ELLIPTIC SYSTEMS WITH UNBALANCED GROWTH AND LOGARITHMIC PERTURBATION: EXISTENCE, UNIQUENESS AND BOUNDEDNESS

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ABSTRACT. In this paper, we study elliptic systems driven by logarithmic double phase operators with unbalanced growth and gradient dependent nonlinearities. The system couples two logarithmic double phase equations and is subject to homogeneous Dirichlet boundary conditions. Due to the presence of convection type terms, the problem is nonvariational in nature and classical critical point methods are not applicable. Under suitable growth and coercivity assumptions on the nonlinearities, we establish the existence of nontrivial weak solutions by means of pseudomonotonicity arguments and eigenvalue estimates for the p_i-Laplacians for $i = 1, 2$. For a special case, we further derive a uniqueness result based on a specific structural condition imposed on the convection terms. In addition, under stronger subcritical growth assumptions, we prove boundedness of weak solutions by adapting a Moser iteration scheme to the logarithmic double phase setting. To the best of our knowledge, these results provide the first existence, uniqueness, and boundedness theory for elliptic systems involving logarithmic double phase operators.

1. Introduction and notation

In recent years the study of double phase problems has become increasingly prominent. These models are typically driven by operators of the form

$$-\operatorname{div}\left(|\nabla\omega|^{p-2}\nabla\omega+\mu(x)|\nabla\omega|^{q-2}\nabla\omega\right)\quad\text{for } 1<p<q,$$

which are associated with the energy functional

$$\omega\mapsto\int_\Omega\left(\frac{|\nabla\omega|^p}{p}+\mu(x)\frac{|\nabla\omega|^q}{q}\right)\mathrm{d}x.$$

Such functionals originate in the work of Zhikov [49] in the context of homogenization theory and elasticity, while their explicit double phase structure was clarified later in [50, 51]. In this framework, the coefficient $\mu(\cdot)$ represents the spatial distribution of two materials whose mechanical response is characterized by different growth exponents p and q. From a variational viewpoint, these models belong to the broader class of problems with nonstandard growth conditions, systematically investigated by Marcellini [36, 35]. The regularity theory developed there applies naturally to the double phase setting. Subsequent refinements and extensions can be found in the work of Cupini–Marcellini–Mascolo [22] and in more recent contributions by Marcellini [33, 34]. A decisive breakthrough was achieved in a sequence

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of papers by Baroni–Colombo–Mingione [6, 7, 8] and Colombo–Mingione [18, 19], where optimal regularity results for solutions of double phase problems were established. Beyond their theoretical interest, these operators arise naturally in several applications, including transonic flow models (see Bahrouni–Rădulescu–Repovš [5]), problems from quantum physics (see Benci–D'Avenia–Fortunato–Pisani [9]), reaction diffusion equations (see Cherfils–Il'yasov [16]), as well as in the analysis of the Lavrentiev phenomenon and thermistor type models (see Zhikov [50, 51]). For further information on the analytic structure of the corresponding function spaces and operators we refer to Colasuonno–Squassina [17], Crespo-Blanco–Gasiński–Harjulehto–Winkert [21], Ho–Winkert [29], Liu–Dai [31], and Perera–Squassina [42].

More recently Arora–Crespo-Blanco–Winkert [2, 3] carried out a detailed analysis of the functional

$$\omega \mapsto \int_\Omega \left(\frac{|\nabla \omega|^p}{p} + \mu(x) \frac{|\nabla \omega|^q}{q} \log(e + |\nabla \omega|) \right) \mathrm{d}x,$$

where e denotes Euler's number, together with the corresponding logarithmic double phase operator $\operatorname{div} \Upsilon(\omega)$, where

$$\Upsilon(\omega) = |\nabla \omega|^{p-2} \nabla \omega + \mu(x) \left[\log(e + |\nabla \omega|) + \frac{|\nabla \omega|}{q(e + |\nabla \omega|)} \right] |\nabla \omega|^{q-2} \nabla \omega, \qquad (1.1)$$

acting on functions ω belonging to some function space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. This operator is generated by

$$\mathcal{H}_{\log}(x,t) = t^p + \mu(x) t^q \log(e+t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty),$$

which combines polynomial growth with a logarithmic perturbation. Here one usually assumes $1 < p < N$, $p < q$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. Several related models had already been studied in earlier works. In particular, Baroni–Colombo–Mingione [7] considered the borderline case $p = q$ through the functional

$$\omega \mapsto \int_\Omega \Big[|\nabla \omega|^p + \mu(x) |\nabla \omega|^p \log(e + |\nabla \omega|) \Big] \mathrm{d}x,$$

under the assumption that $0 \leq \mu(\cdot) \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, and $1 < p < \infty$. These energies are closely connected to models with almost linear growth such as

$$\omega \mapsto \int_\Omega |\nabla \omega| \log(1 + |\nabla \omega|) \, \mathrm{d}x,$$

which appear in plasticity theory with logarithmic hardening. This connection is discussed for instance in Fuchs–Mingione [25], Marcellini–Papi [37], Seregin–Frehse [44] and Fuchs–Seregin [26]. Another closely related example is given by

$$\omega \mapsto \int_\Omega (1 + |\nabla \omega|^2)^{\frac{p}{2}} \log(1 + |\nabla \omega|) \, \mathrm{d}x,$$

which was already investigated by Marcellini [35].

In this paper we are interested in the solvability and uniqueness of elliptic systems with unbalanced growth and logarithmic perturbations of the form

$$\begin{aligned} -\operatorname{div} \Upsilon_1(u_1) &= \Psi_1(x, u_1, u_2, \nabla u_1, \nabla u_2) && \text{in } \Omega, \\ -\operatorname{div} \Upsilon_2(u_2) &= \Psi_2(x, u_1, u_2, \nabla u_1, \nabla u_2) && \text{in } \Omega, \\ u_1 = u_2 &= 0 && \text{on } \partial\Omega, \end{aligned} \qquad (1.2)$$

where

$$\begin{aligned}\Upsilon_i(u_i) &= |\nabla u_i|^{p_i-2}\nabla u_i \\ &\quad + \mu_i(x)\left[\log(e+|\nabla u_i|)+\frac{|\nabla u_i|}{q_i(e+|\nabla u_i|)}\right]|\nabla u_i|^{q_i-2}\nabla u_i\end{aligned} \tag{1.3}$$

for $i=1,2$, and $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. We impose the following assumptions:

(H_1) For $i=1,2$, let $1<p_i<N$, $p_i<q_i<p_i^*:=\frac{Np_i}{N-p_i}$, $0\leq\mu_i(\cdot)\in L^\infty(\Omega)$, and let $\kappa=\frac{e}{e+t_0}$ with $t_0>0$ satisfying $t_0=e\log(e+t_0)$, where e denotes again Euler's number.

(H_2) The functions $\Psi_1,\Psi_2\colon\Omega\times\mathbb{R}^2\times(\mathbb{R}^N)^2\to\mathbb{R}$ are of Carathéodory type with $\Psi_1(\cdot,0,0,0,0)\neq 0$ or $\Psi_2(\cdot,0,0,0,0)\neq 0$ on a subset $\hat\Omega\subset\Omega$ with $|\hat\Omega|>0$, where $|\hat\Omega|$ denotes the Lebesgue measure of $\hat\Omega$, such that the following hold:

(i) There exist $\omega_i\in L^{\frac{\gamma_i}{\gamma_i-1}}(\Omega)$ $(i=1,2)$ such that, for a.a. $x\in\Omega$, for all $s=(s_1,s_2)\in\mathbb{R}^2$ and $\eta=(\eta_1,\eta_2)\in(\mathbb{R}^N)^2$,

$$|\Psi_1(x,s,\eta)|\leq\kappa_1|s_1|^{\alpha_1}+\kappa_2|s_2|^{\alpha_2}+\kappa_3|\eta_1|^{\alpha_3}+\kappa_4|\eta_2|^{\alpha_4}+|\omega_1(x)|,$$

$$|\Psi_2(x,s,\eta)|\leq\nu_1|s_1|^{\beta_1}+\nu_2|s_2|^{\beta_2}+\nu_3|\eta_1|^{\beta_3}+\nu_4|\eta_2|^{\beta_4}+|\omega_2(x)|.$$

Here, $\kappa_i,\nu_i\geq 0$, $i=1,\dots,4$, are constants, and $1<\gamma_i<p_i^*$ for $i=1,2$, where the exponents α_ℓ,β_ℓ for $\ell=1,\dots,4$, are supposed to be nonnegative and satisfy the following assumptions:

$$\begin{array}{llll} (\mathrm{A}_1) & \alpha_1\leq\gamma_1-1, & (\mathrm{A}_2) & \alpha_2\leq\dfrac{\gamma_1-1}{\gamma_1}\gamma_2, \\ (\mathrm{A}_3) & \alpha_3\leq\dfrac{\gamma_1-1}{\gamma_1}p_1, & (\mathrm{A}_4) & \alpha_4\leq\dfrac{\gamma_1-1}{\gamma_1}p_2, \\ (\mathrm{A}_5) & \beta_1\leq\dfrac{\gamma_2-1}{\gamma_2}\gamma_1, & (\mathrm{A}_6) & \beta_2\leq\gamma_2-1, \\ (\mathrm{A}_7) & \beta_3\leq\dfrac{\gamma_2-1}{\gamma_2}p_1, & (\mathrm{A}_8) & \beta_4\leq\dfrac{\gamma_2-1}{\gamma_2}p_2. \end{array}$$

(ii) There exist constants $\chi_1,\chi_2\geq 0$ and $\phi\in L^1(\Omega)$ such that

$$\begin{aligned}&\Psi_1(x,s,\eta)s_1+\Psi_2(x,s,\eta)s_2\\ &\leq\chi_1\left(|\eta_1|^{p_1}+|\eta_2|^{p_2}\right)+\chi_2\left(|s_1|^{p_1}+|s_2|^{p_2}\right)+\phi(x),\end{aligned} \tag{1.4}$$

for a.a. $x\in\Omega$, for all $s=(s_1,s_2)\in\mathbb{R}^2$ and $\eta=(\eta_1,\eta_2)\in(\mathbb{R}^N)^2$ and with

$$\chi_1+\chi_2\max\left\{\lambda_{1,p_1}^{-1},\lambda_{1,p_2}^{-1}\right\}<1, \tag{1.5}$$

where λ_{1,p_i} is the first eigenvalue of the p_i-Laplacian, see (2.1) below.

Under hypothesis (H_1), let

$$\mathcal{H}^i_{\log}(x,t)=t^{p_i}+\mu_i(x)t^{q_i}\log(e+t)\quad\text{for all }(x,t)\in\overline{\Omega}\times[0,\infty)$$

and set $\mathcal{W}:=W_0^{1,\mathcal{H}^1_{\log}}(\Omega)\times W_0^{1,\mathcal{H}^2_{\log}}(\Omega)$ (details can be found in Section 2). A function $(u_1,u_2)\in\mathcal{W}$ is said to be a weak solution of the system (1.2) if

$$\int_\Omega\Upsilon_i(u_i)\cdot\nabla\varphi_i\,\mathrm{d}x=\int_\Omega\Psi_i(x,u_1,u_2,\nabla u_1,\nabla u_2)\varphi_i\,\mathrm{d}x,\quad i=1,2,$$

are fulfilled for all $(\varphi_1, \varphi_2) \in \mathcal{W}$, where Υ_i is defined as in (1.3), i.e.

$$\Upsilon_i(u_i) = |\nabla u_i|^{p_i-2}\nabla u_i + \mu_i(x)\left[\log(e+|\nabla u_i|) + \frac{|\nabla u_i|}{q_i(e+|\nabla u_i|)}\right]|\nabla u_i|^{q_i-2}\nabla u_i,$$

for $i = 1, 2$. Note that this definition is well-defined under the hypotheses (H$_1$) and (H$_2$) (i).

The first result of this paper reads as follows.

Theorem 1.1. *Let assumptions* (H$_1$) *and* (H$_2$) *be satisfied. Then there exists a nontrivial weak solution* $(u_1, u_2) \in \mathcal{W}$ *of the system* (1.2).

Remark 1.2. *The growth condition in* (H$_2$) (i) *for* Ψ_1 *can be extended to include mixed terms such as* $|s_1|^{\tilde{\alpha}_1}|s_2|^{\tilde{\alpha}_2}$ *and* $|\eta_1|^{\tilde{\alpha}_3}|\eta_2|^{\tilde{\alpha}_4}$ *without affecting the analysis. Indeed, under suitable conditions on the exponents, these terms can be absorbed into the pure power terms by means of Young's inequality. More precisely, assume that*

$$\frac{\tilde{\alpha}_1}{\gamma_1} + \frac{\tilde{\alpha}_2}{\gamma_2} \leq \frac{\gamma_1 - 1}{\gamma_1} \tag{1.6}$$

with $\tilde{\alpha}_1, \tilde{\alpha}_2 > 0$. *The cases* $\tilde{\alpha}_1 = 0$ *and/or* $\tilde{\alpha}_2 = 0$ *are trivial. Dividing* (1.6) *by* $\frac{\gamma_1-1}{\gamma_1}$ *and taking*

$$\alpha_1 = \gamma_1 - 1 \qquad \text{and} \qquad \alpha_2 = \frac{\gamma_1 - 1}{\gamma_1}\gamma_2,$$

see (A$_1$) *and* (A$_2$), *condition* (1.6) *is equivalent to*

$$\frac{\tilde{\alpha}_1}{\alpha_1} + \frac{\tilde{\alpha}_2}{\alpha_2} \leq 1. \tag{1.7}$$

From this, since $\tilde{\alpha}_1, \tilde{\alpha}_2 > 0$, *we obtain*

$$\frac{\tilde{\alpha}_1}{\alpha_1} < 1 \quad \text{and} \quad \frac{\tilde{\alpha}_2}{\alpha_2} < 1.$$

In particular, $\tilde{\alpha}_1 < \alpha_1$, *and hence we can choose*

$$r := \frac{\alpha_1}{\tilde{\alpha}_1} > 1.$$

Let $r' = \frac{r}{r-1}$ *be the conjugate exponent of* r. *Then*

$$r' = \frac{\alpha_1}{\alpha_1 - \tilde{\alpha}_1}.$$

Moreover, from (1.7) *we obtain*

$$\frac{\tilde{\alpha}_2}{\alpha_2} \leq 1 - \frac{\tilde{\alpha}_1}{\alpha_1} = \frac{\alpha_1 - \tilde{\alpha}_1}{\alpha_1} = \frac{1}{r'}.$$

Thus,

$$\tilde{\alpha}_2 r' \leq \alpha_2.$$

Consequently, Young's inequality yields

$$\begin{aligned}
|s_1|^{\tilde{\alpha}_1}|s_2|^{\tilde{\alpha}_2} &\leq \frac{1}{r}|s_1|^{\tilde{\alpha}_1 r} + \frac{1}{r'}|s_2|^{\tilde{\alpha}_2 r'} \\
&\leq \frac{1}{r}|s_1|^{\alpha_1} + \frac{1}{r'}\left(1 + |s_2|^{\alpha_2}\right) \\
&\leq C\left(1 + |s_1|^{\alpha_1} + |s_2|^{\alpha_2}\right)
\end{aligned}$$

for some constant $C > 0$. An analogous argument applies to gradient terms $|\eta_1|^{\tilde{\alpha}_3}|\eta_2|^{\tilde{\alpha}_4}$ provided

$$\frac{\tilde{\alpha}_3}{p_1} + \frac{\tilde{\alpha}_4}{p_2} \leq \frac{\gamma_1 - 1}{\gamma_1}.$$

Similarly, mixed terms of the form $|s_1|^{\tilde{\beta}_1}|s_2|^{\tilde{\beta}_2}$ and $|\eta_1|^{\tilde{\beta}_3}|\eta_2|^{\tilde{\beta}_4}$ in the growth of Ψ_2 can be treated under the conditions

$$\frac{\tilde{\beta}_1}{\gamma_1} + \frac{\tilde{\beta}_2}{\gamma_2} \leq \frac{\gamma_2 - 1}{\gamma_2} \quad \text{and} \quad \frac{\tilde{\beta}_3}{p_1} + \frac{\tilde{\beta}_4}{p_2} \leq \frac{\gamma_2 - 1}{\gamma_2}.$$

Therefore, omitting mixed terms in (H_2) (i) does not reduce generality, since they can be incorporated into the existing framework by adjusting constants.

In order to obtain uniqueness of solutions to problem (1.2), we impose additional structural assumptions. For this purpose, let $\Psi\colon \Omega \times \mathbb{R}^2 \times (\mathbb{R}^N)^2 \to \mathbb{R}^2$ be an $\mathbb{R}^2$-valued mapping defined by

$$\Psi(x, s, \eta) = (\Psi_1(x, s, \eta), \Psi_2(x, s, \eta)) \tag{1.8}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}^2$ and $\eta \in (\mathbb{R}^N)^2$. We assume that the following condition is satisfied.

(H_3) There exist constants $\chi_3, \chi_4 \geq 0$ such that

$$\begin{aligned}&(\Psi(x, s, \eta) - \Psi(x, t, \zeta)) \cdot (s - t)\\ &\leq \chi_3 \sum_{i=1}^{2} |s_i - t_i|^{p_i} + \chi_4 \sum_{i=1}^{2} |\eta_i - \zeta_i|^{p_i}\end{aligned} \tag{1.9}$$

for a.a. $x \in \Omega$, for all $s = (s_1, s_2), t = (t_1, t_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \in (\mathbb{R}^N)^2$.

The second main result of this paper can now be stated as follows.

Theorem 1.3. *Let assumptions (H_1), (H_2), and (H_3) be satisfied with $p_i \geq 2$, $i = 1, 2$, and suppose that*

$$\chi_3 \lambda_{1,p_i}^{-1} + \chi_4 < \min\left\{2^{-1}, 2^{2-p_i}\right\}, \quad i = 1, 2. \tag{1.10}$$

Then problem (1.2) admits a unique weak solution in $\mathcal{W}$.

The proof of Theorem 1.1 relies on coercivity and pseudomonotonicity arguments based on the first eigenvalues of the p_i-Laplacians. The main difficulties in our approach arise from the combination of the logarithmic double phase operator with gradient dependence of the right-hand side, so-called convection terms, which prevents the use of variational methods, as well as from the fact that the problem is formulated for elliptic systems. Moreover, the proof of Theorem 1.3 exploits a specific structural property of the convection term. To the best of our knowledge, this is the first work dealing with systems involving the operator (1.1). For systems of double phase type without logarithmic perturbations, we refer to the works of Ahmed–Atigh-Elemine Vall [1], Arora–Dwivedi [4], Benslimane–Benboubker [10], Cen–Marano–Zeng [15], Feng–Bai [23], Frisch–Winkert [24], Georgiev–Kheloufi–Mebarki–Ragusa [27], Guarnotta–Livrea–Winkert [28], Marino–Winkert [38]. We also mention the recent work of Borer–Carl–Winkert [11] concerning systems of p-Laplace type.

Since the operator (1.1) has been introduced only very recently, the available literature on existence results for problems driven by logarithmic operators of this type is still quite limited. The earliest contributions are due to Arora–Crespo-Blanco–Winkert [3], who studied the Dirichlet problem

$$-\operatorname{div}\Upsilon(u)=f(x,u)\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,$$

where Υ is given by (1.1), now involving variable exponents, and f is a Carathéodory nonlinearity of subcritical growth. Under suitable assumptions and by employing the Nehari manifold approach, they established the existence of a sign-changing solution. However, their analysis requires the additional restriction $q+1<p^*$. In a subsequent work [2], the same authors extended their results by proving more general embedding theorems and deriving further existence results via the concentration-compactness principle. As far as we are aware, there are only a few contributions that explicitly focus on the operator (1.1), namely those by Borer–Gasiński–Stapenhorst–Winkert [12], Carranza–Pimenta–Vetro–Winkert [14], Rădulescu–Stapenhorst–Winkert [43], Vetro [45], and Vetro–Winkert [46], see also Borer–Pimenta–Winkert [13].

Furthermore, Lu–Vetro–Zeng [32] investigated existence and uniqueness issues for problems governed by the operator

$$u\mapsto\Delta_{\mathcal{H}_L}u=\operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right),\quad u\in W^{1,\mathcal{H}_L}(\Omega),\tag{1.11}$$

where the associated modular function is defined by

$$\mathcal{H}_L(x,t)=[t^{p(x)}+\mu(x)t^{q(x)}]\log(e+\alpha t)\quad\text{for }\alpha\geq 0,$$

and

$$\mathcal{H}_L'(x,t)=\lim_{h\to 0}\frac{\mathcal{H}_L(x,t+h)-\mathcal{H}_L(x,t)}{h}.$$

Additional contributions in this direction can be found in the works of Zeng–Lu–Rădulescu [48] and Vetro–Zeng [47]. It is important to note that the operator in (1.11) differs structurally from the logarithmic double phase operator introduced in (1.1).

As a result of independent interest, we also study the boundedness of weak solutions to the system (1.2). To this end, we impose the following assumptions.

(H_4) The functions $\Psi_1,\Psi_2\colon\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ are of Carathéodory type and there exist constants $\hat\kappa_i,\hat\nu_i\geq 0$, $i=1,\dots,5$, and nonnegative exponents $\hat\alpha_\ell,\hat\beta_\ell$ for $\ell=1,\dots,4$, with

$$|\Psi_1(x,s,\eta)|\leq\hat\kappa_1|s_1|^{\hat\alpha_1}+\hat\kappa_2|s_2|^{\hat\alpha_2}+\hat\kappa_3|\eta_1|^{\hat\alpha_3}+\hat\kappa_4|\eta_2|^{\hat\alpha_4}+\hat\kappa_5,$$
$$|\Psi_2(x,s,\eta)|\leq\hat\nu_1|s_1|^{\hat\beta_1}+\hat\nu_2|s_2|^{\hat\beta_2}+\hat\nu_3|\eta_1|^{\hat\beta_3}+\hat\nu_4|\eta_2|^{\hat\beta_4}+\hat\nu_5,$$

for a.a. $x \in \Omega$, for all $s = (s_1, s_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2) \in (\mathbb{R}^N)^2$ such that the conditions below are satisfied:

$$
\begin{array}{llll}
(\mathrm{B}_1) & \hat{\alpha}_1 \leq p_1^* - 1, & (\mathrm{B}_2) & \hat{\alpha}_2 < \dfrac{p_1^* - p_1}{p_1^*} p_2^*, \\[2ex]
(\mathrm{B}_3) & \hat{\alpha}_3 \leq p_1 - 1, & (\mathrm{B}_4) & \hat{\alpha}_4 < \dfrac{p_1^* - p_1}{p_1^*} p_2, \\[2ex]
(\mathrm{B}_5) & \hat{\beta}_1 < \dfrac{p_2^* - p_2}{p_2^*} p_1^*, & (\mathrm{B}_6) & \hat{\beta}_2 \leq p_2^* - 1, \\[2ex]
(\mathrm{B}_7) & \hat{\beta}_3 < \dfrac{p_2^* - p_2}{p_2^*} p_1, & (\mathrm{B}_8) & \hat{\beta}_4 \leq p_2 - 1.
\end{array}
$$

Remark 1.4. *Assumption* (H_4) *strengthens the growth conditions imposed in* (H_2) (i)*. While* (H_2) (i) *along with* (ii) *is sufficient to guarantee existence of weak solutions, the sharper subcritical bounds required in* (H_4) *are essential in order to derive* L^∞ *estimates.*

We are now in a position to state the corresponding boundedness result.

Theorem 1.5. *Let assumptions* (H_1) *and* (H_4) *be satisfied. Then any weak solution* $(u_1, u_2) \in \mathcal{W}$ *of the system* (1.2) *belongs to* $L^\infty(\Omega) \times L^\infty(\Omega)$.

Remark 1.6. *As in Remark 1.2, we can also allow mixed terms in the growth assumptions on* Ψ_1 *and* Ψ_2 *in* (H_4)*. More precisely, terms of the form* $|s_1|^{\breve{\alpha}_1}|s_2|^{\breve{\alpha}_2}$ *and* $|\eta_1|^{\breve{\alpha}_3}|\eta_2|^{\breve{\alpha}_4}$ *can be absorbed in the growth of* Ψ_1 *provided*

$$
\frac{\breve{\alpha}_1}{p_1^*} + \frac{\breve{\alpha}_2}{p_2^*} < \frac{p_1^* - p_1}{p_1^*} \quad \text{and} \quad \frac{\breve{\alpha}_3}{p_1} + \frac{\breve{\alpha}_4}{p_2} < \frac{p_1^* - p_1}{p_1^*}.
$$

Similarly, terms of the form $|s_1|^{\breve{\beta}_1}|s_2|^{\breve{\beta}_2}$ *and* $|\eta_1|^{\breve{\beta}_3}|\eta_2|^{\breve{\beta}_4}$ *can be absorbed in the growth of* Ψ_2 *provided*

$$
\frac{\breve{\beta}_1}{p_1^*} + \frac{\breve{\beta}_2}{p_2^*} < \frac{p_2^* - p_2}{p_2^*} \quad \text{and} \quad \frac{\breve{\beta}_3}{p_1} + \frac{\breve{\beta}_4}{p_2} < \frac{p_2^* - p_2}{p_2^*}.
$$

The proof of Theorem 1.5 is based on a Moser iteration scheme adapted from the works of Marino–Winkert [39, 40].

The paper is organized as follows. Section 2 is devoted to the main properties of logarithmic Musielak-Orlicz Sobolev spaces and the associated logarithmic double phase operator, and includes a brief discussion of the eigenvalue problems for the p_i-Laplacians for $i = 1, 2$, which are used later. Section 3 contains the proofs of Theorems 1.1 and 1.3, while Section 4 is dedicated to the proof of Theorem 1.5.

2. Preliminaries

In this section, we collect several basic properties of logarithmic Musielak-Orlicz Sobolev spaces and of the associated logarithmic double phase operator introduced in (1.1). The results presented in this section are mainly inspired by the works of Arora–Crespo-Blanco–Winkert [2, 3].

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we denote by $\mathcal{M}(\Omega)$ the space of all Lebesgue measurable functions $u \colon \Omega \to \mathbb{R}$, where functions that coincide almost everywhere are identified. For $1 \leq r < \infty$, we denote by $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm

$$\|u\|_r = \left(\int_\Omega |u|^r \,\mathrm{d}x\right)^{\frac{1}{r}}$$

for $u \in L^r(\Omega)$ or $u \in L^r(\Omega;\mathbb{R}^N)$, respectively, $|\cdot|$ denoting the Euclidean norm. The space $L^\infty(\Omega)$ is equipped with the norm

$$\|u\|_\infty = \inf\{c > 0\colon |u(x)| \leq c \text{ for a.a.}\, x \in \Omega\}$$

for $u \in L^\infty(\Omega)$. Furthermore, the associated Sobolev spaces $W^{1,r}(\Omega)$ and $W^{1,r}_0(\Omega)$ are both endowed with the norm

$$\|v\|_{1,r} = (\|v\|_r^r + \|\nabla v\|_r^r)^{\frac{1}{r}}$$

for $v \in W^{1,r}(\Omega)$ or $v \in W^{1,r}_0(\Omega)$, respectively. On the space $W^{1,r}_0(\Omega)$, we can introduce the equivalent norm $\|\cdot\|_{1,r,0} := \|\nabla\cdot\|_r$.

For $t \in \mathbb{R}$, we define $t^\pm = \max\{\pm t, 0\}$. For any function $u \in \mathcal{M}(\Omega)$, we set $u^\pm(x) = [u(x)]^\pm$ for a.a. $x \in \Omega$. Then the decompositions $u = u^+ - u^-$ and $|u| = u^+ + u^-$ hold a.e. in Ω. Also, for $s > 1$, we write $s' = \frac{s}{s-1}$ for its conjugate exponent and $|\Omega|$ stands for the Lebesgue measure of Ω.

Suppose that hypothesis (H_1) holds and consider the nonlinear mappings

$$\mathcal{H}^i_{\log}\colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$$

defined by

$$\mathcal{H}^i_{\log}(x,t) = t^{p_i} + \mu_i(x) t^{q_i} \log(e+t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty)$$

for $i = 1,2$. The associated Musielak-Orlicz space is given by

$$L^{\mathcal{H}^i_{\log}}(\Omega) = \left\{u \in \mathcal{M}(\Omega)\colon \rho_{\mathcal{H}^i_{\log}}(u) := \int_\Omega \mathcal{H}^i_{\log}(x,|u|)\,\mathrm{d}x < \infty\right\},$$

where $\rho_{\mathcal{H}^i_{\log}}$ denotes the modular corresponding to $\mathcal{H}^i_{\log}$ and we equip $L^{\mathcal{H}^i_{\log}}(\Omega)$ with the Luxemburg norm

$$\|u\|_{\mathcal{H}^i_{\log}} := \inf\left\{\lambda > 0\colon \rho_{\mathcal{H}^i_{\log}}\left(\frac{u}{\lambda}\right) \leq 1\right\} \quad \text{for } u \in L^{\mathcal{H}^i_{\log}}(\Omega),$$

for $i = 1,2$. The corresponding Musielak-Orlicz Sobolev spaces are defined by

$$W^{1,\mathcal{H}^i_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}^i_{\log}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}^i_{\log}}(\Omega)\right\},$$
$$W^{1,\mathcal{H}^i_{\log}}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{1,\mathcal{H}^i_{\log}}},$$

both endowed with the norm

$$\|\cdot\|_{1,\mathcal{H}^i_{\log}} := \|\cdot\|_{\mathcal{H}^i_{\log}} + \|\nabla\cdot\|_{\mathcal{H}^i_{\log}}.$$

Note that $L^{\mathcal{H}^i_{\log}}(\Omega)$, $W^{1,\mathcal{H}^i_{\log}}_0(\Omega)$, and $W^{1,\mathcal{H}^i_{\log}}(\Omega)$ are separable and reflexive Banach spaces, see [3, Propositions 3.4 and 3.6]. Moreover, on $W^{1,\mathcal{H}^i_{\log}}_0(\Omega)$ we may introduce the equivalent norm

$$\|u\|_{1,\mathcal{H}^i_{\log},0} = \|\nabla u\|_{\mathcal{H}^i_{\log}}$$

for $u \in W_0^{1,\mathcal{H}_{\log}^i}(\Omega)$, see [3, Proposition 3.9]. The next proposition describes the precise relationship between the norm $\|\cdot\|_{1,\mathcal{H}_{\log}^i,0}$ on the space $W_0^{1,\mathcal{H}_{\log}^i}(\Omega)$ and the associated modular function $\rho_{\mathcal{H}_{\log}^i}$. We refer to Arora–Crespo-Blanco–Winkert [3, Proposition 3.6] for details.

Proposition 2.1. *Let hypothesis* (H$_1$) *be satisfied, $\lambda > 0$, and $u, u_n \in W_0^{1,\mathcal{H}_{\log}^i}(\Omega)$. Then the following assertions hold for $i = 1, 2$:*

(i) $\|u\|_{1,\mathcal{H}_{\log}^i,0} = \lambda$ *if and only if* $\rho_{\mathcal{H}_{\log}^i}\left(\frac{\nabla u}{\lambda}\right) = 1$;
(ii) $\|u\|_{1,\mathcal{H}_{\log}^i,0} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_{\mathcal{H}_{\log}^i}(\nabla u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) *if* $\|u\|_{1,\mathcal{H}_{\log}^i,0} < 1$*, then* $\|u\|_{1,\mathcal{H}_{\log}^i,0}^{q_i+\kappa} \leq \rho_{\mathcal{H}_{\log}^i}(\nabla u) \leq \|u\|_{1,\mathcal{H}_{\log}^i,0}^{p_i}$;
(iv) *if* $\|u\|_{1,\mathcal{H}_{\log}^i,0} > 1$*, then* $\|u\|_{1,\mathcal{H}_{\log}^i,0}^{p_i} \leq \rho_{\mathcal{H}_{\log}^i}(\nabla u) \leq \|u\|_{1,\mathcal{H}_{\log}^i,0}^{q_i+\kappa}$;
(v) $\|u_n\|_{1,\mathcal{H}_{\log}^i,0} \to 0$ *if and only if* $\rho_{\mathcal{H}_{\log}^i}(\nabla u_n) \to 0$ *as* $n \to \infty$.

We also recall the following embedding results, which were established in Arora–Crespo-Blanco–Winkert [3, Proposition 3.7].

Proposition 2.2. *Let hypothesis* (H$_1$) *be satisfied. Then the following embedding properties hold for $i = 1, 2$:*

(i) $W_0^{1,\mathcal{H}_{\log}^i}(\Omega) \hookrightarrow W_0^{1,p_i}(\Omega)$ *is continuous;*
(ii) $W_0^{1,\mathcal{H}_{\log}^i}(\Omega) \hookrightarrow L^{p_i^*}(\Omega)$ *is continuous;*
(iii) $W_0^{1,\mathcal{H}_{\log}^i}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact for all* $1 \leq r < p_i^*$.

For $1 < r < \infty$, we consider the eigenvalue problem associated with the r-Laplacian Δ_r, defined by $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, subject to homogeneous Dirichlet boundary conditions given by

$$-\Delta_r u = \lambda |u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

According to Lê [30], the first eigenvalue $\lambda_{1,r}$ of problem (2.1), that is the smallest eigenvalue, is positive, isolated, and simple. Moreover, it admits the variational characterization

$$\lambda_{1,r} = \inf_{u \in W_0^{1,r}(\Omega)} \left\{ \int_\Omega |\nabla u|^r \,\mathrm{d}x \colon \int_\Omega |u|^r \,\mathrm{d}x = 1 \right\}.$$

As a direct consequence, the following inequality holds

$$\|u\|_r^r \leq \lambda_{1,r}^{-1} \|\nabla u\|_r^r \quad \text{for all } u \in W_0^{1,r}(\Omega). \tag{2.2}$$

We now recall several notions that will be used throughout the sequel.

Definition 2.3. *Let X be a reflexive Banach space and let $A \colon X \to X^*$ be an operator. We say that A*

(i) *is bounded if it maps bounded sets of X into bounded sets of X^*;*
(ii) *is pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$ imply $\liminf_{n\to\infty} \langle Au_n, u_n - w\rangle \geq \langle Au, u - w\rangle$ for all $w \in X$;*
(iii) *satisfies the* (S$_+$)*-property if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$ imply $u_n \to u$ in X;*
(iv) *is coercive if*

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty.$$

Remark 2.4. *For bounded operators, the notion of pseudomonotonicity given in Definition 2.3* (ii) *is equivalent to the following characterization, see Proposition 2.3 in Frisch–Winkert [24]: If $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$, then $Au_n \rightharpoonup Au$ in X^* and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$.*

The following surjectivity result for pseudomonotone operators plays a key role in our approach in Section 3. Its proof can be found in Papageorgiou–Winkert [41, Theorem 6.1.57].

Theorem 2.5. *Let X be a real reflexive Banach space, let $A\colon X \to X^*$ be a bounded, coercive, and pseudomonotone operator, and let $b \in X^*$. Then the equation $Au = b$ admits at least one solution $u \in X$.*

Finally, we set $\mathcal{W} := W_0^{1,\mathcal{H}^1_{\log}}(\Omega) \times W_0^{1,\mathcal{H}^2_{\log}}(\Omega)$ and equip $\mathcal{W}$ with the norm

$$\|(u_1,u_2)\|_{\mathcal{W}} := \|u_1\|_{1,\mathcal{H}^1_{\log},0} + \|u_2\|_{1,\mathcal{H}^2_{\log},0}$$

for $(u_1,u_2) \in \mathcal{W}$. Then $\mathcal{W}$ is a reflexive Banach space. Next, we define the operator $\mathcal{A}\colon \mathcal{W} \to \mathcal{W}^*$ by

$$\langle \mathcal{A}(u_1,u_2),(\varphi_1,\varphi_2)\rangle_{\mathcal{W}} = \int_\Omega \Upsilon_1(u_1)\cdot\nabla\varphi_1\,\mathrm{d}x + \int_\Omega \Upsilon_2(u_2)\cdot\nabla\varphi_2\,\mathrm{d}x \tag{2.3}$$

for $(u_1,u_2),(\varphi_1,\varphi_2) \in \mathcal{W}$. Here, $\langle\cdot\,,\cdot\rangle_{\mathcal{W}}$ denotes the duality pairing between $\mathcal{W}$ and its dual space $\mathcal{W}^*$.

The fundamental properties of the operator $\mathcal{A}$ are collected in the following result. Its proof follows along the same lines as the arguments used in Theorem 4.4 of Arora–Crespo-Blanco–Winkert [3], with only minor modifications.

Theorem 2.6. *Let hypothesis* (H_1) *be satisfied. Then the operator $\mathcal{A}\colon \mathcal{W} \to \mathcal{W}^*$ is bounded, continuous, monotone, and fulfills the* (S_+)*-property.*

3. Existence and uniqueness of weak solutions

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Let $\Psi := (\Psi_1,\Psi_2)$, $\gamma_i \in (1,p_i^*)$ be as in hypothesis (H_2) (i) and γ_i' denoting the conjugate exponent of γ_i for $i = 1,2$. We consider the associated Nemytskij operator

$$N_\Psi\colon W_0^{1,\mathcal{H}^1_{\log}}(\Omega) \times W_0^{1,\mathcal{H}^2_{\log}}(\Omega) \subset L^{\gamma_1}(\Omega)\times L^{\gamma_2}(\Omega) \to L^{\gamma_1'}(\Omega)\times L^{\gamma_2'}(\Omega)$$

defined by

$$N_\Psi(u_1,u_2) = (\Psi_1(\,\cdot\,,u_1,u_2,\nabla u_1,\nabla u_2),\Psi_2(\,\cdot\,,u_1,u_2,\nabla u_1,\nabla u_2)).$$

Since $\gamma_i \in (1,p_i^*)$ for $i = 1,2$, Proposition 2.2 (iii) yields the compact embedding

$$j\colon W_0^{1,\mathcal{H}^1_{\log}}(\Omega) \times W_0^{1,\mathcal{H}^2_{\log}}(\Omega) \to L^{\gamma_1}(\Omega)\times L^{\gamma_2}(\Omega).$$

We denote by

$$j^*\colon L^{\gamma_1'}(\Omega)\times L^{\gamma_2'}(\Omega) \to \left(W_0^{1,\mathcal{H}^1_{\log}}(\Omega)\right)^* \times \left(W_0^{1,\mathcal{H}^2_{\log}}(\Omega)\right)^*$$

the adjoint operator of j. Observing that

$$\mathcal{W}^* = \left(W_0^{1,\mathcal{H}^1_{\log}}(\Omega) \times W_0^{1,\mathcal{H}^2_{\log}}(\Omega)\right)^* \cong \left(W_0^{1,\mathcal{H}^1_{\log}}(\Omega)\right)^* \times \left(W_0^{1,\mathcal{H}^2_{\log}}(\Omega)\right)^*,$$

we define the operator $\mathcal{N}_\Psi\colon \mathcal{W} \to \mathcal{W}^*$ by $\mathcal{N}_\Psi = j^* \circ N_\Psi$. Then, for all (u_1,u_2), $(\varphi_1,\varphi_2) \in \mathcal{W}$,

$$\begin{aligned}&\langle \mathcal{N}_\Psi(u_1,u_2),(\varphi_1,\varphi_2)\rangle_{\mathcal{W}}\\&= \int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2)\varphi_1 \,\mathrm{d}x + \int_\Omega \Psi_2(x,u_1,u_2,\nabla u_1,\nabla u_2)\varphi_2 \,\mathrm{d}x.\end{aligned}$$

Note that $\mathcal{N}_\Psi$ is continuous. We now define the operator $\mathcal{B}\colon \mathcal{W} \to \mathcal{W}^*$ by

$$\mathcal{B} := \mathcal{A} - \mathcal{N}_\Psi,$$

where the operator $\mathcal{A}$ was introduced in (2.3). By construction, a function $(u_1,u_2) \in \mathcal{W}$ is a weak solution of (1.2) if and only if $\mathcal{B}(u_1,u_2)=0$.

We apply Theorem 2.5 to the operator $\mathcal{B}$ in order to obtain a solution of the equation $\mathcal{B}(u_1,u_2)=0$. The boundedness of $\mathcal{B}$ follows directly from the boundedness of $\mathcal{A}$, see Theorem 2.6, together with the growth conditions imposed on Ψ_1 and Ψ_2 in hypothesis (H_2) (i).

Next, we verify that $\mathcal{B}$ is pseudomonotone in the sense of Remark 2.4. Let $\{(u_{1,n},u_{2,n})\}_{n\in\mathbb{N}} \subset \mathcal{W}$ be such that

$$(u_{1,n},u_{2,n}) \rightharpoonup (u_1,u_2) \quad \text{in } \mathcal{W},$$

and

$$\limsup_{n\to\infty} \langle \mathcal{B}(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle_{\mathcal{W}} \le 0.$$

Then, componentwise convergence yields

$$u_{1,n} \rightharpoonup u_1 \quad \text{in } W_0^{1,\mathcal{H}^1_{\log}}(\Omega) \quad \text{and} \quad u_{2,n} \rightharpoonup u_2 \quad \text{in } W_0^{1,\mathcal{H}^2_{\log}}(\Omega).$$

Since $\gamma_i \in (1,p_i^*)$ for $i=1,2$, Proposition 2.2 (iii) implies that the embeddings

$$W_0^{1,\mathcal{H}^i_{\log}}(\Omega) \to L^{\gamma_i}(\Omega)$$

are compact for $i \in \{1,2\}$. Consequently, we may assume that

$$u_{1,n} \to u_1 \quad \text{in } L^{\gamma_1}(\Omega) \quad \text{and} \quad u_{2,n} \to u_2 \quad \text{in } L^{\gamma_2}(\Omega). \tag{3.1}$$

Next, we show that

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega \Psi_1(x,u_{1,n},u_{2,n},\nabla u_{1,n},\nabla u_{2,n})(u_{1,n}-u_1)\,\mathrm{d}x = 0,\\&\lim_{n\to\infty} \int_\Omega \Psi_2(x,u_{1,n},u_{2,n},\nabla u_{1,n},\nabla u_{2,n})(u_{2,n}-u_2)\,\mathrm{d}x = 0.\end{aligned} \tag{3.2}$$

We prove the first limit in (3.2). Using the growth assumptions imposed in hypothesis (H_2) (i), we obtain

$$\begin{aligned}&\left| \int_\Omega \Psi_1(x,u_{1,n},u_{2,n},\nabla u_{1,n},\nabla u_{2,n})(u_{1,n}-u_1)\,\mathrm{d}x \right|\\&\le \int_\Omega \Big(\kappa_1|u_{1,n}|^{\alpha_1} + \kappa_2|u_{2,n}|^{\alpha_2} + \kappa_3|\nabla u_{1,n}|^{\alpha_3}\\&\qquad + \kappa_4|\nabla u_{2,n}|^{\alpha_4} + \omega_1(x)\Big)|u_{1,n}-u_1|\,\mathrm{d}x.\end{aligned} \tag{3.3}$$

We estimate each term separately. By Hölder's inequality together with conditions (A_1) and (A_2) in (H_2) (i) and the strong convergence (3.1), we have

$$\begin{aligned}\kappa_1 \int_\Omega |u_{1,n}|^{\alpha_1}|u_{1,n}-u_1|\,\mathrm{d}x &\leq \kappa_1 \left(\int_\Omega |u_{1,n}|^{\alpha_1\gamma_1'}\,\mathrm{d}x\right)^{\frac{1}{\gamma_1'}} \left(\int_\Omega |u_{1,n}-u_1|^{\gamma_1}\,\mathrm{d}x\right)^{\frac{1}{\gamma_1}}\\ &\leq \kappa_1 C_1\left(1+\|u_{1,n}\|_{\gamma_1}^{\gamma_1-1}\right)\|u_{1,n}-u_1\|_{\gamma_1} \to 0\end{aligned}$$

and

$$\begin{aligned}\kappa_2 \int_\Omega |u_{2,n}|^{\alpha_2}|u_{1,n}-u_1|\,\mathrm{d}x &\leq \kappa_2 \left(\int_\Omega |u_{2,n}|^{\alpha_2\gamma_1'}\,\mathrm{d}x\right)^{\frac{1}{\gamma_1'}} \left(\int_\Omega |u_{1,n}-u_1|^{\gamma_1}\,\mathrm{d}x\right)^{\frac{1}{\gamma_1}}\\ &\leq \kappa_2 C_2\left(1+\|u_{2,n}\|_{\gamma_2}^{\frac{\gamma_2}{\gamma_1'}}\right)\|u_{1,n}-u_1\|_{\gamma_1} \to 0\end{aligned}$$

as $n \to \infty$, for some constants $C_1, C_2 > 0$. For the gradient terms, we recall that $W_0^{1,\mathcal{H}_{\log}^i}(\Omega)$ embeds continuously into the space $W_0^{1,p_i}(\Omega)$ for $i = 1,2$, see Proposition 2.2 (i). Hence, using conditions (A_3) and (A_4), Hölder's inequality and the strong convergence (3.1), we conclude that

$$\begin{aligned}\kappa_3 \int_\Omega |\nabla u_{1,n}|^{\alpha_3}|u_{1,n}-u_1|\,\mathrm{d}x &\leq \kappa_3 \left(\int_\Omega |\nabla u_{1,n}|^{\alpha_3\gamma_1'}\right)^{\frac{1}{\gamma_1'}} \left(\int_\Omega |u_{1,n}-u_1|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}\\ &\leq \kappa_3 C_3\left(1+\|\nabla u_{1,n}\|_{p_1}^{\frac{p_1}{\gamma_1'}}\right)\|u_{1,n}-u_1\|_{\gamma_1} \to 0,\\ \kappa_4 \int_\Omega |\nabla u_{2,n}|^{\alpha_4}|u_{1,n}-u_1|\,\mathrm{d}x &\leq \kappa_4 \left(\int_\Omega |\nabla u_{2,n}|^{\alpha_4\gamma_1'}\right)^{\frac{1}{\gamma_1'}} \left(\int_\Omega |u_{1,n}-u_1|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}\\ &\leq \kappa_4 C_4\left(1+\|\nabla u_{2,n}\|_{p_2}^{\frac{p_2}{\gamma_1'}}\right)\|u_{1,n}-u_1\|_{\gamma_1} \to 0,\end{aligned}$$

as $n \to \infty$, for some constants $C_3, C_4 > 0$. Finally, for the last term in (3.3), we use again Hölder's inequality which leads to

$$\int_\Omega |\omega_1(x)||u_{1,n}-u_1|\,\mathrm{d}x \leq \|\omega_1\|_{\gamma_1'}\|u_{1,n}-u_1\|_{\gamma_1} \to 0 \quad \text{as } n\to\infty,$$

since $\omega_1 \in L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)$ by (H_2) (i). Combining all the calculations above, we arrive at

$$\lim_{n\to\infty}\int_\Omega \Psi_1(x,u_{1,n},u_{2,n},\nabla u_{1,n},\nabla u_{2,n})(u_{1,n}-u_1)\,\mathrm{d}x = 0.$$

The second limit in (3.2) can be shown analogously using conditions (A_5)–(A_8) in hypothesis (H_2) (i).

Therefore

$$\lim_{n\to\infty}\langle \mathcal{N}_\Psi(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle_{\mathcal{W}} = 0,$$

which implies

$$\begin{aligned}&\limsup_{n\to\infty}\langle \mathcal{A}(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle_{\mathcal{W}}\\ &= \limsup_{n\to\infty}\langle \mathcal{B}(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle_{\mathcal{W}} \leq 0.\end{aligned}$$

Since $\mathcal{A}$ fulfills the (S_+)-property, it follows that $(u_{1,n},u_{2,n}) \to (u_1,u_2)$ in $\mathcal{W}$. By the continuity of $\mathcal{B}$, we have $\mathcal{B}(u_{1,n},u_{2,n}) \to \mathcal{B}(u_1,u_2)$ in $\mathcal{W}^*$. In particular, this implies

$$\mathcal{B}(u_{1,n},u_{2,n}) \rightharpoonup \mathcal{B}(u_1,u_2) \quad \text{in } \mathcal{W}^*$$

and

$$\langle \mathcal{B}(u_{1,n},u_{2,n}),(u_{1,n},u_{2,n})\rangle_{\mathcal{W}} \to \langle \mathcal{B}(u_1,u_2),(u_1,u_2)\rangle_{\mathcal{W}} \quad \text{as } n \to \infty.$$

Hence, the operator $\mathcal{B}$ is pseudomonotone.

It remains to show that $\mathcal{B}\colon \mathcal{W} \to \mathcal{W}^*$ is coercive. Applying (2.2) with $r = p_1$ and $r = p_2$, respectively, we deduce that

$$\begin{aligned} \|u\|_{p_1}^{p_1} &\le \lambda_{1,p_1}^{-1}\|\nabla u\|_{p_1}^{p_1} \quad \text{for all } u \in W_0^{1,p_1}(\Omega),\\ \|u\|_{p_2}^{p_2} &\le \lambda_{1,p_2}^{-1}\|\nabla u\|_{p_2}^{p_2} \quad \text{for all } u \in W_0^{1,p_2}(\Omega). \end{aligned} \tag{3.4}$$

Note that $W_0^{1,\mathcal{H}^i_{\log}}(\Omega) \subset W_0^{1,p_i}(\Omega)$ for $i=1,2$ by Proposition 2.2 (i). Then, using assumption (1.4) in (H_2) (ii), and the estimates (3.4), we infer that for any $(u_1,u_2) \in \mathcal{W}$

$$\begin{aligned}
&\langle \mathcal{B}(u_1,u_2),(u_1,u_2)\rangle_{\mathcal{W}}\\
&= \int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1\cdot\nabla u_1\,\mathrm{d}x\\
&\quad + \int_\Omega \mu_1(x)\left[\log(e+|\nabla u_1|) + \frac{|\nabla u_1|}{q_1(e+|\nabla u_1|)}\right]|\nabla u_1|^{q_1-2}\nabla u_1\cdot\nabla u_1\,\mathrm{d}x\\
&\quad + \int_\Omega |\nabla u_2|^{p_2-2}\nabla u_2\cdot\nabla u_2\,\mathrm{d}x\\
&\quad + \int_\Omega \mu_2(x)\left[\log(e+|\nabla u_2|) + \frac{|\nabla u_2|}{q_2(e+|\nabla u_2|)}\right]|\nabla u_2|^{q_2-2}\nabla u_2\cdot\nabla u_2\,\mathrm{d}x\\
&\quad - \int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2)u_1\,\mathrm{d}x - \int_\Omega \Psi_2(x,u_1,u_2,\nabla u_1,\nabla u_2)u_2\,\mathrm{d}x\\
&\ge \int_\Omega \left(|\nabla u_1|^{p_1} + \mu_1(x)|\nabla u_1|^{q_1}\log(e+|\nabla u_1|)\right)\mathrm{d}x\\
&\quad + \int_\Omega \left(|\nabla u_2|^{p_2} + \mu_2(x)|\nabla u_2|^{q_2}\log(e+|\nabla u_2|)\right)\mathrm{d}x\\
&\quad - \int_\Omega \left(\chi_1\left(|\nabla u_1|^{p_1}+|\nabla u_2|^{p_2}\right) + \chi_2\left(|u_1|^{p_1}+|u_2|^{p_2}\right) + \phi(x)\right)\mathrm{d}x\\
&\ge (1-\chi_1-\chi_2\lambda_{1,p_1}^{-1})\|\nabla u_1\|_{p_1}^{p_1} + \int_\Omega \mu_1(x)|\nabla u_1|^{q_1}\log(e+|\nabla u_1|)\,\mathrm{d}x\\
&\quad + (1-\chi_1-\chi_2\lambda_{1,p_2}^{-1})\|\nabla u_2\|_{p_2}^{p_2} + \int_\Omega \mu_2(x)|\nabla u_2|^{q_2}\log(e+|\nabla u_2|)\,\mathrm{d}x - \|\phi\|_1,
\end{aligned}$$

where we have used that $\frac{|\nabla u_i|}{q_i(e+|\nabla u_i|)} \ge 0$ for $i=1,2$. Since $C_5 := 1-\chi_1-\chi_2\lambda_{1,p_1}^{-1} > 0$ and $C_6 := 1-\chi_1-\chi_2\lambda_{1,p_2}^{-1} > 0$ by (1.5), we obtain from the above considerations using Proposition 2.1 (iii), (iv)

$$\begin{aligned}
&\langle \mathcal{B}(u_1,u_2),(u_1,u_2)\rangle_{\mathcal{W}}\\
&\ge \min\{1,C_5\}\,\rho_{\mathcal{H}^1_{\log}}(\nabla u_1) + \min\{1,C_6\}\,\rho_{\mathcal{H}^2_{\log}}(\nabla u_2) - \|\phi\|_1
\end{aligned}$$

$$
\begin{aligned}
&\geq \min\{1, C_5\} \min\left\{\|u_1\|_{1,\mathcal{H}^1_{\log},0}^{p_1}, \|u_1\|_{1,\mathcal{H}^1_{\log},0}^{q_1+\kappa}\right\} \\
&\quad + \min\{1, C_6\} \min\left\{\|u_2\|_{1,\mathcal{H}^2_{\log},0}^{p_2}, \|u_2\|_{1,\mathcal{H}^2_{\log},0}^{q_2+\kappa}\right\} - \|\phi\|_1.
\end{aligned}
$$

Due to $1 < p_i < q_i + \kappa$ for $i = 1, 2$, the right-hand side grows faster than linearly with respect to the norm of $\mathcal{W}$, which implies

$$
\lim_{\|(u_1,u_2)\|_{\mathcal{W}} \to \infty} \frac{\langle \mathcal{B}(u_1, u_2), (u_1, u_2)\rangle_{\mathcal{W}}}{\|(u_1, u_2)\|_{\mathcal{W}}} = \infty.
$$

Therefore, $\mathcal{B}$ is coercive.

By Theorem 2.5, there exists $(u_1, u_2) \in \mathcal{W}$ such that $\mathcal{B}(u_1, u_2) = 0$. Moreover, since $\Psi_i(\cdot, 0, 0, 0, 0) \neq 0$ for $i = 1$ or $i = 2$ on $\hat{\Omega} \subset \Omega$ with $|\hat{\Omega}| > 0$, it follows by the fundamental lemma of the calculus of variations that $(u_1, u_2) \neq 0$. This completes the proof. □

Next, we prove Theorem 1.3, which asserts the uniqueness of weak solutions to problem (1.2).

Proof of Theorem 1.3. By Theorem 1.1, problem (1.2) admits at least one nontrivial weak solution. Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{W}$ be two weak solutions of (1.2). Testing the corresponding weak formulations with $\varphi = u - v \in \mathcal{W}$ and subtracting these equations gives

$$
\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega (\Upsilon_i(u_i) - \Upsilon_i(v_i)) \cdot \nabla(u_i - v_i)\,\mathrm{d}x \\
&= \int_\Omega \big(\Psi(x, u, \nabla u) - \Psi(x, v, \nabla v)\big) \cdot (u - v)\,\mathrm{d}x,
\end{aligned}
$$

where Υ_i and Ψ are defined in (1.3) and (1.8), respectively. Explicitly, this leads us to the relation

$$
\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \left(|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla v_i|^{p_i-2}\nabla v_i\right) \cdot \nabla(u_i - v_i)\,\mathrm{d}x \\
&+ \sum_{i=1}^{2} \int_\Omega \mu_i(x)\Bigg(\left[\log(e + |\nabla u_i|) + \frac{|\nabla u_i|}{q_i(e + |\nabla u_i|)}\right] |\nabla u_i|^{q_i-2}\,\nabla u_i \\
&\qquad - \left[\log(e + |\nabla v_i|) + \frac{|\nabla v_i|}{q_i(e + |\nabla v_i|)}\right] |\nabla v_i|^{q_i-2}\,\nabla v_i \Bigg) \cdot \nabla(u_i - v_i)\,\mathrm{d}x \\
&= \int_\Omega (\Psi(x, u, \nabla u) - \Psi(x, v, \nabla v)) \cdot (u - v)\,\mathrm{d}x.
\end{aligned}
$$

Using Lemma 4.2 in Arora–Crespo-Blanco–Winkert [3], we know that the two integrals in the second sum on the left-hand side are nonnegative. Moreover, since $p_i \geq 2$, we have for the first sum the standard monotonicity inequality

$$
\begin{aligned}
&\int_\Omega \left(|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla v_i|^{p_i-2}\nabla v_i\right) \cdot \nabla(u_i - v_i)\,\mathrm{d}x \\
&\geq \min\{2^{-1}, 2^{2-p_i}\}\|\nabla(u_i - v_i)\|_{p_i}^{p_i},
\end{aligned}
$$

which follows from the estimate

$$
(|\eta_i|^{p_i-2}\eta_i - |\zeta_i|^{p_i-2}\zeta_i) \cdot (\eta_i - \zeta_i) \geq \min\{2^{-1}, 2^{2-p_i}\}|\eta_i - \zeta_i|^{p_i}
$$

for all $\eta_i, \zeta_i \in \mathbb{R}^N$, to be found in Lemma 1.1 by Crespo-Blanco [20]. Therefore, we conclude that

$$\begin{aligned}\sum_{i=1}^{2} &\min\{2^{-1}, 2^{2-p_i}\}\|\nabla(u_i - v_i)\|_{p_i}^{p_i}\\ &\le \int_\Omega \big(\Psi(x,u,\nabla u) - \Psi(x,v,\nabla v)\big)\cdot(u-v)\,\mathrm{d}x.\end{aligned}$$

With our assumption (1.9) from (H_3), we can bound the right-hand side and obtain

$$\sum_{i=1}^{2} \min\{2^{-1}, 2^{2-p_i}\}\|\nabla(u_i - v_i)\|_{p_i}^{p_i} \le \chi_3 \sum_{i=1}^{2} \|u_i - v_i\|_{p_i}^{p_i} + \chi_4 \sum_{i=1}^{2} \|\nabla(u_i - v_i)\|_{p_i}^{p_i}.$$

Finally, using the inequality (2.2), this becomes

$$\sum_{i=1}^{2} \min\{2^{-1}, 2^{2-p_i}\}\|\nabla(u_i - v_i)\|_{p_i}^{p_i} \le \sum_{i=1}^{2} (\chi_3\lambda_{1,p_i}^{-1} + \chi_4)\|\nabla(u_i - v_i)\|_{p_i}^{p_i}.$$

Since we assumed in (1.10) that $\chi_3\lambda_{1,p_i}^{-1} + \chi_4 < \min\{2^{-1}, 2^{2-p_i}\}$, this is only possible for $\|\nabla(u_i - v_i)\|_{p_i} = 0$, that is, for $u = v$. □

Now, we give an example for the case $p_1 = p_2 = 2$.

Example 3.1. *Let $N \ge 3$ and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Assume that (H_1) holds with $p_1 = p_2 = 2$. Choose $\gamma_1 = \gamma_2 = 2$ and functions $\omega_i \in L^2(\Omega)$ with $\omega_i \not\equiv 0$ for $i = 1, 2$. Fix unit vectors $a, b \in \mathbb{R}^N$ and let $\varepsilon > 0$. We define*

$$\begin{aligned}\Psi_1(x,s,\eta) &:= \varepsilon\big((a\cdot\eta_1) + (b\cdot\eta_2) + s_2\big) + |\omega_1(x)|,\\ \Psi_2(x,s,\eta) &:= \varepsilon\big((a\cdot\eta_2) + (b\cdot\eta_1) + s_1\big) + |\omega_2(x)|,\end{aligned}$$

for a.a. $x \in \Omega$, for all $s = (s_1, s_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2) \in (\mathbb{R}^N)^2$. Then Ψ_1, Ψ_2 are Carathéodory functions satisfying

$$\Psi_1(\cdot, 0, 0, 0, 0) = |\omega_1(\cdot)| \not\equiv 0, \quad \Psi_2(\cdot, 0, 0, 0, 0) = |\omega_2(\cdot)| \not\equiv 0.$$

Using $|a\cdot\xi| \le |\xi|$ and $|b\cdot\xi| \le |\xi|$ for all $\xi \in \mathbb{R}^N$, we obtain

$$\begin{aligned}|\Psi_1(x,s,\eta)| &\le \varepsilon|\eta_1| + \varepsilon|\eta_2| + \varepsilon|s_2| + |\omega_1(x)|,\\ |\Psi_2(x,s,\eta)| &\le \varepsilon|\eta_2| + \varepsilon|\eta_1| + \varepsilon|s_1| + |\omega_2(x)|.\end{aligned}$$

Hence (H_2) (i) holds with $\kappa_2 = \kappa_3 = \kappa_4 = \varepsilon$, $\nu_1 = \nu_3 = \nu_4 = \varepsilon$, all other coefficients equal to zero, and with exponents $\alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_3 = \beta_4 = 1$, all remaining exponents being zero. For $\gamma_1 = \gamma_2 = 2$, all conditions (A_1)–(A_8) are satisfied.

Let $\delta > 0$. By Young's inequality,

$$\begin{aligned}\varepsilon|\eta_1||s_1| &\le \delta|\eta_1|^2 + \frac{\varepsilon^2}{4\delta}|s_1|^2,\\ \varepsilon|\eta_2||s_1| &\le \delta|\eta_2|^2 + \frac{\varepsilon^2}{4\delta}|s_1|^2,\\ \varepsilon|s_2||s_1| &\le \frac{\varepsilon}{2}\big(|s_1|^2 + |s_2|^2\big),\end{aligned}$$

and analogous estimates hold for the terms in Ψ_2. Moreover,

$$|\omega_1(x)||s_1| \le \delta|s_1|^2 + \frac{1}{4\delta}|\omega_1(x)|^2, \quad |\omega_2(x)||s_2| \le \delta|s_2|^2 + \frac{1}{4\delta}|\omega_2(x)|^2.$$

Summing all these estimates yields

$$\begin{aligned}&\Psi_1(x,s,\eta)s_1+\Psi_2(x,s,\eta)s_2\\&\quad\le \chi_1\big(|\eta_1|^2+|\eta_2|^2\big)+\chi_2\big(|s_1|^2+|s_2|^2\big)+\phi(x),\end{aligned}$$

where

$$\chi_1=2\delta,\quad \chi_2=\varepsilon+\frac{\varepsilon^2}{2\delta}+\delta,\quad \phi(x)=\frac{1}{4\delta}\big(|\omega_1(x)|^2+|\omega_2(x)|^2\big)$$

with $\phi\in L^1(\Omega)$. *Choosing first* $\delta>0$ *sufficiently small and then* $\varepsilon>0$ *ensures that* (1.5) *holds. To verify the condition* (1.9), *we write*

$$\begin{aligned}&(\Psi(x,s,\eta)-\Psi(x,t,\zeta))\cdot(s-t)\\&=\varepsilon\big(a\cdot(\eta_1-\zeta_1)+b\cdot(\eta_2-\zeta_2)+s_2-t_2\big)(s_1-t_1)\\&\quad+\varepsilon\big(a\cdot(\eta_2-\zeta_2)+b\cdot(\eta_1-\zeta_1)+s_1-t_1\big)(s_2-t_2).\end{aligned}$$

Using the Cauchy–Schwarz inequality to estimate the inner products and applying then Young's inequality gives us

$$\begin{aligned}(\Psi(x,s,\eta)-\Psi(x,t,\zeta))\cdot(s-t)&\le\frac{\varepsilon}{2}(|a|+|b|+2)(|s_1-t_1|^2+|s_2-t_2|^2)\\&\quad+\frac{\varepsilon}{2}(|a|+|b|)(|\eta_1-\zeta_1|^2+|\eta_2-\zeta_2|^2),\end{aligned}$$

which is (1.9) *with* $\chi_3=\frac{\varepsilon}{2}(|a|+|b|+2)=2\varepsilon$ *and* $\chi_4=\frac{\varepsilon}{2}(|a|+|b|)=\varepsilon$.

With these constants, condition (1.10) *is satisfied for* $\varepsilon>0$ *sufficiently small. Consequently, assumptions* (H_1)–(H_3) *and* (1.10) *are fulfilled. By Theorems 1.1 and 1.3, the system* (1.2) *admits a unique nontrivial weak solution in* $\mathcal{W}$.

4. Boundedness of weak solutions

In this section, we establish Theorem 1.5.

Proof of Theorem 1.5. Let $(u_1,u_2)\in\mathcal{W}$ be a weak solution of (1.2).

Claim: For every $r\in(1,\infty)$, we have $(u_1,u_2)\in L^r(\Omega)\times L^r(\Omega)$. We restrict the proof to the component u_1, since the argument for u_2 is completely analogous. Without loss of generality, we may assume that $u_1,u_2\ge 0$. Otherwise, the argument can be applied separately to the positive and negative parts u_i^+ and u_i^- for $i\in\{1,2\}$. Throughout the proof, C_i, $i\in\mathbb{N}$, denote positive constants that may depend on finite norms of u_1 and u_2, but are independent of L and θ introduced below. Also recall, by Proposition 2.2, that norms of the form $\|\nabla u_1\|_{p_1}$, $\|u_1\|_{r_1}$, $\|\nabla u_2\|_{p_2}$, and $\|u_2\|_{r_2}$ are finite for every $r_1\in[1,p_1^*]$ and $r_2\in[1,p_2^*]$. In the following, without loss of generality, we suppose that the coefficients and exponents in (H_4) are positive. If some are zero, the calculations become easier.

Taking $L>0$ and $\theta>0$, we define $u_{1,L}:=\min\{u_1,L\}$ and take $\varphi_1=u_1u_{1,L}^{\theta p_1}\in W_0^{1,\mathcal{H}^1_{\log}}(\Omega)$ as a test function in the weak formulation. Since

$$\nabla\varphi_1=u_{1,L}^{\theta p_1}\nabla u_1+\theta p_1u_1u_{1,L}^{\theta p_1-1}\nabla u_{1,L},$$

we obtain

$$\begin{aligned}&\int_\Omega|\nabla u_1|^{p_1-2}\nabla u_1\cdot\Big(u_{1,L}^{\theta p_1}\nabla u_1+\theta p_1u_1u_{1,L}^{\theta p_1-1}\nabla u_{1,L}\Big)\,\mathrm{d}x\\&+\int_\Omega\mu_1(x)\left[\log(e+|\nabla u_1|)+\frac{|\nabla u_1|}{q_1(e+|\nabla u_1|)}\right]|\nabla u_1|^{q_1-2}\nabla u_1\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(u_{1,L}^{\theta p_1}\nabla u_1 + \theta p_1 u_1 u_{1,L}^{\theta p_1 -1}\nabla u_{1,L}\right) \mathrm{d}x\\
&= \int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2) u_1 u_{1,L}^{\theta p_1}\,\mathrm{d}x.
\end{aligned}
$$

Note that $\mu_1 \geq 0$ and $\log(e+t) + \frac{t}{q_1(e+t)} \geq 0$ for $t \geq 0$, and moreover $\nabla u_{1,L} = \mathbf{1}_{\{x\in\Omega\colon\, u_1\leq L\}}\nabla u_1$, where $\mathbf{1}_{\{x\in\Omega\colon\, u_1\leq L\}}$ denotes the characteristic function of the set $\{x\in\Omega\colon u_1\leq L\}$. Hence, the second integral on the left-hand side is nonnegative.

It follows that

$$
\begin{aligned}
&\int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x + \theta p_1 \int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1\cdot\nabla u_{1,L}\big(u_1 u_{1,L}^{\theta p_1-1}\big)\,\mathrm{d}x\\
&\leq \int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2)u_1 u_{1,L}^{\theta p_1}\,\mathrm{d}x.
\end{aligned}
$$

Note that $u_{1,L} = u_1$ and $\nabla u_{1,L} = \nabla u_1$ on $\{x\in\Omega\colon u_1(x)\leq L\}$. Therefore,

$$
|\nabla u_1|^{p_1-2}\nabla u_1\cdot\nabla u_{1,L} = |\nabla u_1|^{p_1} \quad\text{and}\quad u_1 u_{1,L}^{\theta p_1-1} = u_{1,L}^{\theta p_1}
$$

a.e. on $\{x\in\Omega\colon u_1(x)\leq L\}$. This yields

$$
\begin{aligned}
&\int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x + \theta p_1 \int_{\{x\in\Omega\colon\, u_1(x)\leq L\}} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x\\
&\leq \int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2)u_1 u_{1,L}^{\theta p_1}\,\mathrm{d}x.
\end{aligned}
\tag{4.1}
$$

Using the fact that $u_1 \geq 0$, we can estimate the right-hand side of (4.1) via the growth assumptions in (H_4) and obtain

$$
\begin{aligned}
&\int_\Omega \Psi_1(x,u_1,u_2,\nabla u_1,\nabla u_2)u_1 u_{1,L}^{\theta p_1}\,\mathrm{d}x\\
&\leq \int_\Omega \Big(\hat\kappa_1|u_1|^{\hat\alpha_1} + \hat\kappa_2|u_2|^{\hat\alpha_2} + \hat\kappa_3|\nabla u_1|^{\hat\alpha_3} + \hat\kappa_4|\nabla u_2|^{\hat\alpha_4} + \hat\kappa_5\Big)u_1 u_{1,L}^{\theta p_1}\,\mathrm{d}x.
\end{aligned}
$$

Now, we look at each integral separately via the conditions (B_1)–(B_4) in (H_4). First, by (B_1), we have $\hat\alpha_1 \leq p_1^*-1$ and so

$$
\hat\kappa_1 \int_\Omega u_1^{\hat\alpha_1+1}u_{1,L}^{\theta p_1}\,\mathrm{d}x \leq \hat\kappa_1\int_\Omega u_1^{p_1^*}u_{1,L}^{\theta p_1}\,\mathrm{d}x + \hat\kappa_1|\Omega|
$$

as $u_1^{\hat\alpha_1+1}u_{1,L}^{\theta p_1} \leq u_1^{p_1^*}u_{1,L}^{\theta p_1}+1$ a.e. in Ω. From (B_2), we can choose $s_1>1$ such that $\hat\alpha_2 s_1 = p_2^*$. Then, using Hölder's inequality, it follows that

$$
\begin{aligned}
\hat\kappa_2\int_\Omega u_2^{\hat\alpha_2}u_1u_{1,L}^{\theta p_1}\,\mathrm{d}x &\leq \hat\kappa_2\left(\int_\Omega u_2^{\hat\alpha_2 s_1}\,\mathrm{d}x\right)^{\frac{1}{s_1}}\left(\int_\Omega (u_1u_{1,L}^{\theta p_1})^{s_1'}\,\mathrm{d}x\right)^{\frac{1}{s_1'}}\\
&\leq \hat\kappa_2\|u_2\|_{p_2^*}^{\hat\alpha_2}\left(\int_\Omega (u_1^{p_1}u_{1,L}^{\theta p_1})^{s_1'}\,\mathrm{d}x + |\Omega|\right)^{\frac{1}{s_1'}}\\
&\leq C_1\left(1+\|u_1u_{1,L}^{\theta}\|_{p_1s_1'}^{p_1}\right).
\end{aligned}
$$

Note that $s_1 = \frac{p_2^*}{\hat\alpha_2}$ and $s_1' = \frac{s_1}{s_1-1}$. By (B_2) we have

$$
\hat\alpha_2 < \frac{p_1^*-p_1}{p_1^*}p_2^*.
$$

Dividing this by p_2^* and inverting yields

$$s_1 = \frac{p_2^*}{\hat{\alpha}_2} > \frac{p_1^*}{p_1^* - p_1}.$$

Since the function $s \mapsto \frac{s}{s-1}$ is strictly decreasing for $s > 1$, from $s_1 > \frac{p_1^*}{p_1^* - p_1}$, we conclude that

$$s_1' = \frac{s_1}{s_1 - 1} < \frac{\frac{p_1^*}{p_1^* - p_1}}{\frac{p_1^*}{p_1^* - p_1} - 1} = \frac{p_1^*}{p_1}. \tag{4.2}$$

By (B$_3$), we can apply Young's inequality with $\frac{p_1}{\hat{\alpha}_3} > 1$ to get

$$\begin{aligned}
&\hat{\kappa}_3 \int_\Omega |\nabla u_1|^{\hat{\alpha}_3} u_1 u_{1,L}^{\theta p_1} \,\mathrm{d}x \\
&= \hat{\kappa}_3 \int_\Omega \left(\left(\frac{1}{2\hat{\kappa}_3} \right)^{\frac{\hat{\alpha}_3}{p_1}} |\nabla u_1|^{\hat{\alpha}_3} u_{1,L}^{\theta \hat{\alpha}_3} \right) \left(\left(\frac{1}{2\hat{\kappa}_3} \right)^{-\frac{\hat{\alpha}_3}{p_1}} u_1 u_{1,L}^{\theta(p_1 - \hat{\alpha}_3)} \right) \mathrm{d}x \\
&\leq \hat{\kappa}_3 \int_\Omega \frac{1}{2\hat{\kappa}_3} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1} \,\mathrm{d}x + \hat{\kappa}_3 \int_\Omega \left(\frac{1}{2\hat{\kappa}_3} \right)^{-\frac{\hat{\alpha}_3}{p_1 - \hat{\alpha}_3}} u_1^{\frac{p_1}{p_1 - \hat{\alpha}_3}} u_{1,L}^{\theta p_1} \,\mathrm{d}x \\
&\leq \frac{1}{2} \int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1} \,\mathrm{d}x + \hat{\kappa}_3 \left(\frac{1}{2\hat{\kappa}_3} \right)^{-\frac{\hat{\alpha}_3}{p_1 - \hat{\alpha}_3}} \left(\int_\Omega u_1^{p_1^*} u_{1,L}^{\theta p_1} \,\mathrm{d}x + |\Omega| \right) \\
&\leq \frac{1}{2} \int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1} \,\mathrm{d}x + C_2 \left(\int_\Omega u_1^{p_1^*} u_{1,L}^{\theta p_1} \,\mathrm{d}x + 1 \right),
\end{aligned}$$

where we used the fact that $\frac{p_1}{p_1 - \hat{\alpha}_3} \leq p_1^*$ since $\hat{\alpha}_3 \leq p_1 - 1$. Now, we take $s_2 = \frac{p_2}{\hat{\alpha}_4} > 1$ by (B$_4$) and apply Hölder's inequality. This leads to

$$\begin{aligned}
\hat{\kappa}_4 \int_\Omega |\nabla u_2|^{\hat{\alpha}_4} u_1 u_{1,L}^{\theta p_1} \,\mathrm{d}x &\leq \hat{\kappa}_4 \left(\int_\Omega |\nabla u_2|^{\hat{\alpha}_4 s_2} \,\mathrm{d}x \right)^{\frac{1}{s_2}} \left(\int_\Omega (u_1 u_{1,L}^{\theta p_1})^{s_2'} \,\mathrm{d}x \right)^{\frac{1}{s_2'}} \\
&\leq \hat{\kappa}_4 \|\nabla u_2\|_{p_2}^{\hat{\alpha}_4} \left(\int_\Omega (u_1^{p_1} u_{1,L}^{\theta p_1})^{s_2'} \,\mathrm{d}x + |\Omega| \right)^{\frac{1}{s_2'}} \\
&\leq C_3 \left(1 + \|u_1 u_{1,L}^\theta\|_{p_1 s_2'}^{p_1} \right).
\end{aligned}$$

Recall that $s_2 = \frac{p_2}{\hat{\alpha}_4}$ and $s_2' = \frac{s_2}{s_2 - 1} = \frac{p_2}{p_2 - \hat{\alpha}_4}$. Moreover, in (B$_4$) we have

$$\hat{\alpha}_4 < \frac{p_1^* - p_1}{p_1^*} p_2,$$

which is equivalent to

$$p_2 - \hat{\alpha}_4 > p_2 \left(1 - \frac{p_1^* - p_1}{p_1^*} \right) = p_2 \frac{p_1}{p_1^*}.$$

Hence,

$$s_2' = \frac{p_2}{p_2 - \hat{\alpha}_4} < \frac{p_1^*}{p_1}. \tag{4.3}$$

Finally, we have

$$\hat{\kappa}_5 \int_\Omega u_1 u_{1,L}^{\theta p_1} \,\mathrm{d}x \leq \hat{\kappa}_5 \int_\Omega u_1^{p_1^*} u_{1,L}^{\theta p_1} \,\mathrm{d}x + \hat{\kappa}_5 |\Omega|.$$

From (4.2) and (4.3) it follows that

$$s_1', s_2' < \frac{p_1^*}{p_1}.$$

Since $s_1', s_2' > 1$, we may define

$$s := \max\{s_1', s_2'\} \in \left(1, \frac{p_1^*}{p_1}\right).$$

From the above estimates we arrive at

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x + \theta p_1 \int_{\{x\in\Omega\colon\, u_1(x)\le L\}} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x \\ &\le C_4 \int_\Omega u_1^{p_1^*} u_{1,L}^{\theta p_1}\,\mathrm{d}x + C_5 \|u_1 u_{1,L}^\theta\|_{p_1 s}^{p_1} + C_6, \end{aligned} \tag{4.4}$$

where C_4, C_5, and C_6 are independent of L and θ. Note that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x + \theta p_1 \int_{\{x\in\Omega\colon\, u_1(x)\le L\}} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\{x\in\Omega\colon\, u_1(x)> L\}} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x \\ &\quad + \left(\theta p_1 + \frac{1}{2}\right)\int_{\{x\in\Omega\colon\, u_1(x)\le L\}} |\nabla u_1|^{p_1} u_{1,L}^{\theta p_1}\,\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\{x\in\Omega\colon\, u_1(x)> L\}} |\nabla (u_1 u_{1,L}^{\theta})|^{p_1}\,\mathrm{d}x \\ &\quad + \frac{\theta p_1 + 1}{2(\theta+1)^{p_1}}\int_{\{x\in\Omega\colon\, u_1(x)\le L\}} |\nabla (u_1 u_{1,L}^{\theta})|^{p_1}\,\mathrm{d}x \\ &\ge \frac{\theta p_1 + 1}{2(\theta+1)^{p_1}}\int_\Omega |\nabla (u_1 u_{1,L}^{\theta})|^{p_1}\,\mathrm{d}x, \end{aligned} \tag{4.5}$$

since $(\theta+1)^{p_1} \ge \theta p_1 + 1$ which implies that $\frac{\theta p_1+1}{2(\theta+1)^{p_1}} \le \frac{1}{2}$.

Combining (4.4) and (4.5) yields

$$\begin{aligned} &\frac{\theta p_1 + 1}{2(\theta+1)^{p_1}}\int_\Omega |\nabla (u_1 u_{1,L}^{\theta})|^{p_1}\,\mathrm{d}x \\ &\le C_4 \int_\Omega u_1^{p_1^*} u_{1,L}^{\theta p_1}\,\mathrm{d}x + C_5 \|u_1 u_{1,L}^\theta\|_{p_1 s}^{p_1} + C_6. \end{aligned}$$

From this point on, we may proceed as in Marino–Winkert [39, Theorem 3.1], starting from (3.12), with only minor notational changes. More precisely, the exponents p, q in [39] correspond to p_1, p_2 here, and their variables (u, v) correspond to (u_1, u_2). This completes the proof of the claim.

Now one shows that $(u_1, u_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Since this step follows a standard Moser iteration argument, we omit the details and refer to Theorem 3.2 in Marino–Winkert [39], which applies in exactly the same way. $\square$

We modify Example 3.1 so that (H_4) is satisfied as well.

Example 4.1. *We consider the same functions Ψ_1, Ψ_2 as in Example 3.1. Assume in addition that $N = 3$ and that $\omega_i \in L^\infty(\Omega) \cap L^2(\Omega)$ with $\omega_i \not\equiv 0$ for $i = 1, 2$. Then*

(H_4) *holds for the same* Ψ_1, Ψ_2*. Indeed, using* $|a \cdot \xi| \leq |\xi|$ *and* $|b \cdot \xi| \leq |\xi|$ *for all* $\xi \in \mathbb{R}^N$*, we have*

$$|\Psi_1(x, s_1, s_2, \eta_1, \eta_2)| \leq \varepsilon|\eta_1| + \varepsilon|\eta_2| + \varepsilon|s_2| + \|\omega_1\|_\infty,$$
$$|\Psi_2(x, s_1, s_2, \eta_1, \eta_2)| \leq \varepsilon|\eta_2| + \varepsilon|\eta_1| + \varepsilon|s_1| + \|\omega_2\|_\infty.$$

Thus (H_4) *is satisfied with* $\hat{\kappa}_3 = \hat{\kappa}_4 = \hat{\kappa}_2 = \hat{\nu}_4 = \hat{\nu}_3 = \hat{\nu}_1 = \varepsilon$*,* $\hat{\kappa}_5 = \|\omega_1\|_\infty$*,* $\hat{\nu}_5 = \|\omega_2\|_\infty$*, and with exponents* $\hat{\alpha}_3 = \hat{\alpha}_4 = \hat{\alpha}_2 = \hat{\beta}_4 = \hat{\beta}_3 = \hat{\beta}_1 = 1$*. All remaining coefficients and exponents are equal to* 0*. Since* $N = 3$ *and* $p_1 = p_2 = 2$*, the strict constraints* (B_2)*,* (B_4)*,* (B_5)*, and* (B_7) *allow exponent* 1*, and the remaining conditions are immediate.*

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