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Critical logarithmic double phase problems of Brezis–Nirenberg type with nonlinear boundary condition

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Abstract. In this paper, we investigate the existence and multiplicity of solutions to a class of nonlinear elliptic problems governed by the logarithmic double phase operator and subject to nonlinear critical Neumann boundary conditions. By employing variational methods in combination with topological tools such as truncation techniques and Krasnosel'skii's genus theory, we establish the existence of infinitely many weak solutions with negative energy sign. The results highlight the rich solution structure arising from the interplay between the logarithmic double phase operator and the critical boundary growth.

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Keywords. Critical growth on the boundary, Equivalent norm, Logarithmic double phase operator, Krasnosel'skii's genus theory, Multiplicity results.

1. Introduction

In numerous physical models, ranging from nonlinear diffusion processes and material mechanics to field theory, partial differential equations involving nonlinear terms may exhibit critical growth. This critical growth is associated with the lack of compactness in Sobolev embedding, reflecting, in physical terms, the possibility of extreme energy accumulation or the formation of singularities. A similar critical behavior arises in geometry, particularly in Riemannian geometry, as illustrated by the Yamabe problem, see Yamabe [47], or in the existence of nonminimal solutions to Yang–Mills fields, as studied by Taubes [42, 43]. The pioneering work of Brézis–Nirenberg [14] marked the beginning of the systematic study of critical elliptic equations. Since that time, critical problems have become a central research theme, leading to significant developments and opening new directions in the study of nonlinear elliptic partial differential equations.

A major advancement in the study of critical elliptic problems was achieved by García Azorero–Peral Alonso [28] who extended the framework of critical problems to include the p-Laplace operator. In particular, they investigated the following critical p-Laplace problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u + \lambda |u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\lambda > 0$, $1 < p < N$, $1 < q < p < p^*$, with p^* denoting the critical Sobolev exponent given by $p^* = \frac{Np}{N-p}$. By employing variational methods and building upon the pioneering ideas of Benci–Fortunato [10] as well as García Azorero–Peral Alonso [27], the authors established that problem (1.1) admits infinitely many solutions for all $\lambda \in (0, \overline{\lambda})$, where $\overline{\lambda} > 0$ is sufficiently small.

A further variant of the critical problem was introduced by Fernández Bonder–Rossi [23] who employed variational techniques combined with topological arguments to establish a multiplicity result for the

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following problem

$$
\begin{aligned}
\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= |u|^{p-2}u && \text{in } \Omega,\\
|\nabla u|^{p-2}\nabla u\cdot\nu &= \lambda|u|^{r-2}u+|u|^{p_*-2}u && \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

where $1<r<p$, $\lambda>0$ is a real parameter, $\nu(x)$ denotes the outer unit normal of Ω at $x\in\partial\Omega$, and p_* on the boundary of (1.2) is the critical exponent in the Sobolev trace embedding corresponding to $1<p<\infty$, defined by $p_*=\frac{(N-1)p}{N-p}$.

In a more recent development, Farkas–Fiscella–Winkert [22] extended the results of García Azorero–Peral Alonso [28] by establishing a multiplicity result for the following quasilinear elliptic equation with critical growth

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) &= \lambda|u|^{\vartheta-2}u+|u|^{p^*-2}u && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

where $\lambda>0$ is a real parameter, $1<\vartheta<p<q<N$, $q<p^*$, $0\le\mu(\cdot)\in L^\infty(\Omega)$ and

$$
\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)
$$

is the so-called double phase operator with its energy functional

$$
\omega\mapsto\int_\Omega\left(\frac{|\nabla\omega|^p}{p}+\mu(x)\frac{|\nabla\omega|^q}{q}\right)\mathrm{d}x,
\tag{1.3}
$$

where $\Omega\subseteq\mathbb{R}^N$, $N\ge 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. Double phase energy functionals as in (1.3) appeared for the first time in a work by Zhikov [50] in order to describe models for strongly anisotropic materials in the context of homogenization and elasticity theory, see also Zhikov [51,52]. Concerning regularity results of local minimizers of (1.3), we refer to the outstanding papers by Baroni–Colombo–Mingione [6–8] and Colombo–Mingione [16,17], see also the references therein. We remark that (1.3) represents a particular case of integral functionals with nonstandard growth, as originally investigated in the seminal works of Marcellini [33,34]. An overview about the recent developments for double phase problems and related functionals can be found in the paper by Mingione–Rădulescu [36].

Recently, Arora–Crespo-Blanco–Winkert [4] introduced a new class of logarithmic double phase operators, defined by

$$
\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right),
\tag{1.4}
$$

and studied the associated energy functional

$$
I(u)=\int_\Omega\left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\log(e+|\nabla u|)\right)\mathrm{d}x
\tag{1.5}
$$

defined on the logarithmic Musielak–Orlicz–Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ generated by the nonlinear function

$$
\mathcal{H}_{\log}(x,t):=t^p+\mu(x)t^q\log(e+t),
$$

where e is Euler's number, $1<p<N$, $p<q$, and $0\le\mu(\cdot)\in L^\infty(\Omega)$. They established several fundamental properties of the operator (1.4), including boundedness, continuity, strict monotonicity, and the (S_+) property, and further investigated the existence and multiplicity of solutions to equations associated with this operator. Regularity properties of local minimizers of different functionals involving a logarithmic perturbation as in (1.5) can be found in the papers by Baroni–Colombo–Mingione [7], De Filippis–Mingione [20], Fuchs–Mingione [25], Marcellini [33], Marcellini–Papi [35], see also the references

therein. One motivation for studying functionals of the type (1.5) with logarithmic perturbations, for example

$$u \mapsto \int_\Omega |\nabla u| \log(1+|\nabla u|)\,\mathrm{d}x,$$

arises from plasticity theory, in particular from models incorporating logarithmic hardening. For a comprehensive treatment of these issues and further background, we refer to the works of Fuchs–Seregin [26] and Seregin–Frehse [38].

In this paper, inspired by the contributions in [4,22,23,28], we study logarithmic double phase problems with critical growth on the Neumann boundary given by

$$\begin{aligned} \operatorname{div} \mathcal{G}(u) &= |u|^{p-2}u && \text{in } \Omega,\\ \mathcal{G}(u)\cdot \nu &= \lambda |u|^{\alpha-2}u + |u|^{p_*-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $\operatorname{div} \mathcal{G}$ is the logarithmic double phase operator defined in (1.4), $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$, $\lambda > 0$ is a real parameter to be specified and

$$\mathcal{G}(u) = |\nabla u|^{p-2}\nabla u + \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u. \tag{1.7}$$

Further, we assume the following assumptions:

(H) $1<\alpha<p<N$, $p<q<q+\kappa<p_*=\frac{(N-1)p}{N-p}$ and $0 \le \mu(\cdot) \in L^\infty(\Omega)$, where $\kappa = \frac{e}{e+t_0}$ with t_0 being the unique positive solution of $t_0 = e\log(e+t_0)$.

A function $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ is said to be a weak solution of problem (1.6) if

$$\begin{aligned} &\int_\Omega \left[|\nabla u|^{p-2}\nabla u + \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right]\cdot \nabla v\,\mathrm{d}x\\ &\quad + \int_\Omega |u|^{p-2}uv\,\mathrm{d}x = \int_{\partial\Omega} \left(\lambda|u|^{\alpha-2}u + |u|^{p_*-2}u\right) v\,\mathrm{d}\sigma \end{aligned}$$

is satisfied for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$.

Our main result is stated as follows.

Theorem 1.1. *Let hypotheses (H) be satisfied. Then, there exists a constant $\lambda^* > 0$ such that, for every $\lambda \in (0,\lambda^*)$, problem (1.6) admits infinitely many weak solutions with negative energy sign.*

The proof of theorem 1.1 combines variational methods with topological tools, notably truncation techniques and Krasnosel'skii's genus theory. The primary analytical difficulty stems from the presence of a nonlinear term with critical growth, which severely hinders the compactness properties of the Sobolev trace embedding. This lack of compactness prevents the direct application of classical critical point theory, especially in variational frameworks involving critical Sobolev exponents. To overcome this difficulty, we perform a careful convergence analysis of the gradients, inspired by the approach of Boccardo–Murat [11], who developed a method to deal with weak convergence in Sobolev spaces under limited compactness. By adapting their approach to our context involving logarithmic double phase structures, we are able to recover a suitable form of compactness sufficient for the variational framework.

Moreover, in attempting to apply Krasnosel'skii's genus theory which is a fundamental topological method for establishing the existence of infinitely many solutions, one encounters an additional difficulty since the associated energy functional of problem (1.6) is unbounded from below. This lack of boundedness prevents the direct use of standard minimax theorems and requires the development of a refined variational framework adapted to this situation. In order to overcome this difficulty, we make use of a truncation method first introduced by García Azorero–Peral Alonso [28]. The idea is to suitably modify the nonlinearity in such a way that boundedness from below is restored, while the essential structural features of the original problem are preserved. This approach has been successfully adapted in various settings, including the works of Farkas–Fiscella–Winkert [22], Fernández Bonder–Rossi [23], Figueiredo–Santos Júnior–Suárez [24], Zhang–Fiscella–Liang [48], and more recently by Arora–Crespo-Blanco–Winkert [3]. On the other hand, in the context of problems involving the double phase operator with logarithmic perturbation, we highlight the recent contributions by Bahrouni–Fiscella–Winkert [5], Borer–Gasiński–Stapenhorst–Winkert [12], Carranza–Pimenta–Vetro–Winkert [15], Guo–Ling–Lin–Pucci [29], Rădulescu–Stapenhorst–Winkert [39], Tran–Nguyen [44], Vetro [45], Vetro–Winkert [46], Zeng–Lu–Rădulescu [49], and Zhong–Vetro–Chen [53], in which new analytical techniques have been developed to address the complex role of logarithmic terms both in the differential operator and in the associated variational formulation. Finally, we also mention the papers by Ambrosio [1], Ambrosio–Isernia [2], and Isernia [31], which deal with Kirchhoff-type problems involving different leading operators.

2. Preliminaries

We begin by recalling the basic properties of the logarithmic Musielak–Orlicz spaces $L^{\mathcal{H}_{\log}}(\Omega)$ and $W^{1,\mathcal{H}_{\log}}(\Omega)$, together with the main analytical aspects of the associated logarithmic double phase operator. The presentation follows closely the recent contribution of Arora–Crespo-Blanco–Winkert [4]. For a comprehensive background on Musielak–Orlicz spaces and related topics, we refer to the monographs by Diening–Harjulehto–Hästö–Růžička [21], Harjulehto–Hästö [30], and Papageorgiou–Winkert [37], as well as to the work of Crespo-Blanco–Gasiński–Harjulehto–Winkert [19].

For this purpose, we denote by $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces equipped with the norm $\|\cdot\|_r$ for $1 \leq r \leq \infty$ and $W^{1,r}(\Omega)$ stands for the corresponding Sobolev space endowed with the equivalent norm $\|\cdot\|_{1,r} = (\|\nabla\cdot\|_r^r + \|\cdot\|_r^r)^{\frac{1}{r}}$ for $1 < r < \infty$. Furthermore, let σ denote the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$, and consider the boundary Lebesgue space $L^r(\partial\Omega)$ for $1 \leq r < \infty$ endowed with the norm

$$\|u\|_{r,\partial\Omega} = \left(\int_{\partial\Omega} |u|^r \,\mathrm{d}\sigma\right)^{\frac{1}{r}}, \quad u \in L^r(\partial\Omega).$$

In the following, we avoid explicit reference to the trace operator γ and understand all restrictions of Sobolev functions to $\partial\Omega$ in the sense of traces.

Let us assume that the assumptions (H) are satisfied. Then, we consider the nonlinear function $\mathcal{H}_{\log}\colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) := t^p + \mu(x) t^q \log(e+t),$$

where e denotes Euler's number. Note that $\mathcal{H}_{\log}(\cdot,t)$ is measurable for all $t \geq 0$, $\mathcal{H}_{\log}(x,0) = 0$ and further $\mathcal{H}_{\log}(\cdot,t) > 0$ for all $t > 0$. Moreover, one can readily verify that $\mathcal{H}_{\log}$ satisfies the Δ_2-condition. Denoting by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, the Musielak–Orlicz Lebesgue space $L^{\mathcal{H}_{\log}}(\Omega)$ is defined as

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{u \in M(\Omega)\colon \rho_{\mathcal{H}_{\log}}(u) < \infty\right\},$$

endowed with the norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf\left\{\beta > 0\colon \rho_{\mathcal{H}_{\log}}\left(\frac{u}{\beta}\right) \leq 1\right\},$$

where the modular function is given by

$$\rho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q \log(e+|u|)\right)\mathrm{d}x.$$

Note that $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable and reflexive Banach space. Next, we can introduce the logarithmic Musielak–Orlicz–Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ defined by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}_{\log}}(\Omega) \colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}.$$

From Arora–Crespo-Blanco–Winkert [4, Proposition 3.6], we know that $W^{1,\mathcal{H}_{\log}}(\Omega)$ is a separable, reflexive Banach space.

The following embedding results can be found in [4, Proposition 3.7].

Proposition 2.1. *Let (H) be satisfied, then the following hold:*

(i) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ *is continuous;*
(ii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *is continuous and* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r}(\Omega)$ *is compact for all* $1 \leq r < p^*$*;*
(iii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ *is continuous and* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{s}(\partial\Omega)$ *is compact for all* $1 \leq s < p_*$*.*

We endow the space $W^{1,\mathcal{H}_{\log}}(\Omega)$ with the following equivalent norm, see Borer–Gasiński–Stapenhorst–Winkert [12, Proposition 3.1],

$$\begin{aligned}\|u\| &= \inf\left\{\beta > 0 \colon \rho\left(\frac{u}{\beta}\right) \leq 1\right\}\\ &= \inf\left\{\beta > 0 \colon \int_\Omega \left(\left|\frac{\nabla u}{\beta}\right|^p + \mu(x)\left|\frac{\nabla u}{\beta}\right|^q \log\left(e + \left|\frac{\nabla u}{\beta}\right|\right)\right) \mathrm{d}x\right.\\ &\qquad\left. + \int_\Omega \left|\frac{u}{\beta}\right|^p \mathrm{d}x \leq 1\right\},\end{aligned}$$

where the associated modular function ρ is given by

$$\rho(u) = \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q \log(e + |\nabla u|)\right) \mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x.$$

The next result, due to Arora–Crespo-Blanco–Winkert [4] and later extended by Borer–Gasiński–Stapenhorst–Winkert [12], provides a useful characterization of the relation between this norm and the associated modular function.

Proposition 2.2. *Let hypotheses (H) be satisfied,* $\beta > 0$ *and* $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$*. Then the following hold:*

(i) $\|u\| = \beta$ *if and only if* $\rho\left(\frac{u}{\beta}\right) = 1$ *for* $u \neq 0$*;*
(ii) $\|u\| < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) *if* $\|u\| < 1$ *then* $\|u\|^{q+\kappa} \leq \rho(u) \leq \|u\|^p$*;*
(iv) *if* $\|u\| > 1$ *then* $\|u\|^p \leq \rho(u) \leq \|u\|^{q+\kappa}$*;*
(v) $\|u\| \to 0$ *if and only if* $\rho(u) \to 0$*.*

Throughout the paper, for any $s \in [1, p_*]$, we denote by $C_s > 0$ the constant given by Proposition 2.1 (iii), such that

$$\|u\|_{s,\partial\Omega}^s \leq C_s \|u\|^s \tag{2.1}$$

for any $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$.

The lemma below is essential for the arguments presented in the subsequent section, see Arora–Crespo-Blanco–Winkert [4, Lemma 5.4].

Lemma 2.3. *Let $Q > 1$ and define the function*

$$h\colon [0,\infty) \to [0,\infty), \quad h(t) = \frac{t}{Q(e+t)\log(e+t)}.$$

Then h attains its maximum at a point $t_0 > 0$ and this maximum value is $\frac{\kappa}{Q}$, where t_0 and κ are the same as in (H).

Next, we recall the following classical inequalities, which will be used later, see Crespo-Blanco [18] or Simon [41]:

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi-\eta) \geq \begin{cases} K_r|\xi-\eta|^r & \text{if } r \geq 2, \\ K_r\dfrac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-r}} & \text{if } 1 < r < 2, \end{cases} \tag{2.2}$$

for any ξ, $\eta \in \mathbb{R}^N$ with a constant $K_r > 0$. The inequality (2.2) has been generalized by Arora–Crespo-Blanco–Winkert [4, Lemma 4.2] in the following way.

Proposition 2.4. *Let $f\colon [0,\infty) \to [0,\infty)$ be an increasing function and $r > 1$. Then, for any $\xi, \eta \in \mathbb{R}^N$*

$$\left(f(|\xi|)|\xi|^{r-2}\xi - f(|\eta|)|\eta|^{r-2}\eta\right)\cdot(\xi-\eta) \geq C_r|\xi-\eta|^r f(m)$$

if $r \geq 2$, and

$$(|\xi|+|\eta|)^{2-r}\left(f(|\xi|)|\xi|^{r-2}\xi - f(|\eta|)|\eta|^{r-2}\eta\right)\cdot(\xi-\eta) \geq C_r|\xi-\eta|^2 f(m)$$

if $1 < r < 2$, where $m = \min\{|\xi|,|\eta|\}$ and

$$C_r = \begin{cases} \min\{2^{2-r}, 2^{-1}\} & \text{if } r \geq 2, \\ r-1 & \text{if } 1 < r < 2. \end{cases}$$

In the last part of this section, we recall some results developed by Krasnosel'skii [32] and concepts that will be needed later as well. For this purpose, let X be a Banach space and let Σ denote the class of all closed subsets $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2.5. Let $A \in \Sigma$. The Krasnosel'skii's genus $\gamma(A)$ of A is defined as the least positive integer n such that there is an odd mapping $\phi \in C(A,\mathbb{R}^n)$ such that $\phi(x) \neq 0$ for any $x \in A$. If n does not exist, we set $\gamma(A) = \infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

The following proposition summarizes the main properties of Krasnosel'skii's genus that are essential for our approach in Sect. 3, see Rabinowitz [40].

Proposition 2.6. *Let $A, B \in \Sigma$. Then the following hold:*

(i) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$;*
(ii) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$;*
(iii) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
(iv) *The n-dimensional sphere S^n has genus $n+1$ by the Borsuk–Ulam theorem;*
(v) *If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$, with $N_\delta(A) = \{x \in X\colon \operatorname{dist}(x,A) \leq \delta\}$.*

Let X be a Banach space with its dual space X^* and $\varphi \in C^1(X)$. We say that $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega)$ is a Palais–Smale sequence for the functional φ at level $c \in \mathbb{R}$ if

$$\varphi(u_n) \to c \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega)^* \quad \text{as } n \to \infty. \tag{2.3}$$

We say that φ satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if any Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ of φ at level c admits a strongly convergent subsequence in $W^{1,\mathcal{H}_{\log}}(\Omega)$.

3. Proof of the main result

In this section, we present the proof of Theorem 1.1, which establishes the existence and multiplicity of weak solutions to problem (1.6). To this end, we begin by analyzing the corresponding energy functional $\Psi_\lambda\colon W^{1,\mathcal{H}_{\log}}(\Omega)\to\mathbb{R}$ of (1.6) which is defined as

$$\Psi_\lambda(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x+\frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\log(e+|\nabla u|)\,\mathrm{d}x \\ +\frac{1}{p}\|u\|_p^p-\frac{\lambda}{\alpha}\|u\|_{\alpha,\partial\Omega}^{\alpha}-\frac{1}{p_*}\|u\|_{p_*,\partial\Omega}^{p_*}.$$

From the work by Arora–Crespo-Blanco–Winkert [4, Theorem 4.1] we know that $\Psi_\lambda\in C^1(W^{1,\mathcal{H}_{\log}}(\Omega))$ and it is clear that the weak solutions of problem (1.6) correspond precisely to the critical points of the functional Ψ_λ.

We next study the compactness properties of the energy functional Ψ_λ, specifically the Palais–Smale condition.

Lemma 3.1. *Let* (H) *be satisfied and* $\{u_n\}_{n\in\mathbb{N}}\subset W^{1,\mathcal{H}_{\log}}(\Omega)$ *be a bounded* $(\mathrm{PS})_c$ *sequence with* $c\in\mathbb{R}$. *Then, up to a subsequence,* $\nabla u_n(x)\to\nabla u(x)$ *a.e. in* Ω *as* $n\to\infty$.

Proof. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$, by the reflexivity of $W^{1,\mathcal{H}_{\log}}(\Omega)$, Proposition 2.1 (iii) and Brézis [13, Theorem 4.9], there exists a subsequence, not relabeled, and $u\in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\begin{aligned} &u_n\rightharpoonup u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega), && \nabla u_n\rightharpoonup\nabla u \quad \text{in } \left[L^{\mathcal{H}_{\log}}(\Omega)\right]^N,\\ &u_n\to u \quad \text{in } L^r(\Omega), && u_n(x)\to u(x) \quad \text{a.e. in } \Omega,\\ &u_n\to u \quad \text{in } L^s(\partial\Omega), && u_n(x)\to u(x) \quad \text{a.e. in } \partial\Omega,\\ &|u_n(x)|\le h(x) \quad \text{a.e. in } \Omega, \end{aligned} \tag{3.1}$$

as $n\to\infty$ with $r\in[1,p^*)$, $s\in[1,p_*)$ and $h\in L^q(\Omega)$.

Next, for $k\in\mathbb{N}$, we define a truncation function $T_k\colon\mathbb{R}\to\mathbb{R}$ by

$$T_k(t):=\begin{cases} t & \text{if } |t|\le k,\\ k\dfrac{t}{|t|} & \text{if } |t|>k.\end{cases}$$

Fixing $k\in\mathbb{N}$ and using the fact that $\{u_n\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence for Ψ_λ, yields

$$\begin{aligned} o_n(1)&=\langle\Psi_\lambda'(u_n),T_k(u_n-u)\rangle\\ &=\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla T_k(u_n-u)\,\mathrm{d}x\\ &\quad+\int_\Omega \mu(x)\left[\log(e+|\nabla u_n|)+\frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right]\\ &\qquad\times|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla T_k(u_n-u)\,\mathrm{d}x\\ &\quad+\int_\Omega |u_n|^{p-2}u_nT_k(u_n-u)\,\mathrm{d}x-\lambda\int_{\partial\Omega}|u_n|^{\alpha-2}u_nT_k(u_n-u)\,\mathrm{d}\sigma\\ &\quad-\int_{\partial\Omega}|u_n|^{p_*-2}u_nT_k(u_n-u)\,\mathrm{d}\sigma \quad \text{as } n\to\infty, \end{aligned} \tag{3.2}$$

where we used the fact that $\{T_k(u_n-u)\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded. Next, we introduce the functional

$$\Phi\colon f \in \left[L^{\mathcal{H}_{\log}}(\Omega)\right]^N \mapsto \int_\Omega \mathcal{G}(u)\cdot f \,\mathrm{d}x,$$

where $\mathcal{G}$ is defined as in (1.7). It is clear that Φ is linear and bounded.

Now, observe from (3.1) that $\nabla T_k(u_n-u) \rightharpoonup 0$ in $\left[L^{\mathcal{H}_{\log}}(\Omega)\right]^N$, so we obtain

$$\lim_{n\to\infty} \int_\Omega \mathcal{G}(u)\cdot \nabla T_k(u_n-u)\,\mathrm{d}x = 0. \tag{3.3}$$

Using the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ along with Proposition 2.1 (iii), leads to

$$\left|\int_{\partial\Omega} |u_n|^{p_*-2}u_n T_k(u_n-u)\,\mathrm{d}\sigma\right| \le k\int_{\partial\Omega} |u_n|^{p_*-1}\,\mathrm{d}\sigma \le Ck \tag{3.4}$$

for each $n\in\mathbb{N}$ with a constant $C>0$ independent of n and k. Therefore, combining (3.1), (3.2), and (3.3), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\Bigg[\int_\Omega \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right]\cdot\nabla T_k(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\Bigg(\left[\log(e+|\nabla u_n|)+\frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right]|\nabla u_n|^{q-2}\nabla u_n\\
&\quad-\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right]|\nabla u|^{q-2}\nabla u\Bigg)\cdot\nabla T_k(u_n-u)\,\mathrm{d}x\Bigg]\\
&=\limsup_{n\to\infty}\int_{\partial\Omega}|u_n|^{p_*-2}u_nT_k(u_n-u)\,\mathrm{d}\sigma.
\end{aligned}\tag{3.5}$$

Since the function $t\to\log(e+t)+\frac{t}{q(e+t)}$ is increasing, using Proposition 2.4, (3.4) and (3.5), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla T_k(u_n-u)\,\mathrm{d}x\\
&\le \limsup_{n\to\infty}\int_{\partial\Omega}|u_n|^{p_*-2}u_nT_k(u_n-u)\,\mathrm{d}\sigma\\
&\le Ck.
\end{aligned}\tag{3.6}$$

For

$$e_n(x) := \left(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right)\cdot\nabla(u_n(x)-u(x)),$$

we see, due to (2.2), that $e_n(x)\ge 0$ a.e. in Ω. Let $n,k\in\mathbb{N}$ be fixed. We decompose the domain Ω as follows

$$S_n^k = \{x\in\Omega\colon |u_n(x)-u(x)|\le k\} \quad\text{and}\quad G_n^k = \{x\in\Omega\colon |u_n(x)-u(x)|>k\}.$$

Let $\omega\in(0,1)$ be fixed. By applying Hölder's inequality, and using both the boundedness of $\{e_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega)$ and estimate (3.6), we obtain

$$\int_\Omega e_n^\omega\,\mathrm{d}x \le \left(\int_{S_n^k} e_n\,\mathrm{d}x\right)^\omega |S_n^k|^{1-\omega} + \left(\int_{G_n^k} e_n\,\mathrm{d}x\right)^\omega |G_n^k|^{1-\omega}$$

$$\leq (kC)^\omega |S_n^k|^{1-\omega} + \widetilde{C}^\omega |G_n^k|^{1-\omega}.$$

Since $|G_n^k| \to 0$ as $n \to \infty$, we deduce that

$$0 \leq \limsup_{n\to\infty} \int_\Omega e_n^\omega \,\mathrm{d}x \leq (kC)^\omega |\Omega|^{1-\omega}.$$

Since the right-hand side tends to zero as $k \to 0^+$, we conclude that $e_n^\omega \to 0$ in $L^1(\Omega)$ as $n \to \infty$. This implies, up to a subsequence, that $e_n(x) \to 0$ a.e. in Ω. Finally, using (2.2), we get the desired conclusion of the lemma. □

In the following, let S be the best constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$, that is,

$$S := \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_p^p + \|u\|_p^p}{\|u\|_{p_*,\partial\Omega}^p}. \tag{3.7}$$

Lemma 3.2. *Let* (H) *be satisfied and $c < 0$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, the functional Ψ_λ fulfills the* $(\mathrm{PS})_c$ *condition.*

Proof. From the assumptions (H), we have $\alpha < q + \kappa < p_*$. Therefore, we can take $\lambda_0 > 0$ sufficiently small such that

$$|\partial\Omega| \left(\frac{1}{q+\kappa} - \frac{1}{p_*}\right)^{\frac{-p_*}{p_*-\alpha}} \left[\lambda_0 \left(\frac{1}{\alpha} - \frac{1}{q+\kappa}\right)\right]^{\frac{p_*}{p_*-\alpha}} < S^{\frac{p_*}{p_*-p}}, \tag{3.8}$$

where S is given in (3.7).

Claim: Any $(\mathrm{PS})_c$ sequence for Ψ_λ is bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$.

To this end, let $\lambda \in (0, \lambda_0)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega)$ be a $(\mathrm{PS})_c$ sequence for Ψ_λ. Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is not bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$. Then, we may assume, up to a subsequence, not relabeled, that $\lim_{n\to\infty} \|u_n\| = \infty$. Then, there exists $k \in \mathbb{N}$ sufficiently large, such that

$$\|u_n\| \geq 1 \quad \text{for all } n \geq k. \tag{3.9}$$

Applying Lemma 2.3 and (2.1), we deduce that

$$\begin{aligned}
&\Psi_\lambda(u_n) - \frac{1}{p_*}\langle \Psi_\lambda'(u_n), u_n\rangle \\
&= \frac{1}{p}\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q \log(e+|\nabla u_n|)\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |u_n|^p \,\mathrm{d}x - \frac{\lambda}{\alpha}\|u_n\|_{\alpha,\partial\Omega}^\alpha - \frac{1}{p_*}\|u_n\|_{p_*,\partial\Omega}^{p_*} \\
&\quad - \frac{1}{p_*}\Bigg[\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega \mu(x)\left(\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right)|\nabla u_n|^q \,\mathrm{d}x \\
&\qquad + \int_\Omega |u_n|^p \,\mathrm{d}x - \lambda\|u_n\|_{\alpha,\partial\Omega}^\alpha - \|u_n\|_{p_*,\partial\Omega}^{p_*}\Bigg] \\
&\geq \frac{1}{p}\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q \log(e+|\nabla u_n|)\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |u_n|^p \,\mathrm{d}x - \frac{\lambda}{\alpha}\|u_n\|_{\alpha,\partial\Omega}^\alpha - \frac{1}{p_*}\|u_n\|_{p_*,\partial\Omega}^{p_*}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{p_*}\Bigg[\int_\Omega |\nabla u_n|^p\,\mathrm{d}x+\left(\frac{q+\kappa}{q}\right)\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x\\
&\qquad+\int_\Omega |u_n|^p\,\mathrm{d}x-\lambda\|u_n\|^\alpha_{\alpha,\partial\Omega}-\|u_n\|^{p_*}_{p_*,\partial\Omega}\Bigg]\\
&\geq\left(\frac{1}{q}-\frac{q+\kappa}{qp_*}\right)\rho(u_n)-\lambda\left(\frac{1}{\alpha}-\frac{1}{p_*}\right)C_\alpha\|u_n\|^\alpha.
\end{aligned}
$$

Thus, by (2.3), Proposition 2.2 (iv) along with (3.9), there exist constants $c_1, c_2>0$ such that, as $n\to\infty$,

$$
c_1+c_2\|u_n\|+o_n(1)\geq\left(\frac{1}{q}-\frac{q+\kappa}{qp_*}\right)\|u_n\|^p-\lambda\left(\frac{1}{\alpha}-\frac{1}{p_*}\right)C_\alpha\|u_n\|^\alpha,
$$

which is a contradiction since $p_*>q+\kappa>q>p>\alpha>1$. This shows the Claim.

From the Claim, Proposition 2.1 (ii), (iii), Lemma 3.1, Brézis [13, Theorem 4.9] and the reflexivity of $W^{1,\mathcal{H}_{\log}}(\Omega)$, there exists a subsequence, not relabeled, and $u\in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$
\begin{aligned}
u_n&\rightharpoonup u\quad\text{in } W^{1,\mathcal{H}_{\log}}(\Omega), & \nabla u_n&\rightharpoonup\nabla u\quad\text{in }\left[L^{\mathcal{H}_{\log}}(\Omega)\right]^N,\\
\nabla u_n(x)&\to\nabla u(x)\quad\text{a.e. in }\Omega, & u_n&\to u\quad\text{in } L^r(\Omega),\\
u_n&\to u\quad\text{in } L^s(\partial\Omega), & u_n(x)&\to u(x)\quad\text{a.e. in }\partial\Omega,\\
u_n(x)&\to u(x)\quad\text{a.e. in }\Omega, & \|u_n-u\|_{p_*,\partial\Omega}&\to\vartheta,
\end{aligned}
\tag{3.10}
$$

as $n\to\infty$ with $r\in[1,p^*)$ and $s\in[1,p_*)$. Note that, by (3.10) and applying the Brézis–Lieb-type lemma due to Arora–Crespo-Blanco–Winkert [3, Lemma 4.3], we have

$$
\begin{aligned}
&\frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x+\frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x\\
&-\Bigg[\frac{1}{p}\int_\Omega |\nabla u_n-\nabla u|^p\,\mathrm{d}x\\
&\qquad+\frac{1}{q}\int_\Omega \mu(x)|\nabla u_n-\nabla u|^q\log(e+|\nabla u_n-\nabla u|)\,\mathrm{d}x\Bigg]\\
&=\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x+\frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\log(e+|\nabla u|)\,\mathrm{d}x+o_n(1),\\
&\int_\Omega |u_n|^p\,\mathrm{d}x-\int_\Omega |u_n-u|^p\,\mathrm{d}x=\int_\Omega |u|^p\,\mathrm{d}x+o_n(1),\\
&\int_{\partial\Omega} |u_n|^{p_*}\,\mathrm{d}\sigma-\int_{\partial\Omega} |u_n-u|^{p_*}\,\mathrm{d}\sigma=\int_{\partial\Omega} |u|^{p_*}\,\mathrm{d}\sigma+o_n(1),
\end{aligned}
\tag{3.11}
$$

as $n \to \infty$. By proceeding as in Arora–Crespo-Blanco–Winkert [4, pp. 34/35], we obtain the following inequality

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\,\mathrm{d}x \\
&+ \int_\Omega \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right]|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x \\
&\quad + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x - \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\log(e+|\nabla u|)\,\mathrm{d}x.
\end{aligned} \tag{3.12}$$

Now, due to Young's inequality, we get

$$\begin{aligned}
\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x &\geq \frac{1}{p}\int_\Omega |u_n|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x, \\
\int_{\partial\Omega} |u_n|^{p_*-2}u_n(u_n-u)\,\mathrm{d}\sigma &\geq \frac{1}{p_*}\int_{\partial\Omega} |u_n|^{p_*}\,\mathrm{d}\sigma - \frac{1}{p_*}\int_{\partial\Omega} |u|^{p_*}\,\mathrm{d}\sigma.
\end{aligned} \tag{3.13}$$

Therefore, combining (3.10), (3.11), (3.12), and (3.13) it follows that

$$\begin{aligned}
o_n(1) &= \langle \Psi_\lambda'(u_n), u_n - u\rangle \\
&= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right]|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x \\
&\quad + \int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x - \lambda\int_{\partial\Omega}|u_n|^{\alpha-2}u_n(u_n-u)\,\mathrm{d}\sigma \\
&\quad - \int_{\partial\Omega}|u_n|^{p_*-2}u_n(u_n-u)\,\mathrm{d}\sigma \\
&\geq \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega|\nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x \\
&\quad - \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\log(e+|\nabla u|)\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u_n|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x \\
&\quad - \lambda\int_{\partial\Omega}|u_n|^{\alpha-2}u_n(u_n-u)\,\mathrm{d}\sigma - \frac{1}{p_*}\int_{\partial\Omega}|u_n|^{p_*}\,\mathrm{d}\sigma + \frac{1}{p_*}\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\sigma \\
&= \frac{1}{p}\int_\Omega |\nabla u_n - \nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n - \nabla u|^q\log(e+|\nabla u_n - \nabla u|)\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |u_n-u|^p\,\mathrm{d}x - \lambda\int_{\partial\Omega}|u_n|^{\alpha-2}u_n(u_n-u)\,\mathrm{d}\sigma - \frac{1}{p_*}\int_{\partial\Omega}|u_n-u|^{p_*}\,\mathrm{d}\sigma
\end{aligned}$$

$$\geq \frac{1}{p_*}\rho(u_n-u) - \frac{1}{p_*}\|u_n-u\|_{p_*,\partial\Omega}^{p_*} + o_n(1)$$

as $n\to\infty$. Consequently, by (3.10) and (3.11), we infer that

$$\vartheta^{p_*} + o_n(1) \geq \rho(u_n-u) \geq \int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x + \int_\Omega |u_n-u|^p \,\mathrm{d}x, \tag{3.14}$$

as $n\to\infty$.

We suppose now that $\vartheta>0$. Then, using Proposition 2.1 (i), together with (3.7) and (3.14), we obtain $\vartheta^{p_*}\geq S\vartheta^p$, which implies the estimate

$$\vartheta \geq S^{\frac{1}{p_*-p}}. \tag{3.15}$$

For any $n\in\mathbb{N}$, by applying Lemma 2.3, we obtain

$$\begin{aligned}
&\Psi_\lambda(u_n) - \frac{1}{\kappa+q}\langle \Psi_\lambda'(u_n),u_n\rangle \\
&= \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |u_n|^p\,\mathrm{d}x - \frac{\lambda}{\alpha}\|u_n\|_{\alpha,\partial\Omega}^{\alpha} - \frac{1}{p_*}\|u_n\|_{p_*,\partial\Omega}^{p_*} \\
&\quad - \frac{1}{\kappa+q}\Bigg[\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega \mu(x)\left(\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right)|\nabla u_n|^q\,\mathrm{d}x \\
&\qquad\qquad + \int_\Omega |u_n|^p\,\mathrm{d}x - \lambda\|u_n\|_{\alpha,\partial\Omega}^{\alpha} - \|u_n\|_{p_*,\partial\Omega}^{p_*}\Bigg] \\
&\geq \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q\log(e+|\nabla u_n|)\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |u_n|^p\,\mathrm{d}x - \frac{\lambda}{\alpha}\|u_n\|_{\alpha,\partial\Omega}^{\alpha} - \frac{1}{p_*}\|u_n\|_{p_*,\partial\Omega}^{p_*} \\
&\quad - \frac{1}{\kappa+q}\Bigg[\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \left(\frac{q+\kappa}{q}\right)\int_\Omega \mu(x)\log(e+|\nabla u_n|)|\nabla u_n|^q\,\mathrm{d}x \\
&\qquad\qquad + \int_\Omega |u_n|^p\,\mathrm{d}x - \|u_n\|_{\alpha,\partial\Omega}^{\alpha} - \|u_n\|_{p_*,\partial\Omega}^{p_*}\Bigg] \\
&= \left(\frac{1}{p} - \frac{1}{\kappa+q}\right)\|\nabla u_n\|_p^p + \left(\frac{1}{p} - \frac{1}{\kappa+q}\right)\|u_n\|_p^p \\
&\quad - \lambda\left(\frac{1}{\alpha} - \frac{1}{\kappa+q}\right)\|u_n\|_{\alpha,\partial\Omega}^{\alpha} + \left(\frac{1}{\kappa+q} - \frac{1}{p_*}\right)\|u_n\|_{p_*,\partial\Omega}^{p_*}.
\end{aligned}$$

Therefore, as $n\to\infty$, by (2.3), (3.10), (3.11), and using Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
c &\geq \left(\frac{1}{q+\kappa} - \frac{1}{p_*}\right)\left(\vartheta^{p_*} + \|u\|_{p_*,\partial\Omega}^{p_*}\right) - \lambda\left(\frac{1}{\alpha} - \frac{1}{q+\kappa}\right)\|u\|_{\alpha,\partial\Omega}^{\alpha} \\
&\geq \left(\frac{1}{q+\kappa} - \frac{1}{p_*}\right)\left(\vartheta^{p_*} + \|u\|_{p_*,\partial\Omega}^{p_*}\right) - \lambda\left(\frac{1}{\alpha} - \frac{1}{q+\kappa}\right)|\partial\Omega|^{\frac{p_*-\alpha}{p_*}}\|u\|_{p_*,\partial\Omega}^{\alpha}
\end{aligned}$$

$$\geq \left(\frac{1}{q+\kappa}-\frac{1}{p_*}\right)\left(\vartheta^{p_*}+\|u\|_{p_*,\partial\Omega}^{p_*}\right)-\left(\frac{1}{q+\kappa}-\frac{1}{p_*}\right)\|u\|_{p_*,\partial\Omega}^{p_*}$$
$$-|\partial\Omega|\left(\frac{1}{q+\kappa}-\frac{1}{p_*}\right)^{-\frac{\alpha}{p_*-\alpha}}\left[\lambda\left(\frac{1}{\alpha}-\frac{1}{q+\kappa}\right)\right]^{\frac{p_*}{p_*-\alpha}}.$$

Finally, by (3.15) we get

$$\begin{aligned} 0 &> c \\ &\geq \left(\frac{1}{q+\kappa}-\frac{1}{p_*}\right)S^{\frac{p_*}{p_*-p}}-|\partial\Omega|\left(\frac{1}{q+\kappa}-\frac{1}{p_*}\right)^{-\frac{\alpha}{p_*-\alpha}}\left[\lambda\left(\frac{1}{\alpha}-\frac{1}{q+\kappa}\right)\right]^{\frac{p_*}{p_*-\alpha}}>0, \end{aligned}$$

where the last inequality follows from (3.8). This leads to a contradiction and thus implies that $\vartheta=0$. Hence, from (3.14) it follows that $\rho(u_n-u)\to 0$ as $n\to\infty$. Consequently, the result follows by Proposition 2.2 (v). □

As pointed out in the introduction, the energy functional $\Psi_\lambda\colon W^{1,\mathcal{H}_{\log}}(\Omega)\to\mathbb{R}$ is not bounded from below. To overcome this issue, we employ a truncation technique to obtain a lower bound for a suitably defined truncated functional associated with Ψ_λ. Next, we consider $u\in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that $\|u\|\leq 1$. From (2.1) and Proposition 2.2 (iii), it follows that

$$\begin{aligned} \Psi_\lambda(u) &\geq \frac{1}{q}\|u\|^{q+\kappa}-\frac{\lambda}{\alpha}C_\alpha\|u\|^{\alpha}-\frac{1}{p_*}C_{p_*}\|u\|^{p_*} \\ &= C_1\|u\|^{\alpha}(C_2\|u\|^{q+\kappa-\alpha}-C_3\|u\|^{p_*-\alpha}-\lambda)=h_\lambda(\|u\|), \end{aligned} \tag{3.16}$$

where

$$C=\max\{C_\alpha,C_{p_*}\},\quad C_1=\frac{C}{\alpha},\quad C_2=\frac{\alpha}{Cq},\quad C_3=\frac{\alpha}{p_*}.$$

We now study the behavior of $h_\lambda\colon[0,\infty)\to\mathbb{R}$. To this end, we define

$$\beta(t)=C_2t^{q+\kappa-\alpha}-C_3t^{p_*-\alpha}.$$

Note that β attains its maximum at $t_*=\left[\frac{(q+\kappa-\alpha)C_2}{(p_*-\alpha)C_3}\right]^{\frac{1}{p_*-(q+\kappa)}}$ and the maximum value is given by

$$\overline{\lambda}:=\beta(t_*)=C_2\left(\frac{(q+\kappa-\alpha)C_2}{(p_*-\alpha)C_3}\right)^{\frac{q+\kappa-\alpha}{p_*-(q+\kappa)}}\frac{p_*-q-\kappa}{p_*-\alpha}>0,$$

due to $\alpha<q+\kappa<p_*$, see (H). Then, for any $\lambda\in(0,\overline{\lambda})$, the function h_λ has exactly two positive roots, $R_1(\lambda)$ and $R_2(\lambda)$ satisfying $0<R_1(\lambda)<t_*<R_2(\lambda)$. Moreover,

$$R_1(\lambda)\to 0\quad\text{as }\lambda\to 0 \tag{3.17}$$

and

$$h_\lambda(t)\begin{cases} >0 & \text{if } t\in(R_1(\lambda),R_2(\lambda)),\\ <0 & \text{if } t\in(0,R_1(\lambda))\cup(R_2(\lambda),\infty). \end{cases} \tag{3.18}$$

By (3.17), there exists a sufficiently small constant $\widetilde{\lambda}>0$ such that for all $\lambda\in(0,\widetilde{\lambda})$, we have

$$R_1(\lambda)^p<\min\left\{\frac{1}{2q},\frac{t_*^{q+\kappa}}{2q}\right\}. \tag{3.19}$$

We define

$$\lambda_*:=\min\{\lambda_0,\overline{\lambda},\widetilde{\lambda}\}.$$

Next, for each $\lambda \in (0, \lambda_*)$, we introduce the truncated functional $\widehat{\Psi}_\lambda \colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ given by

$$\widehat{\Psi}_\lambda(u) = J_1(u) - \tau(J_1(u)) \left[\frac{\lambda}{\alpha} \int_{\partial\Omega} |u|^\alpha \, \mathrm{d}\sigma + \frac{1}{p_*} \int_{\partial\Omega} |u|^{p_*} \, \mathrm{d}\sigma \right],$$

where

$$J_1(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{q} \int_\Omega \mu(x) |\nabla u|^q \log(e + |\nabla u|) \, \mathrm{d}x + \frac{1}{p} \int_\Omega |u|^p \, \mathrm{d}x,$$

and $\tau \in C_c^\infty(\mathbb{R})$ is a smooth cutoff function defined by

$$\tau(t) = \begin{cases} 1 & \text{if } |t| \leq R_1(\lambda)^p, \\ 0 & \text{if } |t| \geq 2R_1(\lambda)^p. \end{cases}$$

Clearly, $0 \leq \tau \leq 1$. It is easy to verify that $\widehat{\Psi}_\lambda \in C^1(W^{1,\mathcal{H}_{\log}}(\Omega), \mathbb{R})$ and that

$$\Psi_\lambda(u) \leq \widehat{\Psi}_\lambda(u) \quad \text{for all } u \in W^{1,\mathcal{H}_{\log}}(\Omega). \tag{3.20}$$

Moreover, for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$,

$$\widehat{\Psi}_\lambda(u) = \begin{cases} J_1(u) & \text{if } J_1(u) \geq 2R_1(\lambda)^p, \\ \Psi_\lambda(u) & \text{if } J_1(u) \leq R_1(\lambda)^p. \end{cases} \tag{3.21}$$

Lemma 3.3. *Let* (H) *be satisfied and* $\lambda \in (0, \lambda_*)$. *Then the following hold:*

(i) *If* $\widehat{\Psi}_\lambda(u) < 0$, *then* $\Psi_\lambda(u) = \widehat{\Psi}_\lambda(u)$;
(ii) $\widehat{\Psi}_\lambda$ *satisfies the* $(\mathrm{PS})_c$ *condition for* $c < 0$.

Proof. Let $\lambda \in (0, \lambda_*)$ and suppose that $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ is such that $\widehat{\Psi}_\lambda(u) < 0$. We first claim that $\|u\| < 1$. Indeed, assume by contradiction that $\|u\| \geq 1$, then, by Proposition 2.2 (ii), it follows that $\rho(u) \geq 1$. Hence, we obtain

$$\begin{aligned} J_1(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{q} \int_\Omega \mu(x) |\nabla u|^q \log(e + |\nabla u|) \, \mathrm{d}x + \frac{1}{p} \int_\Omega |u|^p \, \mathrm{d}x \\ &\geq \frac{1}{q} \rho(u) \geq 2R_1(\lambda)^p, \end{aligned}$$

by (3.19). Then $\tau(J_1(u)) = 0$ gives a contradiction to (3.21) and $\widehat{\Psi}_\lambda(u) < 0$. Moreover, from (3.20) and (3.16) it follows that $h_\lambda(\|u\|) < 0$. Then, by (3.18), we have either $\|u\| < R_1(\lambda)$ or $\|u\| > R_2(\lambda) > t_*$. We now prove that the second inequality cannot hold. Indeed, assuming $\|u\| > R_2(\lambda) > t_*$ and using Proposition 2.2, we obtain

$$J_1(u) \geq \frac{1}{q} \rho(u) \geq \frac{1}{q} \|u\|^{q+\kappa} \geq \frac{1}{q} t_*^{q+\kappa} \geq 2R_1(\lambda)^p,$$

by (3.19). This contradicts the fact that $\widehat{\Psi}_\lambda(u) < 0$ since, by (3.21), we have $\widehat{\Psi}_\lambda(u) = J_1(u)$ for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that $J_1(u) \geq 2R_1(\lambda)^p$. Therefore, we must have $\|u\| < R_1(\lambda)$. Applying Proposition 2.2 (iii), it follows that

$$J_1(u) \leq \frac{1}{p} \rho(u) \leq \frac{1}{p} \|u\|^p \leq R_1(\lambda)^p.$$

The assertion (ii) follows directly from Lemma 3.2. This completes the proof. □

Lemma 3.4. *Let* (H) *be satisfied and* $\lambda \in (0, \lambda_*)$. *Then, for any* $n \in \mathbb{N}$ *there exists* $\varepsilon = \varepsilon(\lambda, n) > 0$ *such that*

$$\gamma\left(\widehat{\Psi}_\lambda^{-\varepsilon}\right) \geq n,$$

where $\widehat{\Psi}_\lambda^{-\varepsilon} = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon \widehat{\Psi}_\lambda(u) \leq -\varepsilon\right\}$.

Proof. Let $\lambda \in (0, \lambda_*)$ and $n \in \mathbb{N}$ be fixed and let Y_n be an n-dimensional subspace of $W^{1,\mathcal{H}_{\log}}(\Omega)$. Since the norms of $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $L^\alpha(\partial\Omega)$ are equivalent on Y_n, we can choose a constant $\delta_n > R_1(\lambda)^{-p} > 1$ large enough such that

$$\delta_n^{-1}\|u\|^\alpha \leq \|u\|_{\alpha,\partial\Omega}^\alpha \quad \text{for all } u \in Y_n.$$

Thus, for any $u \in Y_n$ such that $\|u\|^p \leq \delta_n^{-1} < 1$, it follows from Proposition 2.2 and the inequality above that

$$\widehat{\Psi}_\lambda(u) \leq \frac{1}{p}\|u\|^p - \frac{\lambda}{\alpha}\delta_n^{-1}\|u\|^\alpha. \tag{3.22}$$

Next, we take two positive constants r and R such that

$$r < R < \min\left\{\delta_n^{-1}, \left(\frac{\lambda\delta_n^{-1}p}{\alpha}\right)^{\frac{1}{p-\alpha}}\right\}. \tag{3.23}$$

Define

$$\mathbb{S}_n = \{u \in Y_n \colon \|u\| = r\}.$$

Note that $\mathbb{S}_n$ is homeomorphic to the $(n-1)$-dimensional sphere S^{n-1}. Hence, by Proposition 2.6 (iv) we have $\gamma(\mathbb{S}_n) = n$. Moreover, from (3.22) and (3.23), it follows that, for $u \in \mathbb{S}_n$,

$$\widehat{\Psi}_\lambda(u) \leq r^\alpha\left(\frac{1}{p}r^{p-\alpha} - \frac{\lambda}{\alpha}\delta_n^{-1}\right) \leq R^\alpha\left(\frac{1}{p}R^{p-\alpha} - \frac{\lambda}{\alpha}\delta_n^{-1}\right) < 0.$$

Therefore, we can find $\varepsilon > 0$ such that $\widehat{\Psi}_\lambda(u) < -\varepsilon$ for any $u \in \mathbb{S}_n$. Thus, $\mathbb{S}_n \subset \widehat{\Psi}_\lambda^{-\varepsilon}$. It then follows from Proposition 2.6 (i) that

$$\gamma\left(\widehat{\Psi}_\lambda^{-\varepsilon}\right) \geq \gamma(\mathbb{S}_n) = n.$$

□

Next, for each $n \in \mathbb{N}$, we define the sets

$$\Sigma_n = \left\{A \subset W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\} \colon A \text{ is closed, } A = -A \text{ and } \gamma(A) \geq n\right\},$$
$$K_c = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon \widehat{\Psi}_\lambda'(u) = 0 \text{ and } \widehat{\Psi}_\lambda(u) = c\right\}$$

and the corresponding minimax level

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \widehat{\Psi}_\lambda(u).$$

It is easy to see that $c_n \leq c_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 3.5. *Let* (H) *be satisfied and* $\lambda \in (0, \lambda_*)$. *Then, for every* $n \in \mathbb{N}$, *the level* c_n *satisfies*

$$-\infty < c_n < 0.$$

Proof. Let $n \in \mathbb{N}$ be fixed. By Lemma 3.4 there exists $\varepsilon > 0$ such that $\gamma(\widehat{\Psi}_\lambda^{-\varepsilon}) \geq n$. Since $\widehat{\Psi}_\lambda$ is even and continuous, it follows that $\widehat{\Psi}_\lambda^{-\varepsilon} \in \Sigma_n$. Moreover, as $\widehat{\Psi}_\lambda(0) = 0$ we have $0 \notin \widehat{\Psi}_\lambda^{-\varepsilon}$. Using the fact that $\sup_{u \in \widehat{\Psi}_\lambda^{-\varepsilon}} \widehat{\Psi}_\lambda(u) \leq -\varepsilon$ and that $\widehat{\Psi}_\lambda$ is bounded from below, we conclude that

$$-\infty < c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \widehat{\Psi}_\lambda(u) \leq \sup_{u \in \widehat{\Psi}_\lambda^{-\varepsilon}} \widehat{\Psi}_\lambda(u) \leq -\varepsilon < 0.$$

□

The following lemma is crucial to prove the multiplicity result.

Lemma 3.6. *Let* (H) *be satisfied,* $\lambda \in (0, \lambda_*)$, *and* $n \in \mathbb{N}$. *If*

$$c = c_n = c_{n+1} = \ldots = c_{n+\ell}$$

for some $\ell \in \mathbb{N}$, *then*

$$\gamma(K_c) \geq \ell + 1.$$

Proof. From Lemma 3.5 we know that $c = c_n = c_{n+1} = \ldots = c_{n+\ell}$ is negative. From this and Lemma 3.3 (ii), it follows that the set K_c is compact.

Assume that $\gamma(K_c) \leq \ell$. Then, by Proposition 2.6 (v) we can find $\delta > 0$ such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \leq \ell$, where

$$N_\delta(K_c) = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon \operatorname{dist}(u, K_c) \leq \delta\right\}.$$

By a standard deformation lemma (see Benci [9, Theorem 3.4]), there exist $\varepsilon \in (0, -c)$ and an odd homeomorphism $\eta \colon W^{1,\mathcal{H}_{\log}}(\Omega) \to W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\eta\left(\widehat{\Psi}_\lambda^{c+\varepsilon} \setminus N_\delta(K_c)\right) \subset \widehat{\Psi}_\lambda^{c-\varepsilon}. \tag{3.24}$$

On the other hand, by the definition of $c = c_{n+\ell}$ there exists $A \in \Sigma_{n+\ell}$ such that $\sup_{u \in A} \widehat{\Psi}_\lambda(u) < c + \varepsilon$, that is $A \subset \widehat{\Psi}_\lambda^{c+\varepsilon}$. Hence, from (3.24), it follows that

$$\eta\left(A \setminus N_\delta(K_c)\right) \subset \eta\left(\widehat{\Psi}_\lambda^{c+\varepsilon} \setminus N_\delta(K_c)\right) \subset \widehat{\Psi}_\lambda^{c-\varepsilon}. \tag{3.25}$$

Taking Proposition 2.6 (i), (iii) into account we obtain

$$\gamma(\eta(\overline{A \setminus N_\delta(K_c)})) \geq \gamma(\overline{A \setminus N_\delta(K_c)}) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq n.$$

Thus, we have that $\eta(\overline{A \setminus N_\delta(K_c)}) \in \Sigma_n$ and so

$$\sup_{u \in \eta(\overline{A \setminus N_\delta(K_c)})} \widehat{\Psi}_\lambda(u) \geq c_n = c,$$

contradicting (3.25). □

Now, we are in the position to give the proof of theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.5 we know that $c_n < 0$. Hence, from Lemma 3.3 (ii), it follows that the functional $\widehat{\Psi}_\lambda$ satisfies the Palais–Smale condition at level $c_n < 0$. Thus, c_n is a critical value of $\widehat{\Psi}_\lambda$ for any $n \in \mathbb{N}$, see Rabinowitz [40].

We distinguish two possible situations. If

$$-\infty < c_1 < c_2 < \ldots < c_n < c_{n+1} < \ldots,$$

then $\widehat{\Psi}_\lambda$ has infinitely many distinct critical values. If there exist $n, \ell \in \mathbb{N}$ such that

$$c_n = c_{n+1} = \ldots = c_{n+\ell} = c,$$

then $\gamma(K_c) \geq \ell+1 \geq 2$ by Lemma 3.6. Consequently, the set K_c has infinitely many points, see Rabinowitz [40, Remark 7.3], which are infinitely many critical values for $\widehat{\Psi}_\lambda$ by Lemma 3.3 (ii). It then follows from Lemma 3.3 (i) that all these critical values correspond to negative critical values of $\Psi_\lambda = \widehat{\Psi}_\lambda$ as well. Therefore, problem (1.6) admits infinitely many weak solutions. □

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