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Strongly singular problems involving the $(1, q)$-Laplacian

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Abstract
In this paper, we study a strongly singular Dirichlet problem driven by the $(1, q)$-Laplacian of the form

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) - \Delta_q u = \frac{f(x)}{\left(u^+\right)^\gamma} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $1 < q < N$, $\gamma > 1$, and $f \in L^N(\Omega)$ with $f(x) > 0$ for a.e. $x \in \Omega$, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. The interplay between the degenerate 1-Laplacian and a reaction term that blows up at the origin creates significant analytical challenges and precludes the use of classical variational methods. By means of an approximation scheme based on (p, q)-Laplacian problems, we establish the existence of a positive solution to the problem above.

Keywords $(1, q)$-Laplacian · Approximation methods · Degenerate elliptic equations · $p \to 1^+$ asymptotics · Strongly singular problems

Mathematics Subject Classification 35J25 · 35J75

1 Introduction

In recent years, increasing attention has been devoted to elliptic equations driven by operators with nonstandard growth, motivated both by their intrinsic mathematical

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interest and by their relevance in models arising in physics, image processing, material sciences, and geometric analysis. Among these operators, variational problems involving the 1-Laplacian,

$$\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

have received particular interest due to their close connection with functions of bounded variation, least gradient problems, and total variation minimization. The systematic study of the 1-Laplacian began with the pioneering works of Andreu–Ballester–Caselles–Mazón [1–3] and was further developed by Demengel [10]. The interest in this operator stems from its highly degenerate and nonsmooth nature, as well as from its numerous applications, including torsion problems (see, for instance, Kawohl [18]) and image restoration, where total variation-based models such as those introduced by Chen–Levine–Rao [8] and Rudin–Osher–Fatemi [23] play a central role.

The natural functional setting for problems involving the 1-Laplacian is the space $\mathrm{BV}(\Omega)$. However, the lack of reflexivity of $\mathrm{BV}(\Omega)$ and the absence of a classical Euler-Lagrange structure lead to substantial analytical difficulties. These challenges require a refined framework based on the pairing theory of Anzellotti [4]. On the other hand, elliptic problems driven by the (p, q)-Laplacian operator

$$\Delta_p u + \Delta_q u = \operatorname{div}[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u],$$

have been extensively investigated due to their rich mathematical structure and their wide range of applications, including models in biophysics (see Fife [13]), plasma physics (see Struwe [26]), and reaction-diffusion phenomena (see Aris [5]).

To the best of our knowledge, the first work addressing problems involving a $(1, q)$-Laplacian operator is due to Figueiredo–Pimenta–Winkert [14]. In that paper, the authors studied the asymptotic behavior, as $p \to 1^+$, of solutions to the (p, q)-Laplacian problem

$$-\Delta_p u - \Delta_q u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where f is a Carathéodory function with superlinear and subcritical growth. They established the existence of two constant-sign solutions and one sign-changing solution. More recently, weakly singular nonlinearities for problems such as (1.1) were considered by Figueiredo–Pimenta–Stapenhorst–Winkert [15], in the special case in which $0 < \gamma < 1$.

Singular elliptic equations have attracted considerable attention since the seminal work of Crandall–Rabinowitz–Tartar [9]. Problems involving strongly singular nonlinearities emerge naturally in models describing diffusion phenomena with absorption and reaction terms in population dynamics. These nonlinearities produce blow-up effects near the zero set of the solution, implying that standard weak formulations are not directly applicable and that the existence of solutions cannot be obtained via classical variational methods. The interplay between such singular behavior and

degenerate operators leads to a highly delicate analytical setting, where positivity, regularity, and integrability issues must be handled simultaneously. For strongly singular problems involving the classical Laplacian, we refer to the works of Lazer–McKenna [19], Boccardo–Orsina [7], and Sun–Zhang [24]. Concerning singular and strongly singular problems driven by the 1-Laplacian operator, we mention the contributions of De Cicco–Giachetti–Segura de León [11], Ortiz Chata–Pimenta–Segura de León [21], Pimenta–Carranza–Figueiredo [22], Giachetti–Oliva–Petitta [16] and De Cicco–Giachetti–Oliva–Petitta [12], see also the references therein. In the latter work, the authors investigated existence and regularity results for the strongly singular problem

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = h(u)f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $0 \leq f(\cdot) \in L^{N,\infty}(\Omega)$, and $h\colon [0,\infty) \to [0,\infty)$ is a continuous function, bounded at infinity and possibly singular at 0.

Motivated by the contributions of De Cicco–Giachetti–Oliva–Petitta [12] and Figueiredo–Pimenta–Winkert [14], in this paper we study the following strongly singular $(1, q)$-Laplacian problem

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) - \Delta_q u = \frac{f(x)}{\left(u^+\right)^\gamma} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $1 < q < N$, $\gamma > 1$, and $f \in L^N(\Omega)$ with $f(x) > 0$ for a.e. $x \in \Omega$. We say that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is a weak solution of (1.2) if $u > 0$ a.e. in Ω,

$$u^{\frac{\gamma-1+q}{q}} \in W^{1,q}_0(\Omega) \quad \text{and} \quad f u^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega),$$

and if there exists a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that

$$-\operatorname{div}\mathbf{z} - \Delta_q u = f(x)u^{-\gamma} \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$\mathbf{z} \cdot \nabla u = |\nabla u| \quad \text{a.e. in } \Omega.$$

Our main result is the following one.

Theorem 1.1 *Let $1 < q < N$, $\gamma > 1$, and $f \in L^N(\Omega)$ with $f(x) > 0$ for a.e. $x \in \Omega$. Then problem (1.2) admits a positive weak solution u. In particular,*

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad u^{\frac{\gamma-1+q}{q}} \in W^{1,q}_0(\Omega).$$

The proof of Theorem 1.1 relies on an approximation scheme inspired by the technique developed by Molino Salas–Segura de León [20], whereby solutions of (1.2) are obtained as limits of solutions to associated (p, q)-Laplacian problems as $p \to 1^+$. The main analytical arguments are strongly influenced by those introduced in De Cicco–Giachetti–Oliva–Petitta [12].

Finally, we emphasize that the present paper is devoted to the strongly singular regime $\gamma > 1$. In this case, testing the approximate equation directly with u_p would produce terms involving $u_p^{1-\gamma}$, and hence a negative power of u_p. This prevents us from deriving the global estimates available in the weakly singular regime and requires the use of suitable power-type test functions, truncation arguments, and local estimates. By contrast, the case $0 < \gamma < 1$ was recently studied by some of the authors in [15], where stronger global estimates are available. The borderline case $\gamma = 1$ does not exhibit the same negative-power obstruction and lies outside the strongly singular regime considered here. We therefore restrict the present work to $\gamma > 1$. We also retain the assumption $f \in L^N(\Omega)$ because, in our approach, it provides the uniform L^∞-bound for the approximate solutions through Stampacchia's argument, which is subsequently used to justify the power-type test functions.

2 Approximate singular problems

We assume from now on that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. For any measurable set $E \subset \mathbb{R}^N$, we denote by $\mathcal{H}^{N-1}(E)$ the $(N-1)$-dimensional Hausdorff measure of E and by $|E|$ its Lebesgue measure. Moreover, we denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω, and by $\mathcal{M}(\Omega)$ the space of Radon measures on Ω. A fundamental tool in our analysis is truncation, which allows us to localize arguments and control possible singular behaviors of the solutions. Given $k > 0$, let us define

$$T_k(s) = \begin{cases} -k, & s < -k, \\ s, & |s| \leq k, \\ k, & s > k, \end{cases} \qquad G_k(s) = \begin{cases} s + k, & s < -k, \\ 0, & |s| \leq k, \\ s - k, & s > k. \end{cases}$$

In this section, we study the existence and regularity of solutions to the following (p, q)-Laplacian problem

$$-\Delta_p u_p - \Delta_q u_p = \frac{f(x)}{\left(u_p^+\right)^\gamma} \quad \text{in } \Omega, \quad u_p = 0 \quad \text{on } \partial\Omega, \tag{P_p}$$

where $\gamma > 1$ and $1 < p < q < N$. A function $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is said to be a weak solution of problem (P_p), if $u > 0$ a.e. in Ω, $fu^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$, and, for all $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega a(\nabla u) \cdot \nabla\varphi \, \mathrm{d}x = \int_\Omega f u^{-\gamma} \varphi \, \mathrm{d}x, \tag{2.1}$$

where $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$a(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi.$$

The existence of solutions to (P_p) is obtained via a standard approximation argument (see, for instance, De Cicco–Giachetti–Oliva–Petitta [12] or Figueiredo–Pimenta–Stapenhorst–Winkert [15] for details), which we include here for the sake of completeness. For each $n \in \mathbb{N}$, we consider the approximate problem

$$-\Delta_p u_n - \Delta_q u_n = \frac{f_n(x)}{\left(|u_n| + \frac{1}{n}\right)^{\gamma}} \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where $f_n(x) = T_n(f(x))$. A function $u_n \in W_0^{1,q}(\Omega)$ is called a weak solution of (2.2), if for all $\varphi \in W_0^{1,q}(\Omega)$,

$$\int_\Omega a(\nabla u_n) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f_n(x) \left(|u_n| + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x. \tag{2.3}$$

Theorem 2.1 *Let $\gamma > 1$ and $1 < p < q < N$. For each $n \in \mathbb{N}$, there exists a nonnegative weak solution $u_n \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ of problem (2.2). Moreover, there exists a constant $C > 0$, independent of p, such that*

$$\|u_n\|_\infty \le C \quad \textit{for all } n \in \mathbb{N}. \tag{2.4}$$

Proof For each $n \in \mathbb{N}$, the existence of a weak solution $u_n \in W_0^{1,q}(\Omega)$ of (2.2) follows along the very same lines as in the paper by Figueiredo–Pimenta–Stapenhorst–Winkert [15, Theorem 3.1].

Next, by using Stampacchia's truncation argument, we show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. To this end, let $k > 0$ and let $G_k(u_n) = (u_n - k)^+$ be a test function in (2.3). This gives

$$\int_\Omega |\nabla G_k(u_n)|^q \,\mathrm{d}x \le \int_\Omega f(x) \left(u_n + \frac{1}{n}\right)^{-\gamma} G_k(u_n) \,\mathrm{d}x \le k^{-\gamma} \int_\Omega f(x) G_k(u_n) \,\mathrm{d}x.$$

By the Sobolev embedding theorem and Hölder's inequality, we obtain

$$\mathcal{S}_q^{-q} \left(\int_\Omega |G_k(u_n)|^{q^*} \,\mathrm{d}x\right)^{\frac{q}{q^*}} \le k^{-\gamma} \left(\int_{A_k} f(x)^{(q^*)'} \,\mathrm{d}x\right)^{\frac{1}{(q^*)'}} \left(\int_\Omega |G_k(u_n)|^{q^*} \,\mathrm{d}x\right)^{\frac{1}{q^*}},$$

where $\mathcal{S}_q$ is the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ and $A_k = \{x \in \Omega\colon u_n(x) > k\}$. Thus,

$$\left(\int_\Omega |G_k(u_n)|^{q^*} \,\mathrm{d}x\right)^{\frac{q-1}{q^*}} \le \mathcal{S}_q^q k^{-\gamma} \left(\int_{A_k} f(x)^{(q^*)'} \,\mathrm{d}x\right)^{\frac{1}{(q^*)'}}.$$

Since $f \in L^N(\Omega)$, Hölder's inequality implies that

$$\left(\int_\Omega |G_k(u_n)|^{q^*}\,\mathrm{d}x\right)^{\frac{q-1}{q^*}} \le \mathcal{S}_q^q k^{-\gamma}\left(\left(\int_\Omega f(x)^N\,\mathrm{d}x\right)^{\frac{(q^*)'}{N}} |A_k|^{1-\frac{(q^*)'}{N}}\right)^{\frac{1}{(q^*)'}}$$
$$\le \mathcal{S}_q^q \|f\|_{L^N(\Omega)} k^{-\gamma} |A_k|^{\frac{1}{(q^*)'}-\frac{1}{N}}$$

and therefore

$$\int_\Omega |G_k(u_n)|^{q^*}\,\mathrm{d}x \le \left(\mathcal{S}_q^q \|f\|_{L^N(\Omega)} k^{-\gamma}\right)^{\frac{q^*}{q-1}} |A_k|^{\left(\frac{1}{(q^*)'}-\frac{1}{N}\right)\left(\frac{q^*}{q-1}\right)}.$$

Since $A_h \subset A_k$ and $G_k(u_n) \ge h-k$ in A_h for all $h > k$, one has

$$|A_h| \le \frac{1}{(h-k)^{q^*}}\left(\mathcal{S}_q^q \|f\|_{L^N(\Omega)} k^{-\gamma}\right)^{\frac{q^*}{q-1}} |A_k|^{\left(\frac{1}{(q^*)'}-\frac{1}{N}\right)\left(\frac{q^*}{q-1}\right)} \quad \text{for all } h > k > 0.$$

From Lemma 4.1 of Stampacchia [25], applied for $k \ge 1$, it follows that there exists $d > 0$ depending only on $q, \gamma, \|f\|_{L^N(\Omega)}$ and Ω, such that $|A_d| = 0$. Hence, $\|u_n\|_\infty \le d$. □

Lemma 2.2 *There exists $p_0 > 1$ such that, for all $\omega \subset\subset \Omega$, there exists a constant $C = C(\|f\|_{L^N(\Omega)}, \gamma, \omega, |\Omega|)$, independent of n and p, such that*

$$\int_\omega |\nabla u_n|^p\,\mathrm{d}x + \int_\omega |\nabla u_n|^q\,\mathrm{d}x < C, \quad \textit{for all } 1 < p < p_0 \textit{ and } n \in \mathbb{N}.$$

Proof We start by recalling that every $s \in \mathbb{R}$ can be decomposed as $s = T_k(s) + G_k(s)$. Using this identity, for any measurable subset $\omega \subset\subset \Omega$, we obtain

$$\int_\omega |\nabla u_n|^q\,\mathrm{d}x = \int_{\omega\cap\{|u_n|\le k\}} |\nabla u_n|^q\,\mathrm{d}x + \int_{\omega\cap\{|u_n|>k\}} |\nabla u_n|^q\,\mathrm{d}x$$
$$= \int_\omega |\nabla T_k(u_n)|^q\,\mathrm{d}x + \int_\omega |\nabla G_k(u_n)|^q\,\mathrm{d}x.$$

An analogous decomposition holds for the p-term $\int_\omega |\nabla u_n|^p\,\mathrm{d}x$. To estimate $\int_\omega |\nabla G_k(u_n)|^p\,\mathrm{d}x + \int_\omega |\nabla G_k(u_n)|^q\,\mathrm{d}x$, for $k > 0$, we take $G_k(u_n)$ as a test function in (2.3). This is admissible since $G_k(u_n) \in W_0^{1,q}(\Omega)$. Then we have

$$\int_\Omega |\nabla G_k(u_n)|^p\,\mathrm{d}x + \int_\Omega |\nabla G_k(u_n)|^q\,\mathrm{d}x$$
$$= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla G_k(u_n)\,\mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla G_k(u_n)\,\mathrm{d}x$$
$$= \int_\Omega f_n(x)\left(|u_n| + \frac{1}{n}\right)^{-\gamma} G_k(u_n)\,\mathrm{d}x.$$

Since $\operatorname{supp} G_k(u_n) \subset \{x \in \Omega : |u_n(x)| > k\}$, we have $|u_n| > k$ on the support of $G_k(u_n)$, which implies $(|u_n| + 1/n)^{-\gamma} \leq k^{-\gamma}$ and $f_n(x) \leq f(x)$. Applying Hölder's and Sobolev's inequalities, it follows that

$$\begin{aligned}\int_\Omega |\nabla G_k(u_n)|^p \,\mathrm{d}x &\leq \frac{1}{k^\gamma} \int_\Omega f(x) G_k(u_n) \,\mathrm{d}x \\ &\leq \frac{1}{k^\gamma} \|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)} \left(\int_\Omega |G_k(u_n)|^{p^*} \,\mathrm{d}x \right)^{\frac{1}{p^*}} \\ &\leq \frac{1}{k^\gamma} \|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)} S_p \left(\int_\Omega |\nabla G_k(u_n)|^p \,\mathrm{d}x \right)^{1/p},\end{aligned}$$

where S_p denotes the optimal Sobolev embedding constant. Consequently,

$$\left(\int_\Omega |\nabla G_k(u_n)|^p \,\mathrm{d}x \right)^{(p-1)/p} \leq \frac{1}{k^\gamma} S_p \|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)}. \tag{2.5}$$

A similar argument yields

$$\left(\int_\Omega |\nabla G_k(u_n)|^q \,\mathrm{d}x \right)^{(q-1)/q} \leq \frac{1}{k^\gamma} S_q \|f\|_{L^{\frac{q^*}{q^*-1}}(\Omega)}. \tag{2.6}$$

Hence,

$$\|G_k(u_n)\|_{W_0^{1,p}(\Omega)} \leq \left(\frac{1}{k^\gamma} S_p \|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)} \right)^{1/(p-1)} \tag{2.7}$$

and

$$\|G_k(u_n)\|_{W_0^{1,q}(\Omega)} \leq \left(\frac{1}{k^\gamma} S_q \|f\|_{L^{\frac{q^*}{q^*-1}}(\Omega)} \right)^{1/(q-1)}. \tag{2.8}$$

We thus obtain a global estimate for $G_k(u_n)$ provided $k > 0$ is chosen sufficiently large. On the other hand, $T_k(u_n)$ concerns the region of Ω where u_n is bounded. Therefore, the corresponding estimate will only be local.

For $\omega \subset\subset \Omega$, let $\varphi \in C_c^\infty(\Omega)$ be such that

$$\varphi = 1 \quad \text{in } \omega, \quad 0 \leq \varphi \leq 1 \quad \text{and} \quad |\nabla \varphi| \leq C_\omega,$$

where C_ω is a positive constant. Taking $T_k(u_n)\varphi^q \in W_0^{1,q}(\Omega)$ as test function in (2.3) gives

$$\begin{aligned}
&\int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma} T_k(u_n)\varphi^q \,\mathrm{d}x \\
&= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(T_k(u_n)\varphi^q)\,\mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(T_k(u_n)\varphi^q)\,\mathrm{d}x \\
&= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla T_k(u_n)\varphi^q + T_k(u_n)q\varphi^{q-1}\nabla\varphi)\,\mathrm{d}x \\
&\quad + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot (\nabla T_k(u_n)\varphi^q + T_k(u_n)q\varphi^{q-1}\nabla\varphi)\,\mathrm{d}x \\
&= \int_\Omega |\nabla T_k(u_n)|^p\varphi^q \,\mathrm{d}x + q\int_\Omega T_k(u_n)\varphi^{q-1}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x \\
&\quad + \int_\Omega |\nabla T_k(u_n)|^q\varphi^q \,\mathrm{d}x + q\int_\Omega T_k(u_n)\varphi^{q-1}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned}$$

Since $\nabla T_k(u_n) = 0$ whenever $|u_n| > k$, from the last equality it follows that

$$\begin{aligned}
&\int_\Omega (|\nabla T_k(u_n)|^p + |\nabla T_k(u_n)|^q)\varphi^q \,\mathrm{d}x \\
&\le \int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma} |T_k(u_n)|\varphi^q \,\mathrm{d}x + q\int_\Omega |T_k(u_n)|\varphi^{q-1}|\nabla u_n|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\quad + q\int_\Omega |T_k(u_n)|\varphi^{q-1}|\nabla u_n|^{q-1}|\nabla\varphi|\,\mathrm{d}x \\
&\le k\int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma}\varphi^q \,\mathrm{d}x + qk\int_\Omega \varphi^{q-1}|\nabla u_n|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\quad + qk\int_\Omega \varphi^{q-1}|\nabla u_n|^{q-1}|\nabla\varphi|\,\mathrm{d}x.
\end{aligned}$$

Observe that $|\nabla u_n|^{p-1} \le (|\nabla T_k(u_n)| + |\nabla G_k(u_n)|)^{p-1}$ and since the supports of $\nabla T_k(u_n)$ and $\nabla G_k(u_n)$ are a.e. disjoint, we have $|\nabla u_n|^{p-1} = |\nabla T_k(u_n)|^{p-1} + |\nabla G_k(u_n)|^{p-1}$ a.e. in ω. Then it follows that

$$\begin{aligned}
&\int_\Omega (|\nabla T_k(u_n)|^p + |\nabla T_k(u_n)|^q)\varphi^q \,\mathrm{d}x \\
&\le qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x + qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\quad + qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x + qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x \\
&\quad + k\int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma}\varphi^q \,\mathrm{d}x.
\end{aligned} \tag{2.9}$$

On the other hand, by taking $\varphi^q \in W_0^{1,q}(\Omega)$ as test function in (2.3), we obtain

$$\begin{aligned}&\int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma}\varphi^q\,\mathrm{d}x\\&= q\int_\Omega \varphi^{q-1}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + q\int_\Omega \varphi^{q-1}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x.\end{aligned}$$

From this equality, proceeding analogously to the previous estimate, we derive that

$$\begin{aligned}&k\int_\Omega f_n(x)\left(|u_n|+\frac{1}{n}\right)^{-\gamma}\varphi^q\,\mathrm{d}x\\&\le qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x + qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x\\&\quad + qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x + qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x.\end{aligned}\tag{2.10}$$

From (2.9) and (2.10), we deduce

$$\begin{aligned}&\int_\Omega (|\nabla T_k(u_n)|^p + |\nabla T_k(u_n)|^q)\varphi^q\,\mathrm{d}x\\&\le 2qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x + 2qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{p-1}|\nabla\varphi|\,\mathrm{d}x\\&\quad + 2qk\int_\Omega \varphi^{q-1}|\nabla T_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x + 2qk\int_\Omega \varphi^{q-1}|\nabla G_k(u_n)|^{q-1}|\nabla\varphi|\,\mathrm{d}x.\end{aligned}$$

Let $\eta, \xi > 0$ be parameters to be specified later. From Young's inequality

$$ab \le \eta^{p'}\frac{a^{p'}}{p'} + \frac{1}{\eta^p}\frac{b^p}{p},$$

it follows that

$$\begin{aligned}&\int_\Omega (|\nabla T_k(u_n)|^p + |\nabla T_k(u_n)|^q)\varphi^q\,\mathrm{d}x\\&\le (p-1)\eta^{p'}\int_\Omega \varphi^{(\frac{q-1}{p-1})p}|\nabla T_k(u_n)|^p\,\mathrm{d}x + \frac{(2kq)^p}{\eta^p}\int_\Omega |\nabla\varphi|^p\,\mathrm{d}x\\&\quad + (q-1)\xi^{q'}\int_\Omega \varphi^q|\nabla T_k(u_n)|^q\,\mathrm{d}x + \frac{(2kq)^q}{\xi^q}\int_\Omega |\nabla\varphi|^q\,\mathrm{d}x\\&\quad + (p-1)\eta^{p'}\int_\Omega \varphi^{(\frac{q-1}{p-1})p}|\nabla G_k(u_n)|^p\,\mathrm{d}x + \frac{(2kq)^p}{\eta^p}\int_\Omega |\nabla\varphi|^p\,\mathrm{d}x\\&\quad + (q-1)\xi^{q'}\int_\Omega \varphi^q|\nabla G_k(u_n)|^q\,\mathrm{d}x + \frac{(2kq)^q}{\xi^q}\int_\Omega |\nabla\varphi|^q\,\mathrm{d}x.\end{aligned}\tag{2.11}$$

Let η and ξ to be chosen such that

$$1-(p-1)\eta^{p'}=\frac{1}{2} \quad \text{and} \quad 1-(q-1)\xi^{q'}=\frac{1}{2}.$$

Since $0\le\varphi\le 1$, $q<(\frac{q-1}{p-1})p$ and $|\nabla\varphi|\le C_\omega$, from (2.5), (2.6) and (2.11), we get

$$\begin{aligned}
&\int_\Omega (|\nabla T_k(u_n)|^p+|\nabla T_k(u_n)|^q)\varphi^q\,\mathrm{d}x\\
&\le 2\Bigg[2^p(p-1)^{p-1}(2kq)^p\int_\Omega|\nabla\varphi|^p\,\mathrm{d}x\\
&\quad+2^q(q-1)^{q-1}(2kq)^q\int_\Omega|\nabla\varphi|^q\,\mathrm{d}x\\
&\quad+\frac{1}{2}\int_\Omega\varphi^q|\nabla G_k(u_n)|^p\,\mathrm{d}x+\frac{1}{2}\int_\Omega\varphi^q|\nabla G_k(u_n)|^q\,\mathrm{d}x\Bigg]\\
&\le 2\left[C(\omega,k)+\frac{1}{2}\left(\frac{1}{k^\gamma}S_p\|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)}\right)^{\frac{p}{p-1}}+\frac{1}{2}\left(\frac{1}{k^\gamma}S_q\|f\|_{L^{\frac{q^*}{q^*-1}}(\Omega)}\right)^{\frac{q}{q-1}}\right].
\end{aligned}$$

Taking into account that

$$\lim_{p\to 1^+}S_p\|f\|_{L^{\frac{p^*}{p^*-1}}(\Omega)}=S_1\|f\|_{L^N(\Omega)},$$

and choosing $k>0$ such that $\frac{1}{k^\gamma}S_1\|f\|_{L^N(\Omega)}<1$, we can find $p_0>1$ sufficiently close to 1, so that from the last inequality, one has

$$\int_\omega(|\nabla T_k(u_n)|^p+|\nabla T_k(u_n)|^q)\,\mathrm{d}x<C(\|f\|_{L^N(\Omega)},|\Omega|,\omega). \tag{2.12}$$

Finally, from (2.7), (2.8), and (2.12), there exists a constant $C(\|f\|_{L^N(\Omega)},|\Omega|,\omega)>0$, independent of n and p, such that

$$\int_\omega|\nabla u_n|^p\,\mathrm{d}x+\int_\omega|\nabla u_n|^q\,\mathrm{d}x\le C(\|f\|_{L^N(\Omega)},|\Omega|,\omega),\quad \text{for all } p\in(1,p_0).$$

This proves the assertion. □

As an immediate consequence of Lemma 2.2, for every $\omega\subset\subset\Omega$, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to a function u_p in $W^{1,q}(\omega)$ and in $W^{1,p}(\omega)$, and

$u_n \to u_p$ a.e. in Ω. We now show that u_p is a solution of (P_p). For $\varphi \in C_c^\infty(\Omega)$, since u_n is a solution to (2.2), by Lemma 2.2, we obtain

$$\begin{aligned}\int_\Omega f_n(x)\left(u_n+\frac{1}{n}\right)^{-\gamma}\varphi\,\mathrm{d}x &\le \frac{1}{p'}\int_{\operatorname{supp}\varphi}|\nabla u_n|^p\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla\varphi|^p\,\mathrm{d}x\\ &\quad+\frac{1}{q'}\int_{\operatorname{supp}\varphi}|\nabla u_n|^q\,\mathrm{d}x+\frac{1}{q}\int_\Omega|\nabla\varphi|^q\,\mathrm{d}x\\ &\le C.\end{aligned}\tag{2.13}$$

Then, $f_n(x)\left(u_n+\frac{1}{n}\right)^{-\gamma}$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$. Taking into account that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\omega)$ for every $\omega \subset\subset \Omega$, we can apply Theorem 2.1 of Boccardo–Murat [6] to deduce that $\nabla u_n \to \nabla u_p$ a.e. in Ω. In particular, for $\varphi \in C_c^\infty(\Omega)$, the local gradient estimates, the a.e. convergence of the gradients, and Vitali's convergence theorem imply that

$$\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\to\int_\Omega|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x,\tag{2.14}$$

$$\int_\Omega|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\to\int_\Omega|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x.\tag{2.15}$$

Still from (2.13), applied to nonnegative test functions, Fatou's lemma implies that

$$u_p^{-\gamma} f \in L^1_{\mathrm{loc}}(\Omega).\tag{2.16}$$

This, in turn, implies that $u_p > 0$ a.e. in Ω. Now we are going to prove that, for all $\varphi \in C_c^\infty(\Omega)$,

$$\lim_{n\to+\infty}\int_\Omega f_n(x)\left(u_n+\frac{1}{n}\right)^{-\gamma}\varphi\,\mathrm{d}x=\int_\Omega f(x)u_p^{-\gamma}\varphi\,\mathrm{d}x.$$

For that, let us fix $\delta > 0$ and introduce the truncation function $V_\delta\colon \mathbb{R}\to[0,1]$ defined by

$$V_\delta(s)=\begin{cases}1, & \text{for } s\le\delta,\\ \dfrac{2\delta-s}{\delta}, & \text{for } \delta<s<2\delta,\\ 0, & \text{for } s\ge 2\delta.\end{cases}\tag{2.17}$$

First, we consider $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ and $V_\delta(u_n)\varphi$ as a test function in (2.2). This yields

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^p V_\delta'(u_n)\varphi \,\mathrm{d}x + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \, V_\delta(u_n)\,\mathrm{d}x \\
&+ \int_\Omega |\nabla u_n|^q V_\delta'(u_n)\varphi \,\mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \, V_\delta(u_n)\,\mathrm{d}x \\
&= \int_\Omega f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} V_\delta(u_n)\varphi \,\mathrm{d}x.
\end{aligned}$$

Since $V_\delta'(s) \leq 0$ for all $s \in \mathbb{R}$, from the previous equality, we obtain the estimate

$$\begin{aligned}
&\int_{\{u_n \leq \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x \\
&\leq \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \, V_\delta(u_n)\,\mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \, V_\delta(u_n)\,\mathrm{d}x.
\end{aligned}$$

Using (2.14) and (2.15), and passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\{u_n \leq \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x \\
&\leq \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p)\,\mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p)\,\mathrm{d}x.
\end{aligned}$$

Taking into account that $u_p > 0$ a.e. in Ω, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&\lim_{\delta\to 0^+} \limsup_{n\to\infty} \int_{\{u_n \leq \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x \\
&\leq \int_{\{u_p = 0\}} |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi \,\mathrm{d}x + \int_{\{u_p = 0\}} |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\varphi \,\mathrm{d}x = 0.
\end{aligned} \tag{2.18}$$

Consequently,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \int_{\{u_n \leq \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x = 0.$$

To proceed, we decompose the integral as

$$\begin{aligned}
&\int_\Omega f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x \\
&= \int_{\{u_n \leq \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x + \int_{\{u_n > \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \,\mathrm{d}x.
\end{aligned} \tag{2.19}$$

The first integral on the right-hand side of (2.19) corresponds to the singular region where u_n is small, and from (2.18) its contribution vanishes as $\delta \to 0^+$. For the second integral, note that

$$f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \chi_{\{u_n > \delta\}} \leq f(x)\delta^{-\gamma}\varphi \quad \text{with} \quad f\delta^{-\gamma}\varphi \in L^1(\Omega).$$

Moreover, since the set $\{\eta \in \mathbb{R} : |\{u_p = \eta\}| > 0\}$ is at most a countable set, we can assume without loss of generality that δ converges to 0, through values that do not belong to this set. Then, by Lebesgue's dominated convergence theorem, we can pass to the limit and deduce that

$$\lim_{\delta \to 0^+} \limsup_{n \to \infty} \int_{\{u_n > \delta\}} f_n(x)\left(u_n + \frac{1}{n}\right)^{-\gamma} \varphi \, \mathrm{d}x = \int_\Omega f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x. \tag{2.20}$$

Finally, since u_n is a solution to (2.2), passing to the limit in the weak formulation and using (2.14), (2.15), and (2.20), we obtain

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x.$$

For the general function $\varphi \in C_c^\infty(\Omega)$, let $\eta \in C_c^\infty(\Omega)$ satisfy $\eta = 1$ on $\operatorname{supp}\varphi$, and choose $M > 0$ such that $\varphi + M\eta \geq 0$ in Ω. Applying the previous identity to the nonnegative test functions $\varphi + M\eta$ and $M\eta$, and subtracting the two resulting identities, we conclude that the same equality holds for every $\varphi \in C_c^\infty(\Omega)$. Therefore, u_p is a weak solution to (P_p).

3 Proof of Theorem 1.1

In Sect. 2, we showed that for $p > 1$, there exists a function $u_p \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ with $u_p > 0$ a.e. in Ω, which is a solution to problem (P_p).

In what follows, we establish a uniform local estimate for the gradients of the solutions, which will be crucial for passing to the limit as $p \to 1^+$.

Lemma 3.1 *There exists $p_0 > 1$ such that, for all $\omega \subset\subset \Omega$, there exists a constant $C = C(\|f\|_{L^N(\Omega)}, \gamma, \omega, |\Omega|)$, such that*

$$\int_\omega |\nabla u_p|^p \, \mathrm{d}x + \int_\omega |\nabla u_p|^q \, \mathrm{d}x < C, \quad \textit{for all } 1 < p < p_0.$$

Proof From Lemma 2.2, the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to problem (2.2) is uniformly bounded in $W^{1,q}(\omega)$ and $W^{1,p}(\omega)$ for any $\omega \subset\subset \Omega$, by a constant C that is independent of n and $p \in (1, p_0)$. As a consequence, passing to a subsequence if necessary, we have $u_n \rightharpoonup u_p$ weakly in $W^{1,q}(\omega)$ and in $W^{1,p}(\omega)$ as $n \to \infty$.

By the weak lower semicontinuity of the norm in $L^p(\omega)$ and $L^q(\omega)$, we have that

$$\int_\omega |\nabla u_p|^p \, dx + \int_\omega |\nabla u_p|^q \, dx \leq \liminf_{n\to\infty} \int_\omega |\nabla u_n|^p \, dx + \liminf_{n\to\infty} \int_\omega |\nabla u_n|^q \, dx$$
$$\leq \liminf_{n\to\infty} \left(\int_\omega |\nabla u_n|^p \, dx + \int_\omega |\nabla u_n|^q \, dx \right) \leq C.$$

This completes the proof. □

Lemma 3.2 *There exists a constant $C > 0$, independent of p, such that*

$$\|u_p\|_{L^\infty(\Omega)} \leq C, \quad \textit{for all } p > 1.$$

Proof Since $u_n \to u_p$ a.e. in Ω, the result follows directly from (2.4). □

From Lemmas 3.1 and 3.2, it follows that for every $\omega \subset\subset \Omega$, (u_p) is uniformly bounded in $W^{1,q}(\omega)$. Consequently, there exists $u \in W^{1,q}(\omega)$, independent of ω by a diagonal argument, such that, up to a subsequence,

$$\begin{aligned} &u_p \rightharpoonup u \quad \text{weakly in } W^{1,q}(\omega), \\ &u_p \to u \quad \text{strongly in } L^r_{\text{loc}}(\Omega) \text{ for all } 1 \leq r < q^*, \\ &u_p \to u \quad \text{a.e. in } \Omega. \end{aligned}$$

Moreover, from Lemma 3.2, it follows that $u \in L^\infty(\Omega)$.

Because of the strong singularity, we cannot obtain a global estimate for u_p on Ω, since convergence may fail near the boundary, where u_p vanishes. However, by considering a higher power of u_p, we can prove a global estimate.

Lemma 3.3 *There exists a constant $C_1 > 0$, independent of p, such that*

$$\left\| u_p^{\frac{\gamma-1+q}{q}} \right\|_{W_0^{1,q}(\Omega)} + \left\| u_p^{\frac{\gamma-1+p}{p}} \right\|_{W_0^{1,p}(\Omega)} \leq C_1. \tag{3.1}$$

Proof Let u_n be a solution to problem (2.2). Taking u_n^γ as a test function in (2.2), we obtain

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n^\gamma \, dx + \int_\Omega |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla u_n^\gamma \, dx = \int_\Omega \frac{f_n(x) u_n^\gamma}{\left(u_n + \frac{1}{n}\right)^\gamma} \, dx.$$

Since

$$\int_\Omega \frac{f_n(x) u_n^\gamma}{\left(u_n + \frac{1}{n}\right)^\gamma} \, dx \leq \int_\Omega f_n(x) \, dx \leq \int_\Omega f(x) \, dx = C,$$

it follows that

$$\int_\Omega \left| \nabla u_n^{\frac{\gamma-1+q}{q}} \right|^q \mathrm{d}x + \int_\Omega \left| \nabla u_n^{\frac{\gamma-1+p}{p}} \right|^p \mathrm{d}x \le C.$$

Consequently,

$$\left\| u_n^{\frac{\gamma-1+q}{q}} \right\|_{W_0^{1,q}(\Omega)} + \left\| u_n^{\frac{\gamma-1+p}{p}} \right\|_{W_0^{1,p}(\Omega)} \le C_1.$$

Therefore, by weak lower semicontinuity, we conclude that

$$\left\| u_p^{\frac{\gamma-1+q}{q}} \right\|_{W_0^{1,q}(\Omega)} + \left\| u_p^{\frac{\gamma-1+p}{p}} \right\|_{W_0^{1,p}(\Omega)} \le C_1.$$

□

From the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \le r < q^*$, it follows from (3.1) that

$$u_p^{\frac{\gamma-1+p}{p}} \to u^\gamma \quad \text{strongly in } L^r(\Omega) \text{ for all } 1 \le r < q^*. \tag{3.2}$$

Lemma 3.4 *As $p \to 1^+$, it holds that*

$$\nabla u_p \to \nabla u \quad a.e.\ in\ \Omega.$$

Proof To prove the almost everywhere convergence of the gradients, we proceed by showing the strong convergence of the truncations.

Let $k > 0$ and choose a cut-off function $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$. Let us consider $\Phi_{p,k} = (T_k(u_p) - T_k(u))\varphi$ as test function in (P_p). Although the weak formulation of (P_p) is initially given for test functions in $C_c^1(\Omega)$, it can be extended to functions in $W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ with compact support by a standard density argument.

Then, we get

$$\begin{aligned} &\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla \Phi_{p,k}\,\mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla \Phi_{p,k}\,\mathrm{d}x \\ &= \int_\Omega f(x) u_p^{-\gamma} \Phi_{p,k}\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

We first estimate the right-hand side of (3.3). For any $\delta > 0$, we split the integral as follows:

$$\begin{aligned} \int_\Omega f(x) u_p^{-\gamma} \Phi_{p,k}\,\mathrm{d}x = &\int_{\{u_p \le \delta\}} f(x) u_p^{-\gamma} (T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x \\ &+ \int_{\{u_p > \delta\}} f(x) u_p^{-\gamma} (T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

Since $|T_k(u_p) - T_k(u)| \leq 2k$, the first integral in the right-hand side of (3.4) can be bounded by

$$\left| \int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} (T_k(u_p) - T_k(u)) \varphi \, \mathrm{d}x \right| \leq 2k \int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x.$$

On the other hand,

$$\lim_{\delta \to 0^+} \limsup_{p \to 1^+} \int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x = 0. \tag{3.5}$$

Indeed, to establish this limit, we introduce the auxiliary cut-off function V_δ defined as in (2.17). By taking $V_\delta(u_p)\varphi$ as a test function in (P_p), we have

$$\begin{aligned} &\int_\Omega |\nabla u_p|^p V_\delta'(u_p) \varphi \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi V_\delta(u_p) \, \mathrm{d}x \\ &+ \int_\Omega |\nabla u_p|^q V_\delta'(u_p) \varphi \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi V_\delta(u_p) \, \mathrm{d}x \\ &= \int_\Omega f(x) u_p^{-\gamma} V_\delta(u_p) \varphi \, \mathrm{d}x. \end{aligned} \tag{3.6}$$

Since $V_\delta'(s) \leq 0$ for all $s \in \mathbb{R}$, the first and third integrals on the left-hand side of (3.6) are non-positive. By dropping these terms and noticing that $V_\delta(u_p) = 1$ on the set $\{u_p \leq \delta\}$, we deduce the following upper estimate:

$$\begin{aligned} &\int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x \\ &\leq \int_\Omega f(x) u_p^{-\gamma} V_\delta(u_p) \varphi \, \mathrm{d}x \\ &\leq \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi V_\delta(u_p) \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi V_\delta(u_p) \, \mathrm{d}x. \end{aligned} \tag{3.7}$$

By Lemma 3.1, the sequences $\{|\nabla u_p|^{p-2} \nabla u_p\}_p$ and $\{|\nabla u_p|^{q-2} \nabla u_p\}_p$ are uniformly bounded in $L^s(\omega, \mathbb{R}^N)$ (for any $s \in [1, \infty)$) and $L^{\frac{q}{q-1}}(\omega, \mathbb{R}^N)$, respectively, for any $\omega \subset\subset \Omega$. Hence, up to a subsequence, there exist vector fields $\mathbf{z}$ and $\mathbf{w}$ such that $|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup \mathbf{z}$ weakly in $L^s(\omega, \mathbb{R}^N)$ and $|\nabla u_p|^{q-2} \nabla u_p \rightharpoonup \mathbf{w}$ weakly in $L^{\frac{q}{q-1}}(\omega, \mathbb{R}^N)$ as $p \to 1^+$.

Moreover, since $u_p \to u$ a.e. in Ω and V_δ is a bounded continuous function, $V_\delta(u_p) \nabla \varphi \to V_\delta(u) \nabla \varphi$ strongly in any Lebesgue space over the support of φ by Lebesgue's dominated convergence theorem. Combining this strong convergence with the weak convergence of the gradient terms, we can pass to the limit in (3.7) as $p \to 1^+$,

obtaining

$$\limsup_{p\to 1^+}\int_{\{u_p\le\delta\}} f(x)u_p^{-\gamma}\varphi\,\mathrm{d}x \le \int_\Omega \mathbf{z}\cdot\nabla\varphi V_\delta(u)\,\mathrm{d}x + \int_\Omega \mathbf{w}\cdot\nabla\varphi V_\delta(u)\,\mathrm{d}x. \quad (3.8)$$

We now pass to the limit as $\delta\to 0^+$. By (2.16) and Fatou's Lemma, $f(x)u^{-\gamma}\in L^1_{\mathrm{loc}}(\Omega)$. Since $f(x)>0$ a.e. in Ω, this implies that $u>0$ a.e. in Ω. Consequently, as $\delta\to 0^+$, we have $V_\delta(u(x))\to 0$ a.e. in Ω. Hence, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}&\lim_{\delta\to 0^+}\left(\int_\Omega \mathbf{z}\cdot\nabla\varphi V_\delta(u)\,\mathrm{d}x + \int_\Omega \mathbf{w}\cdot\nabla\varphi V_\delta(u)\,\mathrm{d}x\right)\\ &= \int_{\{u=0\}}\mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x + \int_{\{u=0\}}\mathbf{w}\cdot\nabla\varphi\,\mathrm{d}x = 0.\end{aligned}$$

Taking $\delta\to 0^+$ in (3.8), we conclude that (3.5) holds.

Now we estimate the second integral in the right-hand side of (3.4). Since $u_p>\delta$, the integrand is dominated by $2k\delta^{-\gamma}f(x)\varphi\in L^1(\Omega)$. Since $T_k(u_p)-T_k(u)\to 0$ a.e. in Ω, by Lebesgue's dominated convergence theorem, it follows that

$$\lim_{p\to 1^+}\int_{\{u_p>\delta\}} f(x)u_p^{-\gamma}(T_k(u_p)-T_k(u))\varphi\,\mathrm{d}x = 0.$$

Letting $\delta\to 0^+$, we conclude that the right-hand side of (3.3) tends to zero as $p\to 1^+$.

Now, we analyze the left-hand side of (3.3). Using

$$\nabla\Phi_{p,k} = \nabla(T_k(u_p)-T_k(u))\varphi + (T_k(u_p)-T_k(u))\nabla\varphi,$$

and the strong convergence

$$T_k(u_p)\to T_k(u)\quad \text{in } L^r_{\mathrm{loc}}(\Omega)\quad \text{for all } 1\le r<q^*,$$

together with the local boundedness of the corresponding gradient terms, we obtain

$$\int_\Omega \left(|\nabla u_p|^{p-2}\nabla u_p + |\nabla u_p|^{q-2}\nabla u_p\right)\cdot(T_k(u_p)-T_k(u))\nabla\varphi\,\mathrm{d}x\to 0.$$

Hence, from (3.3) and the convergence of its right-hand side proved above,

$$\int_\Omega \varphi\left(|\nabla u_p|^{p-2}\nabla u_p + |\nabla u_p|^{q-2}\nabla u_p\right)\cdot\nabla(T_k(u_p)-T_k(u))\,\mathrm{d}x\to 0.$$

Moreover, since $\nabla T_k(u_p) = \nabla u_p$ on $\{u_p \leq k\}$ and $\nabla T_k(u_p) = 0$ on $\{u_p > k\}$, the difference between the last integral and

$$\int_\Omega \varphi\Big(|\nabla T_k(u_p)|^{p-2}\nabla T_k(u_p) + |\nabla T_k(u_p)|^{q-2}\nabla T_k(u_p)\Big) \cdot \nabla(T_k(u_p) - T_k(u))\,\mathrm{d}x$$

is given by

$$-\int_{\{u_p>k\}} \varphi\Big(|\nabla u_p|^{p-2}\nabla u_p + |\nabla u_p|^{q-2}\nabla u_p\Big) \cdot \nabla T_k(u)\,\mathrm{d}x.$$

This term tends to zero, since

$$\chi_{\{u_p>k\}}\nabla T_k(u) \to 0 \quad \text{in } L^q_{\mathrm{loc}}(\Omega)$$

and the gradient terms are uniformly bounded in the corresponding dual Lebesgue spaces. Therefore,

$$\begin{aligned}\lim_{p\to 1^+}\Bigg[&\int_\Omega |\nabla T_k(u_p)|^{p-2}\nabla T_k(u_p)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x \\ &+ \int_\Omega |\nabla T_k(u_p)|^{q-2}\nabla T_k(u_p)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x\Bigg] = 0.\end{aligned} \tag{3.9}$$

By the monotonicity of the p-Laplacian operator, we have

$$\begin{aligned}&\int_\Omega |\nabla T_k(u_p)|^{p-2}\nabla T_k(u_p)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x \\ &\geq \int_\Omega |\nabla T_k(u)|^{p-2}\nabla T_k(u)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Notice that the vector field $|\nabla T_k(u)|^{p-2}\nabla T_k(u)\varphi$ converges strongly in $L^{q'}(\Omega, \mathbb{R}^N)$ (by the Dominated Convergence Theorem, since $p \to 1^+$ and it is dominated by $C(1 + |\nabla T_k(u)|^q)$), while $\nabla(T_k(u_p) - T_k(u)) \rightharpoonup 0$ weakly in $L^q(\Omega, \mathbb{R}^N)$. Hence,

$$\lim_{p\to 1^+}\int_\Omega |\nabla T_k(u)|^{p-2}\nabla T_k(u)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x = 0. \tag{3.11}$$

Then, from (3.10) and (3.11),

$$\liminf_{p\to 1^+}\int_\Omega |\nabla T_k(u_p)|^{p-2}\nabla T_k(u_p)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x \geq 0. \tag{3.12}$$

Combining (3.9) and (3.12), we have

$$\limsup_{p\to 1^+}\int_\Omega |\nabla T_k(u_p)|^{q-2}\nabla T_k(u_p)\cdot\nabla(T_k(u_p) - T_k(u))\varphi\,\mathrm{d}x \leq 0. \tag{3.13}$$

Again, since $\nabla(T_k(u_p) - T_k(u)) \rightharpoonup 0$ weakly in $L^q(\Omega, \mathbb{R}^N)$, we have that

$$\lim_{p \to 1^+} \int_\Omega |\nabla T_k(u)|^{q-2} \nabla T_k(u) \cdot \nabla(T_k(u_p) - T_k(u))\varphi \, \mathrm{d}x = 0. \tag{3.14}$$

Hence, from (3.13) and (3.14), it follows that

$$\limsup_{p \to 1^+} \int_\Omega \Big(|\nabla T_k(u_p)|^{q-2} \nabla T_k(u_p) - |\nabla T_k(u)|^{q-2} \nabla T_k(u) \Big) \cdot \nabla(T_k(u_p) - T_k(u))\varphi \, \mathrm{d}x \le 0.$$

Due to the strict monotonicity of the q-Laplacian operator for fixed $q > 1$, the integrand is non-negative. Hence, the limit is zero and $\nabla T_k(u_p) \to \nabla T_k(u)$ strongly in $L^q_{\mathrm{loc}}(\Omega)$. Since this holds for any $k > 0$ and any test function $\varphi \ge 0$, we conclude that $\nabla u_p \to \nabla u$ a.e. in Ω. □

Lemma 3.5 *There exists* $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ *such that* $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ *and*

$$-\operatorname{div} \mathbf{z} - \Delta_q u = \frac{f(x)}{u^\gamma} \quad \text{in } \mathcal{D}'(\Omega). \tag{3.15}$$

Proof First, let us prove that $f u^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$. For $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$, from (2.1), we have

$$\begin{aligned}
&\int_\Omega f(x) u_p^{-\gamma} \varphi \, \mathrm{d}x \\
&= \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2} \nabla u_p \cdot \nabla \varphi \, \mathrm{d}x \\
&\le \int_\Omega |\nabla u_p|^{p-1} |\nabla \varphi| \, \mathrm{d}x + \int_\Omega |\nabla u_p|^{q-1} |\nabla \varphi| \, \mathrm{d}x \\
&\le (p-1) \int_{\operatorname{supp} \varphi} |\nabla u_p|^p \, \mathrm{d}x + \int_\Omega |\nabla \varphi|^p \, \mathrm{d}x + (q-1) \int_{\operatorname{supp} \varphi} |\nabla u_p|^q \, \mathrm{d}x + \int_\Omega |\nabla \varphi|^q \, \mathrm{d}x \\
&\le C.
\end{aligned}$$

Since the right-hand side is uniformly bounded with respect to p, it follows that $f u_p^{-\gamma}$ is locally bounded in $L^1(\Omega)$ with respect to p. By Fatou's lemma as $p \to 1^+$, we deduce that $f u^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$. Since $f > 0$ a.e. in Ω, this implies that $u > 0$ a.e. in Ω.

Now, we prove the existence of $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$. Let $\omega \subset\subset \Omega$ and fix $1 \leq s < \infty$. Consider $1 < p < s'$. By Hölder's inequality and Lemma 3.1, we obtain

$$\begin{aligned}\int_\omega \left||\nabla u_p|^{p-2}\nabla u_p\right|^s \mathrm{d}x &= \int_\omega |\nabla u_p|^{s(p-1)}\,\mathrm{d}x \\ &\leq \left(\int_\omega |\nabla u_p|^p\,\mathrm{d}x\right)^{\frac{s}{p'}} |\omega|^{1-\frac{s}{p'}} \\ &\leq C(\omega)^{\frac{s}{p'}} |\omega|^{1-\frac{s}{p'}} \\ &\leq C(\omega)^{\frac{s}{p'}} |\Omega|^{1-\frac{s}{p'}}.\end{aligned}$$

Therefore

$$\||\nabla u_p|^{p-2}\nabla u_p\|_{L^s(\omega)} \leq C(\omega)^{\frac{1}{p'}} |\Omega|^{\frac{1}{s}-\frac{1}{p'}}. \tag{3.16}$$

Hence, the sequence $(|\nabla u_p|^{p-2}\nabla u_p)_{1<p<p_0}$ is bounded in $L^s(\omega, \mathbb{R}^N)$. This yields the existence of $\mathbf{z}_s \in L^s(\omega; \mathbb{R}^N)$, independent of ω by a diagonal argument, such that, up to a subsequence,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}_s \quad \text{in } L^s(\omega, \mathbb{R}^N) \text{ as } p \to 1^+.$$

Moreover, once more, a diagonal argument shows that $\mathbf{z}_s$ does not depend on s and hence can be denoted simply by $\mathbf{z}$. Hence,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z} \quad \text{in } L^s(\omega; \mathbb{R}^N) \quad \text{for all } s \in [1, +\infty), \tag{3.17}$$

as $p \to 1^+$. Consequently, by the lower semicontinuity of the norm in $L^s(\omega; \mathbb{R}^N)$ and from (3.16), we have

$$\|\mathbf{z}\|_{L^s(\omega)} \leq |\Omega|^{\frac{1}{s}}. \tag{3.18}$$

Finally, by letting $s \to \infty$ in (3.18), we conclude that $\mathbf{z} \in L^\infty(\omega, \mathbb{R}^N)$ and $\|\mathbf{z}\|_{L^\infty(\omega)} \leq 1$.

By Lemma 3.4, we have

$$\nabla u_p \to \nabla u \quad \text{a.e. in } \Omega.$$

Moreover, from Lemma 3.1,

$$\int_\omega \left||\nabla u_p|^{q-2}\nabla u_p\right|^{\frac{q}{q-1}}\,\mathrm{d}x = \int_\omega |\nabla u_p|^q\,\mathrm{d}x \leq C(\omega),$$

where $C(\omega)$ does not depend on p. Hence, $\left(|\nabla u_p|^{q-2}\nabla u_p\right)_p$ is bounded in $L^{\frac{q}{q-1}}_{\text{loc}}(\Omega)^N$. From Lemma 4.8 by Kavian [17], we have

$$|\nabla u_p|^{q-2}\nabla u_p \rightharpoonup |\nabla u|^{q-2}\nabla u \quad \text{in } L^{\frac{q}{q-1}}_{\text{loc}}(\Omega)^N \text{ as } p \to 1^+.$$

Hence, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\varphi \,\mathrm{d}x \to \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x. \tag{3.19}$$

Now, let us prove that

$$\int_\Omega \frac{f(x)}{u_p^\gamma}\varphi \,\mathrm{d}x \to \int_\Omega \frac{f(x)}{u^\gamma}\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in C_c^1(\Omega).$$

For $\delta > 0$, let $V_\delta(s)$ be defined as in (2.17). First, let $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. Taking $V_\delta(u_p)\varphi$ as a test function in (2.1) yields

$$\begin{aligned}
&\int_\Omega |\nabla u_p|^p V_\delta'(u_p)\varphi \,\mathrm{d}x + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p) \,\mathrm{d}x \\
&+ \int_\Omega |\nabla u_p|^q V_\delta'(u_p)\varphi \,\mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p) \,\mathrm{d}x \\
&= \int_\Omega f(x) u_p^{-\gamma} \, V_\delta(u_p)\varphi \,\mathrm{d}x.
\end{aligned}$$

Since $V_\delta'(s) \leq 0$ for all $s \in \mathbb{R}$, the first and third integrals on the left-hand side are nonpositive. By neglecting these terms, we derive the upper estimate

$$\begin{aligned}
&\int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} \, \varphi \,\mathrm{d}x \\
&\leq \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p) \,\mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\varphi \, V_\delta(u_p) \,\mathrm{d}x.
\end{aligned}$$

Next, we pass to the limit as $p \to 1^+$. Using the weak convergences in (3.17) and (3.19), together with the boundedness of $V_\delta(u_p)$ and φ, we obtain

$$\limsup_{p \to 1^+} \int_{\{u_p \leq \delta\}} f(x) u_p^{-\gamma} \, \varphi \,\mathrm{d}x \leq \int_\Omega \mathbf{z} \cdot \nabla\varphi \, V_\delta(u) \,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi \, V_\delta(u) \,\mathrm{d}x.$$

Taking into account that $u > 0$ a.e. in Ω, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} &\lim_{\delta\to 0^+} \limsup_{p\to 1^+} \int_{\{u_p\le\delta\}} f(x)u_p^{-\gamma}\,\varphi\,\mathrm{d}x \\ &\le \int_{\{u=0\}} \mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x + \int_{\{u=0\}} |\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = 0. \end{aligned} \tag{3.20}$$

Moreover, by considering the positive and negative parts of φ, and by density, (3.20) can be extended to any test function $\varphi \in C_c^1(\Omega)$.

To proceed, we decompose the integral term as

$$\int_\Omega f(x)u_p^{-\gamma}\,\varphi\,\mathrm{d}x = \int_{\{u_p\le\delta\}} f(x)u_p^{-\gamma}\,\varphi\,\mathrm{d}x + \int_{\{u_p>\delta\}} f(x)u_p^{-\gamma}\,\varphi\,\mathrm{d}x.$$

The first integral corresponds to the singular region where u_p is small and we already know from (3.20) that its contribution vanishes as $\delta \to 0^+$. For the second integral, note that $|f(\cdot)u_p^{-\gamma}\varphi\chi_{u_p>\delta}| \le f(\cdot)\delta^{-\gamma}|\varphi| \in L^1(\Omega)$. Moreover, as in Sect. 2, since the set $\{\eta\in\mathbb{R}\colon |\{u=\eta\}|>0\}$ is at most a countable set, we can assume without loss of generality that δ converges to 0, through values that do not belong to this set. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0^+} \limsup_{p\to 1^+} \int_{\{u_p>\delta\}} f(x)u_p^{-\gamma}\,\varphi\,\mathrm{d}x = \int_\Omega f(x)u^{-\gamma}\,\varphi\,\mathrm{d}x. \tag{3.21}$$

On the other hand, taking $\varphi \in C_c^1(\Omega)$ as test function in (P_p), we obtain

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega f(x)u_p^{-\gamma}\varphi\,\mathrm{d}x.$$

Passing to the limit as $p \to 1^+$, and using the convergence established in (3.17) and (3.19), together with (3.21), we conclude that

$$\int_\Omega \mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega f(x)u^{-\gamma}\varphi\,\mathrm{d}x,$$

for all $\varphi \in C_c^1(\Omega)$. This identity corresponds exactly to relation (3.15), thereby completing the proof of the desired relation. □

Lemma 3.6 *The vector field* $\mathbf{z}$ *and the function* u *satisfy*

$$\mathbf{z}\cdot\nabla u = |\nabla u| \quad \textit{a.e. in } \Omega.$$

Proof The usual technique would consist in taking $u_p\varphi$ as a test function in (2.1), where $\varphi \in C_c^1(\Omega)$ with $\varphi \geq 0$. However, on the right-hand side, a singular integral involving $u_p^{1-\gamma}$ would appear. To overcome this difficulty, we first prove that

$$\mathbf{z} \cdot \nabla(u^\gamma) = |\nabla(u^\gamma)|, \quad \text{a.e. in } \Omega.$$

The first step is to establish the identity

$$\int_\Omega \mathbf{z} \cdot \nabla(u^\gamma \varphi)\, \mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla(u^\gamma \varphi)\, \mathrm{d}x = \int_\Omega f(x)\varphi\, \mathrm{d}x, \tag{3.22}$$

for all $\varphi \in C_c^1(\Omega)$.

To this end, we take $(\rho_\varepsilon * u^\gamma)\varphi$ as a test function in (3.15), where ρ_ε denotes a standard mollifier. Then, we obtain

$$\begin{aligned}&\int_\Omega \mathbf{z} \cdot \nabla((\rho_\varepsilon * u^\gamma)\varphi)\, \mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla((\rho_\varepsilon * u^\gamma)\,\varphi)\, \mathrm{d}x \\ &= \int_\Omega f(x)u^{-\gamma}(\rho_\varepsilon * u^\gamma)\,\varphi\, \mathrm{d}x.\end{aligned}$$

By passing to the limit as $\varepsilon \to 0$ and applying Lebesgue's dominated convergence theorem, we readily deduce (3.22).

Now, taking $u_p^{\frac{\gamma-1+p}{p}}\varphi$ as a test function in (P_p) gives

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$$\begin{aligned}&\left(\frac{\gamma-1+p}{p}\right)\left(\frac{p^2}{\gamma-1+p^2}\right)^p \int_\Omega \left|\nabla u_p^{\frac{\gamma-1+p^2}{p^2}}\right|^p \varphi\, \mathrm{d}x \\ &+ \int_\Omega u_p^{\frac{\gamma-1+p}{p}} |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi\, \mathrm{d}x \\ &+ \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla\left(u_p^{\frac{\gamma-1+p}{p}}\varphi\right) \mathrm{d}x = \int_\Omega f(x)u_p^{\frac{(\gamma-1)(1-p)}{p}}\varphi\, \mathrm{d}x.\end{aligned} \tag{3.23}$$

By Young's inequality, we have

$$\begin{aligned}&\left(\frac{\gamma-1+p}{p}\right)^{\frac{1}{p}}\left(\frac{p^2}{\gamma-1+p^2}\right)\int_\Omega \left|\nabla u_p^{\frac{\gamma-1+p^2}{p^2}}\right|\varphi\, \mathrm{d}x \\ &\leq \frac{1}{p}\left(\frac{\gamma-1+p}{p}\right)\left(\frac{p^2}{\gamma-1+p^2}\right)^p \int_\Omega \left|\nabla u_p^{\frac{\gamma-1+p^2}{p^2}}\right|^p \varphi\, \mathrm{d}x + \frac{(p-1)}{p}\int_\Omega \varphi\, \mathrm{d}x.\end{aligned} \tag{3.24}$$

Then, from (3.23) and (3.24), it follows that

$$\begin{aligned}
&p\left(\frac{\gamma-1+p}{p}\right)^{\frac{1}{p}}\left(\frac{p^2}{\gamma-1+p^2}\right)\int_\Omega\left|\nabla u_p^{\frac{\gamma-1+p^2}{p^2}}\right|\varphi\,\mathrm{d}x\\
&+\int_\Omega u_p^{\frac{\gamma-1+p}{p}}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x\\
&+\int_\Omega|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\left(u_p^{\frac{\gamma-1+p}{p}}\varphi\right)\mathrm{d}x\\
&\leq\int_\Omega f(x)u_p^{\frac{(\gamma-1)(1-p)}{p}}\varphi\,\mathrm{d}x+(p-1)\int_\Omega\varphi\,\mathrm{d}x.
\end{aligned}\tag{3.25}$$

Now, we pass to the limit in (3.25) as $p\to 1^+$. By similar arguments used at the end of the proof of Lemma 3.5, we can show that

$$\limsup_{p\to1^+}\int_\Omega f(x)u_p^{\frac{(\gamma-1)(1-p)}{p}}\varphi\,\mathrm{d}x=\int_\Omega f(x)\varphi\,\mathrm{d}x.\tag{3.26}$$

Moreover, we split the integral linearly as follows:

$$\begin{aligned}
&\int_\Omega u_p^{\frac{\gamma-1+p}{p}}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x-\int_\Omega u^\gamma\mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x\\
&=\int_\Omega\left(u_p^{\frac{\gamma-1+p}{p}}-u^\gamma\right)|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad+\int_\Omega u^\gamma\left(|\nabla u_p|^{p-2}\nabla u_p-\mathbf{z}\right)\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned}$$

For the first term, fixing $1<r<1^*$ and using Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\int_\Omega\left(u_p^{\frac{\gamma-1+p}{p}}-u^\gamma\right)|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x\right|\\
&\leq\int_{\operatorname{supp}\varphi}\left|u_p^{\frac{\gamma-1+p}{p}}-u^\gamma\right||\nabla u_p|^{p-1}|\nabla\varphi|\,\mathrm{d}x\\
&\leq\left\|u_p^{\frac{\gamma-1+p}{p}}-u^\gamma\right\|_{L^r(\operatorname{supp}\varphi)}\left\||\nabla u_p|^{p-1}\right\|_{L^{\frac{r}{r-1}}(\operatorname{supp}\varphi)}\|\nabla\varphi\|_{L^\infty(\Omega)}\\
&=o_p(1),
\end{aligned}$$

due to (3.2) and (3.17).

For the second term, notice that since $u^\gamma\in L^r_{\mathrm{loc}}(\Omega)$ and φ has compact support, the vector field $u^\gamma\nabla\varphi$ belongs to $L^r(\operatorname{supp}\varphi;\mathbb{R}^N)$. From (3.17), $|\nabla u_p|^{p-2}\nabla u_p\rightharpoonup\mathbf{z}$

weakly in $L^{\frac{r}{r-1}}(\operatorname{supp}\varphi;\mathbb{R}^N)$. Hence,

$$\lim_{p\to 1^+}\int_\Omega u^\gamma\left(|\nabla u_p|^{p-2}\nabla u_p-\mathbf{z}\right)\cdot\nabla\varphi\,\mathrm{d}x=0.$$

Combining both limits, we conclude that

$$\lim_{p\to 1^+}\int_\Omega u_p^{\frac{\gamma-1+p}{p}}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega u^\gamma\mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x. \tag{3.27}$$

Next, we prove that

$$\int_\Omega|\nabla u|^{q-2}\nabla u\cdot\nabla(u^\gamma\varphi)\,\mathrm{d}x\le\liminf_{p\to 1^+}\int_\Omega|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\left(u_p^{\frac{\gamma-1+p}{p}}\varphi\right)\mathrm{d}x. \tag{3.28}$$

First, note that

$$\begin{aligned}&\int_\Omega|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\left(u_p^{\frac{\gamma-1+p}{p}}\varphi\right)\mathrm{d}x\\&=\left(\frac{\gamma-1+p}{p}\right)\int_\Omega u_p^{\frac{\gamma-1}{p}}\varphi|\nabla u_p|^q\,\mathrm{d}x+\int_\Omega u_p^{\frac{\gamma-1+p}{p}}|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x.\end{aligned} \tag{3.29}$$

For the first term in the right-hand side, note that Fatou's lemma implies that

$$\gamma\int_\Omega u^{\gamma-1}\varphi|\nabla u|^q\,\mathrm{d}x\le\liminf_{p\to 1^+}\left(\frac{\gamma-1+p}{p}\right)\int_\Omega u_p^{\frac{\gamma-1}{p}}\varphi|\nabla u_p|^q\,\mathrm{d}x. \tag{3.30}$$

For the second term, since

$$u_p^{\frac{\gamma-1+p}{p}}\to u^\gamma\quad\text{in } L^q_{\mathrm{loc}}(\Omega)\quad\text{and}\quad|\nabla u_p|^{q-2}\nabla u_p\rightharpoonup|\nabla u|^{q-2}\nabla u\quad\text{in } L^{\frac{q}{q-1}}_{\mathrm{loc}}(\Omega),$$

due to the boundedness of the sequence obtained in Lemma 3.1, it follows that

$$\lim_{p\to 1^+}\int_\Omega u_p^{\frac{\gamma-1+p}{p}}|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega u^\gamma|\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x. \tag{3.31}$$

Now, from (3.29), (3.30) and (3.31), it follows (3.28). From (3.26), (3.27) and (3.28), we have that

$$\left(u_p^{\frac{\gamma-1+p^2}{p^2}}\right)_p\quad\text{is bounded in } W^{1,1}_{\mathrm{loc}}(\Omega).$$

Hence, it converges, up to a subsequence, to u^γ a.e. in Ω and in $L^r(\Omega)$ for all $r < \frac{N}{N-1}$. In addition, $\nabla u_p^{\frac{\gamma-1+p^2}{p^2}}$ converges weakly-* as measures to $\nabla(u^\gamma)$ locally in Ω.

Calculating the lim inf as $p \to 1^+$ in (3.24) and taking into account (3.26), (3.27), (3.28) and the lower semicontinuity, it follows that

$$\begin{aligned}&\int_\Omega \varphi|\nabla u^\gamma|\,\mathrm{d}x + \int_\Omega u^\gamma \mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla(u^\gamma\varphi)\,\mathrm{d}x\\ &\le \int_\Omega f(x)\varphi\,\mathrm{d}x = \int_\Omega \mathbf{z}\cdot\nabla(u^\gamma\varphi)\,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla(u^\gamma\varphi)\,\mathrm{d}x.\end{aligned}$$

Hence, for all $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$,

$$\int_\Omega |\nabla u^\gamma|\varphi\,\mathrm{d}x \le \int_\Omega \mathbf{z}\cdot\nabla u^\gamma\varphi\,\mathrm{d}x.$$

Then,

$$|\nabla u^\gamma| \le \mathbf{z}\cdot\nabla u^\gamma, \quad \text{a.e. in } \Omega.$$

Since the opposite inequality follows from the fact that $\|\mathbf{z}\|_\infty \le 1$, we deduce that

$$|\nabla u^\gamma| = \mathbf{z}\cdot\nabla u^\gamma, \quad \text{a.e. in } \Omega.$$

Hence

$$\gamma u^{\gamma-1}|\nabla u| = \gamma u^{\gamma-1}\mathbf{z}\cdot\nabla u, \quad \text{a.e. in } \Omega.$$

Since $u > 0$ a.e. in Ω, it follows that

$$\mathbf{z}\cdot\nabla u = |\nabla u|, \quad \text{a.e. in } \Omega.$$

□

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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