 2, 347–403.
- [4] F. Andreu-Vaillio, V. Caselles, J.M. Mazón, “Parabolic Quasilinear Equations Minimizing Linear Growth Functionals”, Birkhäuser Verlag, Basel, 2004.
- [5] F. Balducci, F. Oliva, F. Petitta, *Finite energy solutions for nonlinear elliptic equations with competing gradient, singular and L^1 terms*, J. Differential Equations **391** (2024), 334–369.
- [6] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), no. 2, 241–273.
- [7] A. Bensoussan, L. Boccardo, F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), no. 4, 347–364.
- [8] L. Boccardo, F. Murat, J.-P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. (4) **152** (1988), 183–196.
- [9] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.
- [10] M. Cicalese, C. Trombetti, *Asymptotic behaviour of solutions to p -Laplacian equation*, Asymptot. Anal. **35** (2003), no. 1, 27–40.
- [11] M.G. Crandall, P.H. Rabinowitz, L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222.

- [12] F. Demengel, *On some nonlinear partial differential equations involving the 1-Laplacian and critical Sobolev exponent*, ESAIM Control Optim. Calc. Var. **4** (1999), 667–686.
- [13] L.M. De Cave, *Nonlinear elliptic equations with singular nonlinearities*, Asymptot. Anal. **84** (2013), no. 3-4, 181–195.
- [14] V. De Cicco, D. Giachetti, S. Segura de León, *Elliptic problems involving the 1-Laplacian and a singular lower order term*, J. Lond. Math. Soc. (2) **99** (2019), no. 2, 349–376.
- [15] V. De Cicco, D. Giachetti, F. Oliva, F. Petitta, *The Dirichlet problem for singular elliptic equations with general nonlinearities*, Calc. Var. Partial Differential Equations **58** (2019), no. 4, Paper No. 129, 40 pp.
- [16] G.M. Figueiredo, M.T.O. Pimenta, P. Winkert, *The asymptotic behavior of constant sign and nodal solutions of (p, q) -Laplacian problems as p goes to 1*, Nonlinear Anal. **251** (2025), Paper No. 113677, 14 pp.
- [17] D. Giachetti, F. Oliva, F. Petitta, *1-Laplacian type problems with strongly singular nonlinearities and gradient terms*, Commun. Contemp. Math. **24** (2022), no. 10, Paper No. 2150081, 40 pp.
- [18] O. Kavian, “Introduction à la Théorie des Points Critiques et Applications’ aux Problèmes Elliptiques”, Springer-Verlag, Paris, 1993.
- [19] B. Kawohl, *On a family of torsional creep problems*, J. Reine Angew. Math. **410** (1990), 1–22.
- [20] M. Latorre, F. Oliva, F. Petitta, S. Segura de León, *The Dirichlet problem for the 1-Laplacian with a general singular term and L^1 -data*, Nonlinearity **34** (2021), no. 3, 1791–1816.
- [21] A.C. Lazer, P.J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 721–730.
- [22] W. Liu, G. Dai, N.S. Papageorgiou, P. Winkert, *Existence of solutions for singular double phase problems via the Nehari manifold method*, Anal. Math. Phys. **12** (2022), no. 3, Paper No. 75, 25 pp.
- [23] A. Mercaldo, S. Segura de León, C. Trombetti, *On the solutions to 1-Laplacian equation with L^1 data*, J. Funct. Anal. **256** (2009), no. 8, 2387–2416.
- [24] J.C. Ortiz Chata, M.T.O. Pimenta, S. Segura de León, *Existence of solutions to a 1-Laplacian problem with a concave-convex nonlinearity*, J. Math. Anal. Appl. **525** (2023), no. 2, Paper No. 127149, 25 pp.
- [25] N.S. Papageorgiou, P. Winkert, “Applied Nonlinear Functional Analysis”, Second Revised Edition, De Gruyter, Berlin, 2024.
- [26] L.I. Rudin, S. Osher, E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D **60** (1992), no. 1-4, 259–268.
- [27] S. Segura de León, *Existence and uniqueness for L^1 data of some elliptic equations with natural growth*, Adv. Differential Equations **8** (2003), no. 11, 1377–1408.
- [28] J. Serrin, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc. **101** (1961), 139–167.
- [29] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), no. 1, 189–258.
- [30] C.A. Stuart, *Existence and approximation of solutions of non-linear elliptic equations*, Math. Z. **147** (1976), no. 1, 53–63.

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