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On $\vec{p}(\cdot)$-Laplacian Equations with Generalized Anisotropic Neumann Boundary Condition

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Abstract

In this paper, we investigate the existence and multiplicity of solutions for an anisotropic variable exponent problem of the form

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u=\lambda k(x)|u|^{\alpha(x)-2}u$$

in Ω, subject to the anisotropic Neumann boundary condition

$$\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\eta_i=h(x)|u|^{\beta(x)-2}u \quad \text{on } \partial\Omega,$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 3$, is a bounded domain with smooth boundary and

$$\Delta_{\vec{p}(\cdot)}u=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)$$

denotes the anisotropic variable exponent $\vec{p}(\cdot)$-Laplace differential operator. Under suitable assumptions on the data, we prove that the above problem admits at least two nontrivial nonnegative weak solutions. The proof is based on the Nehari manifold approach combined with the fibering map technique and the direct method of the calculus of variations.

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1 Introduction and Main Results

In this paper, we study the following nonlinear anisotropic elliptic problem

$$\begin{aligned} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u &= \lambda k(x)|u|^{\alpha(x)-2}u \quad \text{in } \Omega,\\ \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\eta_i &= h(x)|u|^{\beta(x)-2}u \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary and

$$\Delta_{\vec{p}(\cdot)}u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) \tag{1.2}$$

denotes the anisotropic variable exponent $\vec{p}(\cdot)$-Laplace differential operator. The vector-valued function $\vec{p}\colon \overline{\Omega} \to \mathbb{R}^N$ is defined by $\vec{p}(x) = (p_1(x), p_2(x), \dots, p_N(x))$, where $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \dots, N\}$. For $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \sup_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \inf_{x\in\overline{\Omega}} p(x),$$

where

$$C_+(\overline{\Omega}) = \left\{p(x) \in C(\overline{\Omega})\colon\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

In the following, we write

$$P_-^- = \min\{p_1^-, p_2^-, \dots, p_N^-\} \quad \text{and} \quad P_+^+ = \max\{p_1^+, p_2^+, \dots, p_N^+\}. \tag{1.3}$$

Here, $\vec{\eta} = (\eta_1, \eta_2, \dots, \eta_N)$ denotes the outward unit normal vector on $\partial\Omega$. The $\vec{p}(\cdot)$-generalized normal derivative ν of u is defined by

$$\nu = \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\eta_i,$$

see Henríquez-Amador–Vélez-Santiago [27].

Throughout this article, we assume that the following hypotheses hold:

(a1) there exists $\gamma \in C_+(\overline{\Omega})$ such that $k \in L^{\gamma(\cdot)}(\Omega)$ and $k(x) > 0$ for a.a. $x \in \Omega$;
(a2) there exists $\delta \in C_+(\partial\Omega)$ such that $h \in L^{\delta(\cdot)}(\partial\Omega)$ and $h(x) > 0$ for a.a. $x \in \partial\Omega$;
(a3) $\alpha \in C(\overline{\Omega})$ such that

$$1 < \alpha^- \le \alpha(x) \le \alpha^+ < \left(\frac{\gamma(x) - 1}{\gamma(x)}\right) \frac{NP_-^-}{N - P_-^-}$$

for all $x \in \overline{\Omega}$;
(a4) $\beta \in C(\partial\Omega)$ such that

$$1 < \beta^- \le \beta(x) \le \beta^+ < \left(\frac{\delta(x) - 1}{\delta(x)}\right) \frac{(N-1)P_-^-}{N - P_-^-}$$

for all $x \in \partial\Omega$;
(a5) $1 < \alpha^- \le \alpha^+ < P_-^- \le P_+^+ < \beta^- \le \beta^+$;
(a6) $\dfrac{P_-^-}{\beta^+} < \left(\dfrac{P_-^- - \alpha^+}{\beta^+ - \alpha^+}\right)\left(\dfrac{\beta^- - \alpha^-}{P_+^+ - \alpha^-}\right)$.

A function $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is said to be a weak solution of problem (1.1) if

$$\langle Lu, v\rangle = \lambda \int_\Omega k(x)|u|^{\alpha(x)-2} uv \,\mathrm{d}x + \int_{\partial\Omega} h(x)|u|^{\beta(x)-2} uv \,\mathrm{d}\sigma,$$

for all $v \in W^{1,\vec{p}(\cdot)}(\Omega)$, where

$$\langle Lu, v\rangle = \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \,\mathrm{d}x + \int_\Omega \sum_{i=1}^N |u|^{p_i(x)-2} uv \,\mathrm{d}x,$$

for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

The main result of this paper reads as follows.

Theorem 1.1 *Let assumptions* (**a1**)–(**a6**) *hold and define*

$$\tau = \left(\frac{\beta^- - P_+^+}{C_\Omega(\beta^- - \alpha^-)}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}},$$

with C_Ω and $C_{\partial\Omega}$ being the constants given in Lemma 2.8. Then, for any $\lambda \in (0, \tau)$, problem (1.1) *admits at least two nontrivial nonnegative weak solutions.*

The proof of Theorem 1.1 relies on the Nehari manifold approach combined with the fibering map technique. More precisely, we consider the Nehari manifold $\mathcal{N}_\lambda(\Omega)$ associated with problem (1.1), which consists of all nontrivial functions $u \in W^{1,\vec{p}(\cdot)}(\Omega)$

satisfying the Nehari constraint. It provides a natural constraint for the corresponding energy functional. A crucial step in the analysis is the decomposition of the Nehari manifold into three disjoint subsets

$$\mathcal{N}_\lambda(\Omega) = \mathcal{N}_\lambda^+(\Omega) \cup \mathcal{N}_\lambda^-(\Omega) \cup \mathcal{N}_\lambda^0(\Omega),$$

which is obtained through the study of the associated fibering map. We show that the set $\mathcal{N}_\lambda^0(\Omega)$ is empty for suitable values of the parameter λ. Consequently, the Nehari manifold splits into two nonempty components $\mathcal{N}_\lambda^+(\Omega)$ and $\mathcal{N}_\lambda^-(\Omega)$. We then minimize the energy functional on each of these two subsets. The corresponding minimizers yield two distinct nontrivial weak solutions of problem (1.1). Although this method has been used in several works, our approach is inspired in particular by the works of Afrouzi–Rasouli [4, 39].

If $p_i = 2$ for all $i = 1, \dots, N$ in (1.2), then

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \Delta u,$$

which is the classical Laplacian. If $\vec{p}$ is constant, that is, $p_i \equiv p$ for each $i = 1, \dots, N$, then we obtain

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = \tilde{\Delta}_p u,$$

which is known as the pseudo-p-Laplacian, see Belloni–Kawohl [11] or Brasco–Franzina [19]. Further information on anisotropic operators and in particular on anisotropic Sobolev spaces can be found in Kufner–Rákosník [30], Nikol'skiĭ [35], and Rákosník [37, 38]. Anisotropic differential problems arise in various applications. For example, the spread of an epidemic disease in a heterogeneous habitat can be modeled by anisotropic nonlinear systems. In general, anisotropic operators are used in models where partial derivatives exhibit different behavior depending on the spatial direction. Another important example is the anisotropic Laplacian

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \xi_i} \left(\frac{1}{2} F^2 \right) (\nabla u) \right)$$

where

$$F(\xi) = \left(\sum_{i=1}^{N} |\xi_i|^2 \right)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^N.$$

This operator plays an important role in physical models of crystal growth and is closely related to the so-called Wulff shape associated with F, see Wulff [42]. For further applications in different areas, we refer to Almi–Caponi–Friedrich–Solombrino [5], Antontsev–Díaz–Shmarev [7], Bendahmane–Chrif–El Manouni [12], Bendahmane–Langlais–Saad [13], Vétois [41], and the references therein.

The $\vec{p}(\cdot)$-Laplacian is a nonhomogeneous generalization of the classical $p(\cdot)$-Laplacian. In contrast to the isotropic case, it allows different growth conditions for the partial derivatives in different spatial directions. As a consequence, the associated differential operator exhibits a more complicated nonlinear structure and requires the use of anisotropic variable exponent Sobolev spaces. Variational problems involving the $\vec{p}(\cdot)$-Laplacian have attracted considerable attention in recent years due to their wide range of applications in the modeling of physical phenomena. In particular, such operators naturally arise in the study of electrorheological fluids, nonlinear elastic materials, thermorheological fluids, and other processes with spatially heterogeneous properties. The development of the theory of variable exponent Sobolev spaces and related anisotropic frameworks has provided powerful tools for the analysis of these models. We refer to Acerbi–Mingione [1, 2], Antontsev–Rodrigues [8], Mihăilescu–Pucci–Rădulescu [34], Růžička [40], Zhikov [46] for fundamental results and further references. More recently, the theory of variable exponent spaces has been extended in several directions in order to describe increasingly complex systems. These developments include anisotropic settings, fractional operators, and applications to nonlinear problems arising in elasticity and materials science. Such advances illustrate the flexibility of the variable exponent framework and its capability to model phenomena with nonuniform growth and strongly nonlinear behavior.

In general, the literature on anisotropic variable exponent problems is still rather limited. One of the first contributions in this direction is due to Mihăilescu–Pucci–Rădulescu [32], who studied the eigenvalue problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where $\vec{p}(\cdot)$ satisfies the assumptions stated above and $q(x) > 1$ for all $x \in \overline{\Omega}$. The authors established several existence and nonexistence results, which strongly depend on the interplay between the growth rates of the anisotropic exponents. Their results reveal interesting phenomena concerning the structure of the spectrum, including the appearance of a continuous spectrum in various situations. Later, Boureanu–Pucci–Rădulescu [16] considered the problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = f(x,u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an odd Carathéodory function satisfying the Ambrosetti-Rabinowitz condition. By applying the symmetric mountain pass theorem, the authors proved the existence of an unbounded sequence of solutions. Further existence results for anisotropic problems, both with and without variable exponents, can be found for instance in Afrouzi–Mirzapour–Rădulescu [3], Amoroso–Sciammetta–Winkert

[6], Boureanu [14, 15], Eddine–Ragusa–Repovš [23], Fragalà–Gazzola–Kawohl [26], Mihăilescu–Pucci–Rădulescu [33], Rădulescu–Stăncuţ [36], Zhan–Feng [44], see also the references therein. We also mention papers dealing with parabolic anisotropic problems, see Antontsev–Shmarev [9], Arora–Shmarev [10], Liang–Zhao–Zhai [31], and Zhan–Feng [43, 45].

The first work dealing with the anisotropic variable exponent $\vec{p}(\cdot)$-Laplacian with homogeneous Neumann boundary condition is due to Ji [29], who studied the problem

$$\begin{aligned}
-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u &= \lambda|u|^{q(x)-2}u \quad \text{in } \Omega,\\
\frac{\partial u}{\partial \eta} &= 0 \qquad\qquad\quad \text{on } \partial\Omega,
\end{aligned}$$

where the existence of a nontrivial solution was proved by using the mountain pass geometry of the associated energy functional. Moreover, several important embedding results were established. Further results for problems with nonhomogeneous anisotropic Neumann boundary condition have been published by Boureanu–Rădulescu [17], Boureanu–Vélez-Santiago [18], and Eddine–Ragusa [22].

The paper is organized as follows. In Sect. 2, we recall some basic properties of variable exponent Sobolev spaces, introduce weighted Lebesgue spaces both in the domain and on the boundary, and establish several embedding results between these spaces. Section 3 is devoted to a detailed analysis of the associated Nehari manifold and the corresponding fibering map. Finally, Sect. 4 contains the proof of Theorem 1.1.

2 Preliminary Results

In this section, we recall some basic results on variable exponent Sobolev spaces and anisotropic variable exponent Sobolev spaces. Most of these results can be found in the monograph by Diening–Harjulehto–Hästö–Růžička [20] as well as in the papers by Fan [24] and Fan–Zhao [25].

To this end, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $M(\Omega)$ denote the set of all real-valued measurable functions, and let $p \in C_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{u \in M(\Omega)\colon \int_\Omega |u|^{p(x)}\,\mathrm{d}x < +\infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)}\,\mathrm{d}x \le 1\right\}.$$

It is well known that $L^{p(\cdot)}(\Omega)$ is a separable, uniformly convex, and reflexive Banach space whose dual space is given by $(L^{p(\cdot)}(\Omega))^* = L^{p'(\cdot)}(\Omega)$, where $p' \in C_+(\overline{\Omega})$ is the conjugate exponent of p defined by $p'(x) = p(x)/(p(x)-1)$ for all $x \in \overline{\Omega}$. For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following, Hölder-type inequality holds

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

Moreover, if $p_1, p_2 \in C_+(\overline{\Omega})$ satisfy $p_1(x) \leq p_2(x)$ for all $x \in \overline{\Omega}$, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

The modular function $\rho_{p(\cdot)}\colon L^{p(\cdot)}(\Omega) \to [0,\infty)$ associated with $L^{p(\cdot)}(\Omega)$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

In what follows, we write $\rho := \rho_{p(\cdot)}$.

Proposition 2.1 *Let $p \in C_+(\overline{\Omega})$, $u, u_n \in L^{p(\cdot)}(\Omega)$ for all $n \in \mathbb{N}$, and $\lambda > 0$. Then the following hold:*

(i) $\|u\|_{p(\cdot)} = \lambda$ *if and only if* $\rho(\frac{u}{\lambda}) = 1$.
(ii) $\|u\|_{p(\cdot)} < 1$ *(resp.* $=, > 1$*) if and only if* $\rho(u) < 1$ *(resp.* $=, > 1$*).*
(iii) *If* $\|u\|_{p(\cdot)} < 1$, *then* $\|u\|_{p(\cdot)}^{p^+} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^-}$.
(iv) *If* $\|u\|_{p(\cdot)} > 1$, *then* $\|u\|_{p(\cdot)}^{p^-} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^+}$.
(v) $\|u_n - u\|_{p(\cdot)} \to 0$ *if and only if* $\rho(u_n - u) \to 0$.
(vi) *If* $u_n \to u$ *in* $L^{p(\cdot)}(\Omega)$, *then* $\rho(u_n) \to \rho(u)$.
(vii) *The functional* ρ *is weakly sequentially lower semicontinuous.*

Note that if $h \in L^\infty(\Omega)$, $h \geq 0$, $p \in C_+(\overline{\Omega})$, and $|u|^{h(\cdot)} \in L^{p(\cdot)}(\Omega)$, then the inequality

$$\min\left\{\|u\|_{h(\cdot)p(\cdot)}^{h^-}, \|u\|_{h(\cdot)p(\cdot)}^{h^+}\right\} \leq \left\||u|^{h(\cdot)}\right\|_{p(\cdot)} \leq \max\left\{\|u\|_{h(\cdot)p(\cdot)}^{h^-}, \|u\|_{h(\cdot)p(\cdot)}^{h^+}\right\}$$

holds. If $h(x) = h$ is constant, then $\||u|^h\|_{p(\cdot)} = \|u\|_{hp(\cdot)}^h$.

Next, we introduce weighted variable exponent Lebesgue spaces. For this purpose, let $w\colon \Omega \to \mathbb{R}$ be a measurable weight function such that $w(x) > 0$ for a.a. $x \in \Omega$. Then $L^{p(\cdot)}_{w(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}_{w(\cdot)}(\Omega) = \left\{u \in M(\Omega)\colon \int_\Omega w(x)|u|^{p(x)}\,\mathrm{d}x < +\infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{w(\cdot),p(\cdot)} = \left\|(w(x))^{\frac{1}{p(\cdot)}}u\right\|_{p(\cdot)}.$$

The space $L^{p(\cdot)}_{w(\cdot)}(\Omega)$ is a Banach space with properties similar to those of the usual variable exponent Lebesgue spaces. The related modular $\rho_{w(\cdot),p(\cdot)}\colon L^{p(\cdot)}_{w(\cdot)}(\Omega) \to [0,\infty)$ is defined by

$$\rho_{w(\cdot),p(\cdot)}(u) = \rho_{p(\cdot)}\left((w(\cdot))^{\frac{1}{p(\cdot)}}u\right) = \int_\Omega w(x)|u|^{p(x)}\,\mathrm{d}x.$$

The next proposition provides properties analogous to those stated in Proposition 2.1, see Ho–Sim [28, Propositions 2.2 and 2.3].

Proposition 2.2 *Let $p \in C_+(\overline{\Omega})$, $u, u_n \in L^{p(\cdot)}_{w(\cdot)}(\Omega)$ for all $n \in \mathbb{N}$, and $\lambda > 0$. Then the following hold:*

(i) $\|u\|_{w(\cdot),p(\cdot)} = \lambda$ *if and only if* $\rho_{w(\cdot),p(\cdot)}(\frac{u}{\lambda}) = 1$.
(ii) $\|u\|_{w(\cdot),p(\cdot)} < 1$ *(resp.* $=, > 1$*) if and only if* $\rho_{w(\cdot),p(\cdot)}(u) < 1$ *(resp.* $=, > 1$*).*
(iii) *If* $\|u\|_{w(\cdot),p(\cdot)} \leq 1$, *then* $\|u\|^{p^+}_{w(\cdot),p(\cdot)} \leq \rho_{w(\cdot),p(\cdot)}(u) \leq \|u\|^{p^-}_{w(\cdot),p(\cdot)}$.
(iv) *If* $\|u\|_{w(\cdot),p(\cdot)} \geq 1$, *then* $\|u\|^{p^-}_{w(\cdot),p(\cdot)} \leq \rho_{w(\cdot),p(\cdot)}(u) \leq \|u\|^{p^+}_{w(\cdot),p(\cdot)}$.
(v) $\|u_n - u\|_{w(\cdot),p(\cdot)} \to 0$ *if and only if* $\rho_{w(\cdot),p(\cdot)}(u_n - u) \to 0$.
(vi) *If* $u_n \to u$ *in* $L^{p(\cdot)}_{w(\cdot)}(\Omega)$, *then* $\rho_{w(\cdot),p(\cdot)}(u_n) \to \rho_{w(\cdot),p(\cdot)}(u)$.

Furthermore, let σ denote the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$. For $r \in C_+(\partial\Omega)$, we denote by $L^{r(\cdot)}(\partial\Omega)$ the variable exponent Lebesgue space on the boundary $\partial\Omega$ endowed with the norm $\|\cdot\|_{r(\cdot),\partial\Omega}$. Analogously, for $w\colon \partial\Omega \to \mathbb{R}$ being a measurable weight function such that $w(x) > 0$ for a.a. $x \in \partial\Omega$, we define the weighted variable exponent Lebesgue space on the boundary $L^{r(\cdot)}_{w(\cdot)}(\partial\Omega)$ equipped with the norm $\|u\|_{w(\cdot),r(\cdot),\partial\Omega}$.

Now, we introduce the underlying function space. Let $\vec{p}\colon \overline{\Omega} \to \mathbb{R}^N$ be defined by $\vec{p}(x) = (p_1(x), p_2(x), \dots, p_N(x))$, where each $p_i \in C(\overline{\Omega})$ satisfies $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \dots, N\}$. Then the anisotropic variable exponent Sobolev space is defined by

$$W^{1,\vec{p}(\cdot)}(\Omega) = \left\{u \in L^{p_i(\cdot)}(\Omega)\colon \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\right\},$$

endowed with the norm

$$\|u\|_{\vec{p}(\cdot)} = \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(\cdot)} + \sum_{i=1}^N \|u\|_{p_i(\cdot)}.$$

Note that the space $W^{1,\vec{p}(\cdot)}(\Omega)$, equipped with the norm $\|\cdot\|_{\vec{p}(\cdot)}$, is a reflexive Banach space, see Ji [29]. For more details on anisotropic Sobolev spaces, we refer to Boureanu–Rădulescu [17], Henríquez-Amador–Vélez-Santiago [27], Mihăilescu–Pucci–Rădulescu [32], and the references therein.

The following result is a direct consequence of the definition of variable exponent Lebesgue and Sobolev spaces. In what follows, we use the notation introduced in (1.3).

Proposition 2.3 *Let*

$$D(u) = \int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x$$

for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$. Then the following hold:

(i) *If $\|u\|_{\vec{p}(\cdot)} \geq 1$, then $\|u\|_{\vec{p}(\cdot)}^{P_-^-} \leq D(u) \leq \|u\|_{\vec{p}(\cdot)}^{P_+^+}$.*

(ii) *If $\|u\|_{\vec{p}(\cdot)} \leq 1$, then $\|u\|_{\vec{p}(\cdot)}^{P_+^+} \leq D(u) \leq \|u\|_{\vec{p}(\cdot)}^{P_-^-}$.*

The following result can be found in the paper by Ji [29, Theorem 2.2].

Proposition 2.4 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $p_i \in C(\overline{\Omega})$ satisfies $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \ldots, N\}$. If $q \in C_+(\overline{\Omega})$ satisfies $q(x) < \frac{NP_-^-}{N-P_-^-}$ for all $x \in \overline{\Omega}$, then the embedding*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

Next, we establish an analogous result for boundary variable exponent Lebesgue spaces.

Proposition 2.5 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ having the cone property and $p_i \in C(\overline{\Omega})$ fulfills $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \ldots, N\}$. If $q \in C_+(\partial\Omega)$ satisfies $q(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$, then the embedding*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$$

is continuous and compact.

Proof First, the proof of Proposition 2.4 implies that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,P_-^-}(\Omega)$ is continuous. By hypothesis, we have

$$1 < q^+ = \sup_{x \in \partial\Omega} q(x) < \frac{(N-1)P_-^-}{N - P_-^-}.$$

Hence, the trace operator $\gamma\colon W^{1,P_-^-}(\Omega) \to L^{q^+}(\partial\Omega)$ is compact. Moreover, since $q(x) \leq q^+$ for all $x \in \partial\Omega$, the embedding $L^{q^+}(\partial\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ is continuous. Therefore, the following chain of embeddings holds:

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,P_-^-}(\Omega) \overset{c}{\hookrightarrow} L^{q^+}(\partial\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega).$$

Consequently, the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ is continuous and compact. This completes the proof. $\square$

Next we establish suitable embeddings into weighted variable exponent spaces.

Lemma 2.6 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $p_i \in C(\overline{\Omega})$ satisfies $2 \le p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \dots, N\}$. Assume (a1) and (a3). Then the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\alpha(\cdot)}_{k(\cdot)}(\Omega)$ is continuous and compact. In particular, there exists a constant $C_\Omega > 0$ such that*

$$\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \le C_\Omega \left(\|u\|^{\alpha^-}_{\vec{p}(\cdot)} + \|u\|^{\alpha^+}_{\vec{p}(\cdot)} \right)$$

for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof Let γ_0 be the conjugate exponent of γ, $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ and define

$$m(x) = \left(\frac{\gamma(x)}{\gamma(x) - 1} \right) \alpha(x) \quad \text{for all } x \in \overline{\Omega}.$$

It follows that

$$\gamma_0(x) = \frac{\gamma(x)}{\gamma(x) - 1} \quad \text{and} \quad m(x) = \gamma_0(x)\alpha(x) \quad \text{for all } x \in \overline{\Omega}.$$

Using assumption (a3), we obtain

$$m(x) = \gamma_0(x)\alpha(x) < \gamma_0(x) \left(\frac{\gamma(x) - 1}{\gamma(x)} \right) \frac{NP_-^-}{N - P_-^-} = \frac{NP_-^-}{N - P_-^-}$$

for all $x \in \overline{\Omega}$. Hence $m(x) < \frac{NP_-^-}{N-P_-^-}$ for all $x \in \overline{\Omega}$. By Proposition 2.4, the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ is continuous and compact. Consequently, $u \in L^{m(\cdot)}(\Omega) = L^{\gamma_0(\cdot)\alpha(\cdot)}(\Omega)$, which implies $|u|^{\alpha(\cdot)} \in L^{\gamma_0(\cdot)}(\Omega)$. Applying Hölder's inequality yields

$$\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \le 2\|k\|_{\gamma(\cdot)} \left\| |u|^{\alpha(\cdot)} \right\|_{\gamma_0(\cdot)} < +\infty.$$

Thus $u \in L^{\alpha(\cdot)}_{k(\cdot)}(\Omega)$ and therefore $W^{1,\vec{p}(\cdot)}(\Omega) \subset L^{\alpha(\cdot)}_{k(\cdot)}(\Omega)$.

Now, let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\vec{p}(\cdot)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Since the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ is compact, we obtain $u_n \to u$ in $L^{m(\cdot)}(\Omega) = L^{\gamma_0(\cdot)\alpha(\cdot)}(\Omega)$. Hence,

$$\|u_n - u\|_{\gamma_0(\cdot)\alpha(\cdot)} \to 0 \quad \text{as } n \to \infty,$$

which implies

$$\left\| |u_n - u|^{\alpha(\cdot)} \right\|_{\gamma_0(\cdot)} \to 0 \quad \text{as } n \to \infty.$$

Again using Hölder's inequality, one has

$$\int_{\Omega} k(x)|u_n - u|^{\alpha(x)}\,\mathrm{d}x \le 2\|k\|_{\gamma(\cdot)} \left\| |u_n - u|^{\alpha(\cdot)} \right\|_{\gamma_0(\cdot)} \to 0 \quad \text{as } n \to \infty.$$

By Proposition 2.2, it follows that

$$\|u_n - u\|_{k(\cdot),\alpha(\cdot)} \to 0 \quad \text{as } n \to \infty,$$

that is, $u_n \to u$ in $L^{\alpha(\cdot)}_{k(\cdot)}(\Omega)$. Hence, the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\alpha(\cdot)}_{k(\cdot)}(\Omega)$ is compact and therefore continuous.

Finally, from (**a3**) we have

$$\gamma_0(x)\alpha^- \le \gamma_0(x)\alpha(x) \le \gamma_0(x)\alpha^+ < \frac{NP^-_-}{N - P^-_-} \quad \text{for all } x \in \overline{\Omega}.$$

Using Hölder's inequality and Proposition 2.4, we obtain

$$\int_{\Omega} k(x)|u|^{\alpha^-}\,\mathrm{d}x \le 2\|k\|_{\gamma(\cdot)}\|u\|^{\alpha^-}_{\alpha^-\gamma_0(\cdot)} \le \hat{C}_{\Omega}\|u\|^{\alpha^-}_{\vec{p}(\cdot)}, \tag{2.1}$$

$$\int_{\Omega} k(x)|u|^{\alpha^+}\,\mathrm{d}x \le 2\|k\|_{\gamma(\cdot)}\|u\|^{\alpha^+}_{\alpha^+\gamma_0(\cdot)} \le \tilde{C}_{\Omega}\|u\|^{\alpha^+}_{\vec{p}(\cdot)}, \tag{2.2}$$

where $\hat{C}_{\Omega}$ and $\tilde{C}_{\Omega}$ are two positive constants. Consequently, from (2.1) and (2.2), there exists a constant $C_{\Omega} > 0$ such that

$$\begin{aligned}\int_{\Omega} k(x)|u|^{\alpha(x)}\,\mathrm{d}x &\le \int_{\Omega} k(x)|u|^{\alpha^-}\,\mathrm{d}x + \int_{\Omega} k(x)|u|^{\alpha^+}\,\mathrm{d}x \\ &\le C_{\Omega}\left(\|u\|^{\alpha^-}_{\vec{p}(\cdot)} + \|u\|^{\alpha^+}_{\vec{p}(\cdot)}\right).\end{aligned}$$

This concludes the proof. □

We also have an analogous result for weighted variable exponent Lebesgue spaces on the boundary.

Lemma 2.7 *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ having the cone property and $p_i \in C(\overline{\Omega})$ satisfies $2 \le p_i(x) < N$ for all $x \in \overline{\Omega}$ and for each $i \in \{1, 2, \ldots, N\}$. Assume (**a2**) and (**a4**). Then the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega)$ is continuous and compact. In particular, there exists a constant $C_{\partial\Omega} > 0$ such that*

$$\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \le C_{\partial\Omega}\left(\|u\|^{\beta^-}_{\vec{p}(\cdot)} + \|u\|^{\beta^+}_{\vec{p}(\cdot)}\right)$$

for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof Let δ_0 denote the conjugate exponent of δ. For $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, define

$$n(x) = \left(\frac{\delta(x)}{\delta(x)-1}\right)\beta(x) \quad \text{for all } x \in \partial\Omega.$$

Then

$$\delta_0(x) = \frac{\delta(x)}{\delta(x)-1} \quad \text{and} \quad n(x) = \delta_0(x)\beta(x) \quad \text{for all } x \in \partial\Omega.$$

From (**a4**), we obtain

$$n(x) = \delta_0(x)\beta(x) < \delta_0(x)\left(\frac{\delta(x)-1}{\delta(x)}\right)\frac{(N-1)P_-^-}{N-P_-^-} = \frac{(N-1)P_-^-}{N-P_-^-}$$

for all $x \in \partial\Omega$. This implies $n(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$. From Proposition 2.5, we know that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{n(\cdot)}(\partial\Omega)$ is continuous and compact. Thus, $u \in L^{n(\cdot)}(\partial\Omega) = L^{\delta_0(\cdot)\beta(\cdot)}(\partial\Omega)$ and so $|u|^{\beta(\cdot)} \in L^{\delta_0(\cdot)}(\partial\Omega)$. Taking Hölder's inequality into account yields

$$\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \le 2\|h\|_{L^{\delta(\cdot)}(\partial\Omega)} \left\||u|^{\beta(\cdot)}\right\|_{L^{\delta_0(\cdot)}(\partial\Omega)} < +\infty.$$

From this, we infer that $u \in L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega)$ and $W^{1,\vec{p}(\cdot)}(\Omega) \subset L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega)$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\vec{p}(\cdot)}(\Omega)$ be such that $u_n \rightharpoonup u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Because the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{n(\cdot)}(\partial\Omega)$ is compact, it follows that $u_n \to u$ in $L^{n(\cdot)}(\partial\Omega) = L^{\delta_0(\cdot)\beta(\cdot)}(\partial\Omega)$. This gives

$$\|u_n - u\|_{L^{\delta_0(\cdot)\beta(\cdot)}(\partial\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\left\||u_n - u|^{\beta(\cdot)}\right\|_{L^{\delta_0(\cdot)}(\partial\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Further, by using Hölder's inequality, we have

$$\int_{\partial\Omega} h(x)|u_n - u|^{\beta(x)}\,\mathrm{d}\sigma \le 2\|h\|_{L^{\delta(\cdot)}(\partial\Omega)} \left\||u_n - u|^{\beta(\cdot)}\right\|_{L^{\delta_0(\cdot)}(\partial\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Arguing as in Proposition 2.2, it follows that $u_n \to u$ in $L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega)$. This shows that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega)$ is compact and hence, continuous as well.

Now, by using (**a4**), we have

$$\delta_0(x)\beta^- \le \delta_0(x)\beta(x) \le \delta_0(x)\beta^+ < \frac{(N-1)P_-^-}{N-P_-^-} \quad \text{for all } x \in \partial\Omega. \tag{2.3}$$

Due to Hölder's inequality, (2.3) and Proposition 2.5, we get

$$\int_{\partial\Omega} h(x)|u|^{\beta^-}\,\mathrm{d}\sigma \le 2\|h\|_{L^{\delta(\cdot)}(\partial\Omega)}\|u\|^{\beta^-}_{L^{\beta^-\delta_0(\cdot)}(\partial\Omega)} \le \hat{C}_{\partial\Omega}\|u\|^{\beta^-}_{\vec{p}(\cdot)}, \tag{2.4}$$

$$\int_{\partial\Omega} h(x)|u|^{\beta^+}\,\mathrm{d}\sigma \le 2\|h\|_{L^{\delta(\cdot)}(\partial\Omega)}\|u\|^{\beta^+}_{L^{\beta^+\delta_0(\cdot)}(\partial\Omega)} \le \tilde{C}_{\partial\Omega}\|u\|^{\beta^+}_{\vec{p}(\cdot)}, \tag{2.5}$$

where $\hat{C}_{\partial\Omega}$ and $\tilde{C}_{\partial\Omega}$ are positive constants. Therefore, from (2.4) and (2.5), we can find a constant $C_{\partial\Omega}$ such that

$$\begin{aligned}\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma &\le \int_{\partial\Omega} h(x)|u|^{\beta^-}\,\mathrm{d}\sigma + \int_{\partial\Omega} h(x)|u|^{\beta^+}\,\mathrm{d}\sigma\\ &\le C_{\partial\Omega}\left(\|u\|^{\beta^-}_{\vec{p}(\cdot)} + \|u\|^{\beta^+}_{\vec{p}(\cdot)}\right).\end{aligned}$$

□

The following result is a direct consequence of Lemmas 2.6 and 2.7.

Lemma 2.8 *Assume that the hypotheses of Lemmas 2.6 and 2.7 hold. Then, for every $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, the following inequalities are satisfied:*

(i) $\displaystyle\int_{\Omega} k(x)|u|^{\alpha(x)}\,\mathrm{d}x \le \begin{cases} C_\Omega\|u\|^{\alpha^+}_{\vec{p}(\cdot)} & \text{if } \|u\|_{\vec{p}(\cdot)} \ge 1,\\ C_\Omega\|u\|^{\alpha^-}_{\vec{p}(\cdot)} & \text{if } \|u\|_{\vec{p}(\cdot)} \le 1.\end{cases}$

(ii) $\displaystyle\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \le \begin{cases} C_{\partial\Omega}\|u\|^{\beta^+}_{\vec{p}(\cdot)} & \text{if } \|u\|_{\vec{p}(\cdot)} \ge 1,\\ C_{\partial\Omega}\|u\|^{\beta^-}_{\vec{p}(\cdot)} & \text{if } \|u\|_{\vec{p}(\cdot)} \le 1,\end{cases}$

where C_Ω and $C_{\partial\Omega}$ are positive constants.

3 The Nehari Manifold and Fibering Map Approach

In this section, we analyze the Nehari manifold and the associated fibering map, which are used in Sect. 4 to prove Theorem 1.1. To this end, let $\mathcal{J}_\lambda\colon W^{1,\vec{p}(\cdot)}(\Omega) \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$\begin{aligned}\mathcal{J}_\lambda(u) = &\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega \frac{k(x)}{\alpha(x)}|u|^{\alpha(x)}\,\mathrm{d}x\\ &- \int_{\partial\Omega}\frac{h(x)}{\beta(x)}|u|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned}$$

It is easy to verify that $\mathcal{J}_\lambda$ is well-defined and belongs to $C^1(W^{1,\vec{p}(\cdot)}(\Omega), \mathbb{R})$. Its Fréchet derivative is given by

$$\langle \mathcal{J}_\lambda'(u), v\rangle = \langle Lu, v\rangle - \lambda\int_\Omega k(x)|u|^{\alpha(x)-2}uv\,\mathrm{d}x - \int_{\partial\Omega} h(x)|u|^{\beta(x)-2}uv\,\mathrm{d}\sigma,$$

for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$. Moreover, the critical points of $\mathcal{J}_\lambda$ coincide with the weak solutions of (1.1). The corresponding Nehari manifold associated with problem (1.1) is defined by

$$\mathcal{N}_\lambda(\Omega) = \left\{u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \langle \mathcal{J}_\lambda'(u), u\rangle = 0\right\}.$$

Clearly, every nonzero critical point of $\mathcal{J}_\lambda$ belongs to $\mathcal{N}_\lambda(\Omega)$. Furthermore, define $\mathcal{I}_\lambda(u) = \langle \mathcal{J}_\lambda'(u), u\rangle$. Hence, if $u \in \mathcal{N}_\lambda(\Omega)$, then $\mathcal{I}_\lambda(u) = 0$. A straightforward computation shows that for all $u \in \mathcal{N}_\lambda(\Omega)$,

$$\begin{aligned}\langle \mathcal{I}_\lambda'(u), u\rangle = &\int_\Omega \sum_{i=1}^N p_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega k(x)\alpha(x)|u|^{\alpha(x)}\,\mathrm{d}x\\ &- \int_{\partial\Omega} h(x)\beta(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned}$$

The Nehari manifold is closely related to the behavior of the function

$$\Phi_{\lambda,u}\colon [0,+\infty) \to \mathbb{R}, \quad \Phi_{\lambda,u}(t) = \mathcal{J}_\lambda(tu).$$

Such functions are called fibering maps and were introduced by Drábek–Pohozaev [21]. For $u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$, we obtain

$$\begin{aligned}\Phi_{\lambda,u}(t) = &\int_\Omega \sum_{i=1}^N t^{p_i(x)}\frac{1}{p_i(x)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega t^{\alpha(x)}\frac{k(x)}{\alpha(x)}|u|^{\alpha(x)}\,\mathrm{d}x\\ &- \int_{\partial\Omega} t^{\beta(x)}\frac{h(x)}{\beta(x)}|u|^{\beta(x)}\,\mathrm{d}\sigma,\\ \Phi_{\lambda,u}'(t) = &\int_\Omega \sum_{i=1}^N t^{p_i(x)-1}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega t^{\alpha(x)-1}k(x)|u|^{\alpha(x)}\,\mathrm{d}x\\ &- \int_{\partial\Omega} t^{\beta(x)-1}h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma,\\ \Phi_{\lambda,u}''(t) = &\int_\Omega \sum_{i=1}^N (p_i(x)-1)t^{p_i(x)-2}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x\\ &- \lambda\int_\Omega (\alpha(x)-1)t^{\alpha(x)-2}k(x)|u|^{\alpha(x)}\,\mathrm{d}x\\ &- \int_{\partial\Omega} (\beta(x)-1)t^{\beta(x)-2}h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned}$$

Lemma 3.1 *Let $u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ and $t > 0$. Then $tu \in \mathcal{N}_\lambda(\Omega)$ if and only if $\Phi_{\lambda,u}'(t) = 0$. Moreover, the Nehari manifold $\mathcal{N}_\lambda(\Omega)$ is a closed subset of $W^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof It is easy to verify that

$$\Phi'_{\lambda,u}(t) = \langle \mathcal{J}'_\lambda(tu), u\rangle = \frac{1}{t}\langle \mathcal{J}'_\lambda(tu), tu\rangle,$$

for all $u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ and for all $t > 0$. Hence, we infer that $tu \in \mathcal{N}_\lambda(\Omega)$ if and only if $\Phi'_{\lambda,u}(t) = 0$. By the definition of Nehari manifold, we have $\mathcal{N}_\lambda(\Omega) \cup \{0\} = \mathcal{I}_\lambda^{-1}(\{0\})$. Since $\mathcal{I}_\lambda$ is a continuous functional from $W^{1,\vec{p}(\cdot)}(\Omega)$ into $\mathbb{R}$ and $\{0\}$ is a closed set in $\mathbb{R}$, it follows that $\mathcal{I}_\lambda^{-1}(\{0\})$ is closed in $W^{1,\vec{p}(\cdot)}(\Omega)$. Consequently, $\mathcal{N}_\lambda(\Omega)$ is a closed subset of $W^{1,\vec{p}(\cdot)}(\Omega)$. □

By Lemma 3.1, we have $tu \in \mathcal{N}_\lambda(\Omega)$ if and only if $\Phi'_{\lambda,u}(t) = 0$. In particular, $u \in \mathcal{N}_\lambda(\Omega)$ if and only if $\Phi'_{\lambda,u}(1) = 0$. Therefore, it is natural to decompose $\mathcal{N}_\lambda(\Omega)$ into three subsets corresponding to local minima, local maxima, and points of inflection of the fibering map. More precisely, we define:

$$\begin{aligned}
\mathcal{N}_\lambda^+(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) \colon \Phi''_{\lambda,u}(1) > 0\}\\
&= \{tu \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \Phi'_{\lambda,u}(t) = 0, \Phi''_{\lambda,u}(t) > 0\}\\
&= \{u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \langle \mathcal{J}'_\lambda(u), u\rangle = 0, \langle \mathcal{I}'_\lambda(u), u\rangle > 0\},\\
\mathcal{N}_\lambda^-(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) \colon \Phi''_{\lambda,u}(1) < 0\}\\
&= \{tu \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \Phi'_{\lambda,u}(t) = 0, \Phi''_{\lambda,u}(t) < 0\}\\
&= \{u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \langle \mathcal{J}'_\lambda(u), u\rangle = 0, \langle \mathcal{I}'_\lambda(u), u\rangle < 0\},\\
\mathcal{N}_\lambda^0(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) \colon \Phi''_{\lambda,u}(1) = 0\}\\
&= \{tu \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \Phi'_{\lambda,u}(t) = 0, \Phi''_{\lambda,u}(t) = 0\}\\
&= \{u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\} \colon \langle \mathcal{J}'_\lambda(u), u\rangle = 0, \langle \mathcal{I}'_\lambda(u), u\rangle = 0\}.
\end{aligned}$$

First, we show that $\mathcal{J}_\lambda$ is not bounded from below on $W^{1,\vec{p}(\cdot)}(\Omega)$, but it is bounded from below on $\mathcal{N}_\lambda(\Omega)$.

Lemma 3.2 *The functional $\mathcal{J}_\lambda$ is not bounded from below on $W^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof Let $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{\vec{p}(\cdot)} > 1$. By the definition of $\mathcal{J}_\lambda$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(u) \geq{}& \frac{1}{P_+^+}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x - \frac{\lambda}{\alpha^-}\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x\\
&- \frac{1}{\beta^-}\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.
\end{aligned} \tag{3.1}$$

Using Proposition 2.3 and Lemma 2.8, we obtain

$$\mathcal{J}_\lambda(u) \geq \frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{P_+^+} - \frac{\lambda C_\Omega}{\alpha^-}\|u\|_{\vec{p}(\cdot)}^{\alpha^+} - \frac{C_{\partial\Omega}}{\beta^-}\|u\|_{\vec{p}(\cdot)}^{\beta^+}.$$

Since $1 < \alpha^+ < P_-^- < \beta^+$, the last term dominates as $\|u\|_{\vec{p}(\cdot)} \to \infty$. Consequently, $\mathcal{J}_\lambda(u) \to -\infty$ and therefore $\mathcal{J}_\lambda$ is not bounded from below on $W^{1,\vec{p}(\cdot)}(\Omega)$. □

Lemma 3.3 *The functional $\mathcal{J}_\lambda$ is coercive and bounded from below on $\mathcal{N}_\lambda(\Omega)$.*

Proof Let $u \in \mathcal{N}_\lambda(\Omega)$ be such that $\|u\|_{\vec{p}(\cdot)} > 1$. Since $u \in \mathcal{N}_\lambda(\Omega)$, we have $\Phi'_{\lambda,u}(1) = 0$, which yields

$$\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma = \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x. \tag{3.2}$$

Substituting (3.2) into the estimate for $\mathcal{J}_\lambda$ in (3.1), we obtain

$$\begin{aligned}\mathcal{J}_\lambda(u) \geq &\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x\\ &- \lambda\left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x.\end{aligned}$$

Using Proposition 2.3 and Lemma 2.8, we deduce

$$\mathcal{J}_\lambda(u) \geq \left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\|u\|_{\vec{p}(\cdot)}^{P_-^-} - \lambda\left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)C_\Omega\|u\|_{\vec{p}(\cdot)}^{\alpha^+}.$$

Since $P_-^- > \alpha^+$, it follows that $\mathcal{J}_\lambda(u) \to \infty$ as $\|u\|_{\vec{p}(\cdot)} \to \infty$. Hence, $\mathcal{J}_\lambda$ is coercive on $\mathcal{N}_\lambda(\Omega)$. In particular, it is bounded from below on $\mathcal{N}_\lambda(\Omega)$. □

Next, we prove that a local minimizer u of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+(\Omega)$ or $\mathcal{N}_\lambda^-(\Omega)$ is a critical point of $\mathcal{J}_\lambda$ provided $u \notin \mathcal{N}_\lambda^0(\Omega)$.

Lemma 3.4 *Let u be a local minimizer of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+(\Omega)$ or $\mathcal{N}_\lambda^-(\Omega)$ such that $u \notin \mathcal{N}_\lambda^0(\Omega)$. Then u is a critical point of $\mathcal{J}_\lambda$.*

Proof By assumption, u is a local minimizer for $\mathcal{J}_\lambda$ subject to the constraint $\mathcal{I}_\lambda(u) = \langle \mathcal{J}_\lambda'(u), u\rangle = 0$. By the Lagrange multiplier rule, there exists $\zeta \in \mathbb{R}$ such that

$$\mathcal{J}_\lambda'(u) = \zeta\mathcal{I}_\lambda'(u). \tag{3.3}$$

It follows that $\langle \mathcal{J}_\lambda'(u), u\rangle = \zeta\langle \mathcal{I}_\lambda'(u), u\rangle$. Since $u \in \mathcal{N}_\lambda(\Omega)$, we have $\langle \mathcal{J}_\lambda'(u), u\rangle = 0$ and therefore $\zeta\langle \mathcal{I}_\lambda'(u), u\rangle = 0$. Recalling that $\langle \mathcal{I}_\lambda'(u), u\rangle = \Phi''_{\lambda,u}(1)$, we obtain $\zeta\Phi''_{\lambda,u}(1) = 0$. Since $u \notin \mathcal{N}_\lambda^0(\Omega)$, it follows that $\Phi''_{\lambda,u}(1) \neq 0$. Hence, $\zeta = 0$ and by (3.3) we obtain $\mathcal{J}_\lambda'(u) = 0$. Therefore, u is a critical point of $\mathcal{J}_\lambda$. □

Set

$$\tau = \left(\frac{\beta^- - P_+^+}{C_\Omega(\beta^- - \alpha^-)}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}},$$

where C_Ω and $C_{\partial\Omega}$ are the constants given in Lemma 2.8.

Lemma 3.5 *For any $\lambda \in (0, \tau)$, we have $\mathcal{N}_\lambda^0(\Omega) = \emptyset$.*

Proof Assume by contradiction that $\mathcal{N}_\lambda^0(\Omega) \neq \emptyset$. Then there exists $u \in \mathcal{N}_\lambda^0(\Omega)$ for some $\lambda \in (0, \tau)$. Since $u \in \mathcal{N}_\lambda^0(\Omega)$, we have $\Phi'_{\lambda,u}(1) = 0$ and $\Phi''_{\lambda,u}(1) = 0$. From $\Phi'_{\lambda,u}(1) = 0$, it follows that

$$\begin{aligned}
&\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \\
&= \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

Case 1: $\|u\|_{\vec{p}(\cdot)} \le 1$.
Using $\Phi''_{\lambda,u}(1) = 0$ together with (3.4), we obtain

$$\begin{aligned}
0 &= \Phi''_{\lambda,u}(1) \\
&\ge P_-^- \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda\alpha^+ \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \\
&\quad - \beta^+ \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \\
&= \left(P_-^- - \alpha^+\right) \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x \\
&\quad - (\beta^+ - \alpha^+) \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.
\end{aligned}$$

By Proposition 2.3 and Lemma 2.8, it follows that

$$0 \ge (P_-^- - \alpha^+)\|u\|_{\vec{p}(\cdot)}^{P_+^+} - C_{\partial\Omega}(\beta^+ - \alpha^+)\|u\|_{\vec{p}(\cdot)}^{\beta^-}.$$

Hence,

$$\|u\|_{\vec{p}(\cdot)} \ge \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)} \right)^{\frac{1}{\beta^- - P_+^+}}. \tag{3.5}$$

On the other hand, again using $\Phi''_{\lambda,u}(1) = 0$ and (3.4), we get

$$\begin{aligned}
0 &= \Phi''_{\lambda,u}(1) \\
&\le P_+^+ \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda\alpha^- \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \beta^- \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \\
&= -(\beta^- - P_+^+) \int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x \\
&\quad + \lambda(\beta^- - \alpha^-) \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x.
\end{aligned}
$$

By Proposition 2.3 and Lemma 2.8, one has

$$
0 \le -(\beta^- - P_+^+)\|u\|_{\vec{p}(\cdot)}^{P_+^+} + \lambda C_\Omega(\beta^- - \alpha^-)\|u\|_{\vec{p}(\cdot)}^{\alpha^-}.
$$

Therefore,

$$
\|u\|_{\vec{p}(\cdot)} \le \left(\frac{\lambda C_\Omega(\beta^- - \alpha^-)}{\beta^- - P_+^+} \right)^{\frac{1}{P_+^+ - \alpha^-}}. \tag{3.6}
$$

Combining (3.5) and (3.6), we obtain

$$
\left(\frac{\lambda C_\Omega(\beta^- - \alpha^-)}{\beta^- - P_+^+} \right)^{\frac{1}{P_+^+ - \alpha^-}} \ge \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)} \right)^{\frac{1}{\beta^- - P_+^+}}.
$$

Hence,

$$
\lambda \ge \left(\frac{\beta^- - P_+^+}{C_\Omega(\beta^- - \alpha^-)} \right) \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)} \right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}} = \tau,
$$

which contradicts $\lambda \in (0, \tau)$.

Case 2: $\|u\|_{\vec{p}(\cdot)} \ge 1$.

As above, from $\Phi''_{\lambda,u}(1) = 0$ and (3.4), we get

$$
\begin{aligned}
0 &= \Phi''_{\lambda,u}(1) \\
&\ge P_-^- \int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda\alpha^+ \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \\
&\quad - \beta^+ \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \\
&= (P_-^- - \alpha^+) \int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x \\
&\quad - (\beta^+ - \alpha^+) \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.
\end{aligned}
$$

Using Proposition 2.3 and Lemma 2.8, one has

$$0 \geq (P_-^- - \alpha^+)\|u\|_{\vec{p}(\cdot)}^{P_-^-} - C_{\partial\Omega}(\beta^+ - \alpha^+)\|u\|_{\vec{p}(\cdot)}^{\beta^+},$$

and therefore,

$$\|u\|_{\vec{p}(\cdot)} \geq \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{1}{\beta^+ - P_-^-}}. \tag{3.7}$$

Similarly, we can deduce that

$$\begin{aligned}
0 &= \Phi''_{\lambda,u}(1) \\
&\leq P_+^+ \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda\alpha^- \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \\
&\quad - \beta^- \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma \\
&= -(\beta^- - P_+^+) \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x \\
&\quad + \lambda(\beta^- - \alpha^-) \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x.
\end{aligned}$$

Again by Proposition 2.3 and Lemma 2.8, we have

$$0 \leq -(\beta^- - P_+^+)\|u\|_{\vec{p}(\cdot)}^{P_-^-} + \lambda(\beta^- - \alpha^-)C_\Omega\|u\|_{\vec{p}(\cdot)}^{\alpha^+},$$

which implies

$$\|u\|_{\vec{p}(\cdot)} \leq \left(\frac{\lambda C_\Omega(\beta^- - \alpha^-)}{\beta^- - P_+^+}\right)^{\frac{1}{P_-^- - \alpha^+}}. \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$\left(\frac{\lambda C_\Omega(\beta^- - \alpha^-)}{\beta^- - P_+^+}\right)^{\frac{1}{P_-^- - \alpha^+}} \geq \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{1}{\beta^+ - P_-^-}}.$$

Hence,

$$\lambda \geq \left(\frac{\beta^- - P_+^+}{C_\Omega(\beta^- - \alpha^-)}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_-^- - \alpha^+}{\beta^+ - P_-^-}}. \tag{3.9}$$

By choosing $C_{\partial\Omega} > 1$, we obtain from (**a5**) that

$$0 < \frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)} < 1. \tag{3.10}$$

Moreover, from $P_-^- \leq P_+^+, \alpha^- \leq \alpha^+$ and $\beta^- \leq \beta^+$, we have $P_-^- - \alpha^+ \leq P_+^+ - \alpha^-$ and $\beta^- - P_+^+ \leq \beta^+ - P_-^-$. This implies that

$$\frac{P_-^- - \alpha^+}{\beta^+ - P_-^-} \leq \frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}. \tag{3.11}$$

Now, we obtain from (3.10) and (3.11) that

$$\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_-^- - \alpha^+}{\beta^+ - P_-^-}} \geq \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}}. \tag{3.12}$$

Combining (3.9) and (3.12), we get

$$\lambda \geq \left(\frac{\beta^- - P_+^+}{C_\Omega(\beta^- - \alpha^-)}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}} = \tau,$$

again a contradiction. Therefore, $\mathcal{N}_\lambda^0(\Omega) = \emptyset$. This completes the proof. □

Corollary 3.6 *Let $\lambda \in (0, \tau)$. Then the sets $\mathcal{N}_\lambda^\pm(\Omega)$ are closed in $W^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof We prove that $\mathcal{N}_\lambda^-(\Omega)$ is closed. The proof for $\mathcal{N}_\lambda^+(\Omega)$ is analogous. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^-(\Omega)$ and let $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ be such that $u_n \to u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Since $u_n \in \mathcal{N}_\lambda^-(\Omega)$, we have $\Phi'_{\lambda,u_n}(1) = 0$ and $\Phi''_{\lambda,u_n}(1) < 0$ for each $n \in \mathbb{N}$. The identity $\Phi'_{\lambda,u_n}(1) = 0$ gives

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)}\right) \mathrm{d}x - \lambda \int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x \\ &- \int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma = 0.\end{aligned} \tag{3.13}$$

Since

$$u_n \to u \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in } L^{p_i(\cdot)}(\Omega)$$

for each $i \in \{1, \dots, N\}$, Proposition 2.1 implies

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x.\end{aligned} \tag{3.14}$$

Moreover, by Lemmas 2.6 and 2.7, one has

$$u_n \to u \quad \text{in } L^{\alpha(\cdot)}_{k(\cdot)}(\Omega), \text{ in } L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega) \text{ and a.e. in } \overline{\Omega}.$$

Hence, Proposition 2.2 yields

$$\lim_{n\to\infty}\int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x, \tag{3.15}$$

$$\lim_{n\to\infty}\int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma. \tag{3.16}$$

Passing to the limit in (3.13) and using (3.14), (3.15), and (3.16), we obtain

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x - \lambda \int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x \\ &- \int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma = 0.\end{aligned}$$

Thus, $\Phi'_{\lambda,u}(1) = 0$. Next, since $\Phi''_{\lambda,u_n}(1) < 0$, we have

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N p_i(x) \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \\ &< \lambda \int_\Omega \alpha(x)k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x + \int_{\partial\Omega} \beta(x)h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned} \tag{3.17}$$

By (3.14), (3.15), (3.16) and using Fatou's lemma, one can easily verify that

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \sum_{i=1}^N p_i(x) \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^N p_i(x) \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)} \right) \mathrm{d}x,\end{aligned} \tag{3.18}$$

and

$$\lim_{n\to\infty}\int_\Omega \alpha(x)k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega \alpha(x)k(x)|u|^{\alpha(x)}\,\mathrm{d}x, \tag{3.19}$$

$$\lim_{n\to\infty}\int_{\partial\Omega} \beta(x)h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} \beta(x)h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma. \tag{3.20}$$

Passing to the limit in (3.17) and using (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N p_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x\\ &\leq \lambda\int_\Omega \alpha(x)k(x)|u|^{\alpha(x)}\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned}$$

Hence, $\Phi''_{\lambda,u}(1) \leq 0$. Since $\lambda \in (0,\tau)$, Lemma 3.5 gives $\mathcal{N}_\lambda^0(\Omega) = \emptyset$. Therefore, $\Phi''_{\lambda,u}(1) \neq 0$, and thus $\Phi''_{\lambda,u}(1) < 0$. Together with $\Phi'_{\lambda,u}(1) = 0$, this shows that $u \in \mathcal{N}_\lambda^-(\Omega)$. Hence, $\mathcal{N}_\lambda^-(\Omega)$ is closed in $W^{1,\vec{p}(\cdot)}(\Omega)$. □

The following result is a direct consequence of Lemmas 3.3 and 3.5.

Corollary 3.7

(i) *For any $\lambda \in (0,\tau)$, we have $\mathcal{N}_\lambda(\Omega) = \mathcal{N}_\lambda^+(\Omega) \cup \mathcal{N}_\lambda^-(\Omega)$.*
(ii) *The functional $\mathcal{J}_\lambda$ is coercive and bounded from below on both $\mathcal{N}_\lambda^+(\Omega)$ and $\mathcal{N}_\lambda^-(\Omega)$.*

Define

$$\Theta_\lambda = \inf_{u\in\mathcal{N}_\lambda(\Omega)} \mathcal{J}_\lambda(u), \quad \Theta_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u) \quad \text{and} \quad \Theta_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(u).$$

Lemma 3.8 *For any $\lambda \in (0,\tau)$, we have $\Theta_\lambda \leq \Theta_\lambda^+ < 0$.*

Proof Let $u \in \mathcal{N}_\lambda^+(\Omega)$. Then, $\Phi'_{\lambda,u}(1) = 0$ and $\Phi''_{\lambda,u}(1) > 0$. From $\Phi'_{\lambda,u}(1) = 0$, we obtain

$$\begin{aligned}&\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma\\ &= \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x.\end{aligned} \tag{3.21}$$

Using $\Phi''_{\lambda,u}(1) > 0$ together with (3.21), it follows that

$$\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma < \left(\frac{P_+^+ - \alpha^-}{\beta^- - \alpha^-}\right)\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x. \tag{3.22}$$

By the definition of $\mathcal{J}_\lambda$ and identity (3.21), one has

$$\mathcal{J}_\lambda(u) \le \left(\frac{1}{P_-^-} - \frac{1}{\alpha^+}\right)\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x + \left(\frac{1}{\alpha^+} - \frac{1}{\beta^+}\right)\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.$$

Now, using (3.22) in the above inequality, we get

$$\mathcal{J}_\lambda(u) < \left[\frac{(\alpha^+ - P_-^-)\beta^+ + P_-^-(\beta^+ - \alpha^+)\left(\frac{P_+^+ - \alpha^-}{\beta^- - \alpha^-}\right)}{P_-^-\alpha^+\beta^+}\right] \times \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x. \tag{3.23}$$

By assumption (**a6**), we have

$$(\alpha^+ - P_-^-)\beta^+ + P_-^-(\beta^+ - \alpha^+)\left(\frac{P_+^+ - \alpha^-}{\beta^- - \alpha^-}\right) < 0. \tag{3.24}$$

Combining (3.23) and (3.24), we conclude that $\mathcal{J}_\lambda(u) < 0$. Therefore, by the definitions of Θ_λ and Θ_λ^+, it follows that $\Theta_\lambda \le \Theta_\lambda^+ < 0$. This completes the proof. □

Lemma 3.9 *Let* $0 < \lambda < \left(\frac{\alpha^-}{P_+^+}\right)\tau$. *Then there exists* $\zeta > 0$ *such that* $\Theta_\lambda^- \ge \zeta > 0$.

Proof Let $u \in \mathcal{N}_\lambda^-(\Omega)$. Then $\Phi'_{\lambda,u}(1) = 0$ and $\Phi''_{\lambda,u}(1) < 0$. From $\Phi'_{\lambda,u}(1) = 0$, we obtain

$$\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma = \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x - \lambda\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x. \tag{3.25}$$

Case 1: $\|u\|_{\vec{p}(\cdot)} \le 1$.
From Lemma 3.5, we have

$$\|u\|_{\vec{p}(\cdot)} > \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{1}{\beta^- - P_+^+}}. \tag{3.26}$$

Using the definition of $\mathcal{J}_\lambda$ and the identity (3.25), we obtain

$$\begin{aligned}\mathcal{J}_\lambda(u) \geq \left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right) \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x \\ - \lambda \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right) \int_\Omega k(x)|u|^{\alpha(x)} \, \mathrm{d}x.\end{aligned}$$

By Proposition 2.3 and Lemma 2.8, we deduce

$$\mathcal{J}_\lambda(u) \geq \|u\|_{\vec{p}(\cdot)}^{\alpha^-} \left[\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right) \|u\|_{\vec{p}(\cdot)}^{P_+^+ - \alpha^-} - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\right].$$

Using (3.26) in the above inequality, we get

$$\begin{aligned}\mathcal{J}_\lambda(u) > & \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{\alpha^-}{\beta^- - P_+^+}} \\ & \times \left[\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}} \right. \\ & \left. - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\right] := m_1.\end{aligned} \tag{3.27}$$

Since $\lambda < \left(\frac{\alpha^-}{P_+^+}\right)\tau$, it follows that

$$\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}} - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right) > 0. \tag{3.28}$$

Hence, from (3.27) and (3.28), we deduce that $m_1 > 0$.

Case 2: $\|u\|_{\vec{p}(\cdot)} \geq 1$.

Again by Lemma 3.5, we have

$$\|u\|_{\vec{p}(\cdot)} > \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{1}{\beta^+ - P_-^-}}. \tag{3.29}$$

From Lemma 3.3, we also have

$$\mathcal{J}_\lambda(u) \geq \|u\|_{\vec{p}(\cdot)}^{\alpha^+} \left[\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right) \|u\|_{\vec{p}(\cdot)}^{P_-^- - \alpha^+} - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\right].$$

In addition, by using (3.29) in the above inequality, we get

$$\mathcal{J}_\lambda(u) > \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{\alpha^+}{\beta^+ - P_-^-}} \times \left[\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_-^- - \alpha^+}{\beta^+ - P_-^-}} - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\right].$$

Taking $C_{\partial\Omega} > 1$, by Lemma 3.5, one has

$$\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_-^- - \alpha^+}{\beta^+ - P_-^-}} \geq \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}}.$$

This shows that

$$\mathcal{J}_\lambda(u) > \left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{\alpha^+}{\beta^+ - P_-^-}} \left[\left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\left(\frac{P_-^- - \alpha^+}{C_{\partial\Omega}(\beta^+ - \alpha^+)}\right)^{\frac{P_+^+ - \alpha^-}{\beta^- - P_+^+}} - \lambda C_\Omega \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\right] := m_2.$$

By using (3.28), we deduce that $m_2 > 0$. Finally, choosing $\zeta = \min\{m_1, m_2\} > 0$, we get $\mathcal{J}_\lambda(u) > \zeta > 0$ for all $u \in \mathcal{N}_\lambda^-(\Omega)$. Therefore $\Theta_\lambda^- \geq \zeta > 0$. □

The following lemma is immediate.

Lemma 3.10

(i) *If* $u \in \mathcal{N}_\lambda^+(\Omega)$, *then* $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x > 0$.
(ii) *If* $u \in \mathcal{N}_\lambda^-(\Omega)$, *then* $\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma > 0$.
(iii) *If* $u \in \mathcal{N}_\lambda^0(\Omega)$, *then* $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x > 0$ *and* $\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma > 0$.

The following result describes the behavior of the fibering map $\Phi_{\lambda,u}$.

Lemma 3.11 *Let* $u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ *be such that* $\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma > 0$. *Then there exists* $\tau_0 > 0$ *such that, for every* $\lambda \in (0, \tau_0)$, *the following assertions hold.*

(i) *If* $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x = 0$, *then there exists a unique* $t_1 > 0$ *such that* $t_1 u \in \mathcal{N}_\lambda^-(\Omega)$ *and*

$$\mathcal{J}_\lambda(t_1 u) := \sup_{t\geq 0} \mathcal{J}_\lambda(tu).$$

(ii) *If* $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x > 0$*, then there exists a unique* $t^* > 0$ *and unique positive numbers* t_1*,* t_2 *with* $t_2 < t^* < t_1$ *such that* $t_1 u \in \mathcal{N}_\lambda^-(\Omega)$ *and* $t_2 u \in \mathcal{N}_\lambda^+(\Omega)$*. Moreover,*

$$\mathcal{J}_\lambda(t_1 u) := \sup_{t\geq 0} \mathcal{J}_\lambda(tu) \quad \textit{and} \quad \mathcal{J}_\lambda(t_2 u) := \inf_{0\leq t\leq t^*} \mathcal{J}_\lambda(tu).$$

Proof Let $0 < t < 1$ be sufficiently small. Then we have

$$\Phi_{\lambda,u}(t) \geq \frac{t^{P_+^+}}{P_+^+}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \frac{t^{\beta^-}}{\beta^-}\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma > 0,$$

since $P_+^+ < \beta^-$. On the other hand, for $t > 1$ sufficiently large, we have

$$\Phi_{\lambda,u}(t) \leq \frac{t^{P_+^+}}{P_-^-}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x - \frac{t^{\beta^-}}{\beta^+}\int_{\partial\Omega} h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma < 0.$$

Thus, $\Phi_{\lambda,u}$ attains its maximum at some point $t_1 > 0$ and therefore, $\Phi'_{\lambda,u}(t_1) = \langle \mathcal{J}'_\lambda(t_1 u), u\rangle = 0$. Setting $u^* = t_1 u$, we get $\langle \mathcal{J}'_\lambda(u^*), u^*\rangle = 0$, that is, $u^* = t_1 u \in \mathcal{N}_\lambda(\Omega)$. Now, using the fact that $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x = 0$ for each $u \in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}$ and $\langle \mathcal{J}'_\lambda(u^*), u^*\rangle = 0$, we obtain

$$\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u^*}{\partial x_i}\right|^{p_i(x)} + |u^*|^{p_i(x)}\right)\mathrm{d}x = \int_{\partial\Omega} h(x)|u^*|^{\beta(x)}\,\mathrm{d}\sigma. \tag{3.30}$$

Define $\hat{\Phi}_{\lambda,u^*}(t) = \mathcal{J}_\lambda(tu^*)$ for $t \geq 0$. Then $\hat{\Phi}_{\lambda,u^*}(1) = \mathcal{J}_\lambda(u^*)$ and $\hat{\Phi}'_{\lambda,u^*}(1) = \langle \mathcal{J}'_\lambda(u^*), u^*\rangle = 0$. If $0 < t < 1$, then

$$\begin{aligned}
\hat{\Phi}'_{\lambda,u^*}(t) &= \langle \mathcal{J}'_\lambda(tu^*), u^*\rangle = \frac{1}{t}\langle \mathcal{J}'_\lambda(tu^*), tu^*\rangle\\
&= \int_\Omega \sum_{i=1}^N t^{p_i(x)-1}\left(\left|\frac{\partial u^*}{\partial x_i}\right|^{p_i(x)} + |u^*|^{p_i(x)}\right)\mathrm{d}x\\
&\quad - \lambda\int_\Omega t^{\alpha(x)-1}k(x)|u^*|^{\alpha(x)}\,\mathrm{d}x - \int_{\partial\Omega} t^{\beta(x)-1}h(x)|u^*|^{\beta(x)}\,\mathrm{d}\sigma\\
&\geq t^{P_+^+-1}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u^*}{\partial x_i}\right|^{p_i(x)} + |u^*|^{p_i(x)}\right)\mathrm{d}x\\
&\quad - \lambda t^{\alpha^--1}\int_\Omega k(x)|u^*|^{\alpha(x)}\,\mathrm{d}x - t^{\beta^--1}\int_{\partial\Omega} h(x)|u^*|^{\beta(x)}\,\mathrm{d}\sigma.
\end{aligned}$$

Using $\int_\Omega k(x)|u^*|^{\alpha(x)}\,\mathrm{d}x = 0$ and (3.30) in the above inequality, we obtain

$$\hat{\Phi}'_{\lambda,u^*}(t) \geq \left(t^{P_+^+-1} - t^{\beta^- -1}\right)\int_{\partial\Omega} h(x)|u^*|^{\beta(x)}\,\mathrm{d}\sigma > 0.$$

Similarly, for $t > 1$, one has

$$\hat{\Phi}'_{\lambda,u^*}(t) \leq \left(t^{P_+^+-1} - t^{\beta^- -1}\right)\int_{\partial\Omega} h(x)|u^*|^{\beta(x)}\,\mathrm{d}\sigma < 0.$$

Thus, $\hat{\Phi}'_{\lambda,u^*}$ changes sign from positive to negative at $t = 1$. Therefore, 1 is the unique point of maximum of $\hat{\Phi}_{\lambda,u^*}$, which means that u^* is the unique maximizer of $t \mapsto \mathcal{J}_\lambda(tu^*)$. Consequently, t_1 is unique. This proves part (i).

Now, assume that $\int_\Omega k(x)|u|^{\alpha(x)}\,\mathrm{d}x > 0$. Define $g_1, g_2, g_3 \colon [0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}
g_1(t) &= \int_\Omega \sum_{i=1}^N t^{p_i(x)-1}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + |u|^{p_i(x)}\right)\mathrm{d}x,\\
g_2(t) &= \lambda\int_\Omega t^{\alpha(x)-1}k(x)|u|^{\alpha(x)}\,\mathrm{d}x,\\
g_3(t) &= \int_{\partial\Omega} t^{\beta(x)-1}h(x)|u|^{\beta(x)}\,\mathrm{d}\sigma.
\end{aligned}$$

Then $\Phi'_{\lambda,u}(t) = g_1(t) - g_2(t) - g_3(t)$. The functions g_1, g_2 and g_3 are continuous, strictly increasing, and satisfy $g_i(0) = 0$ for $i = 1, 2, 3$. Moreover, the following properties hold:

(v1) $\lim_{t\to 0^+} \frac{g_3(t)}{g_1(t)} = 0$.
(v2) $\lim_{t\to +\infty} g_2(t) = +\infty$.
(v3) $\lim_{t\to 0^+} \frac{g_1(t)-g_3(t)}{g_2(t)} = 0$.
(v4) $g_1 - g_3$ has a unique point of maximum, say $t_{\max}$ and $(g_1 - g_3)(t) \to -\infty$ as $t \to +\infty$.
(v5) There exists $t' \in (0, t_{\max})$ such that $t \mapsto \frac{g_1(t)-g_3(t)}{g_2(t)}$ is strictly increasing on $(0, t')$.

From (v1), we deduce that $(g_1 - g_3)(t) > 0$ for $t > 0$ sufficiently small. By (v2), there exists $\xi_1 \in (0, t_{\max})$ such that $g_2(\xi_1) > (g_1 - g_3)(\xi_1)$. For $\lambda > 0$ sufficiently small, there exists $\xi_2 \in (0, t_{\max})$ such that $g_2(\xi_2) < (g_1 - g_3)(\xi_2)$. Hence, by the intermediate value theorem, there exists $t_2 = t_2(\lambda) \in (0, t_{\max})$ such that

$$\frac{g_1(t_2) - g_3(t_2)}{g_2(t_2)} = 1. \tag{3.31}$$

Since the map $t \mapsto \frac{g_1(t)-g_3(t)}{g_2(t)}$ is strictly increasing on $(0, t')$, the number t_2 is unique. Moreover, $t \mapsto \frac{g_1(t)-g_3(t)}{g_2(t)}$ is strictly increasing in $[t_2, t')$ and from (3.31), we obtain

$$1 = \frac{g_1(t_2) - g_3(t_2)}{g_2(t_2)} < \frac{g_1(t) - g_3(t)}{g_2(t)} \quad \text{for all } t \in (t_2, t').$$

This yields that

$$g_2(t) < g_1(t) - g_3(t) \quad \text{for all } t \in (t_2, t').$$

Fix $\tau_1 > 0$ such that, for each $\lambda \in (0, \tau_1)$,

$$g_2(t) < g_1(t) - g_3(t) \quad \text{for all } t \in (t_2, t_{\max}). \tag{3.32}$$

On the other hand, by (v4), $(g_1 - g_3 - g_2)(t) \to -\infty$ as $t \to \infty$. Hence, there exists $\xi_3 > t_{\max}$ such that $(g_1 - g_3 - g_2)(\xi_3) < 0$, that is,

$$(g_1 - g_3)(\xi_3) < g_2(\xi_3) \quad \text{for } \xi_3 > t_{\max}. \tag{3.33}$$

From (3.32), (3.33), and the intermediate value theorem, it follows that for every $\lambda \in (0, \tau_1)$, there exists a positive number $t_1 > t_{\max}$ such that

$$(g_1 - g_3)(t_1) = g_2(t_1). \tag{3.34}$$

Since $t_{\max}$ is the unique point of maximum for $g_1 - g_3$, the function $g_1 - g_3$ is strictly decreasing in $(t_{\max}, +\infty)$. As g_2 is strictly increasing on $(0, +\infty)$, the function $g_1 - g_2 - g_3$ is strictly decreasing on $(t_{\max}, +\infty)$. Therefore, t_1 is unique.

From (3.31) and (3.34) we conclude that t_1, t_2 with $t_2 < t_1$ are exactly the two nontrivial critical points of $\Phi_{\lambda,u}$. We choose $\tau_0 = \min\{\tau, \tau_1\}$. Then $\Phi_{\lambda,u}(0) = 0$, $\Phi_{\lambda,u}(t) < 0$ for $t > 0$ sufficiently small, $\Phi'_{\lambda,u}(t) < 0$ for all $t \in (0, t_2)$, $\Phi'_{\lambda,u}(t) > 0$ for all $t \in (t_2, t_{\max})$ and $\Phi'_{\lambda,u}(t_2) = 0$. By Lemma 3.5, we have $\mathcal{N}_\lambda^0(\Omega) = \emptyset$. Hence, we deduce that $\Phi_{\lambda,u}$ attains a local minimum at t_2 and $\Phi''_{\lambda,u}(t_2) > 0$. This implies that $t_2 u \in \mathcal{N}_\lambda^+(\Omega)$. Similarly, we have $\Phi'_{\lambda,u}(t) > 0$ for all $t \in (t_{\max}, t_1)$, $\Phi'_{\lambda,u}(t) < 0$ for all $t > t_1$ and $\Phi'_{\lambda,u}(t_1) = 0$. Again using Lemma 3.5, we conclude that $\Phi_{\lambda,u}$ attains a local maximum at t_1 and $\Phi''_{\lambda,u}(t_1) < 0$. It follows that $t_1 u \in \mathcal{N}_\lambda^-(\Omega)$.

By Lemmas 3.8 and 3.9, we deduce that

$$\Phi_{\lambda,u}(t_2) = \mathcal{J}_\lambda(t_2 u) < 0 \quad \text{and} \quad \Phi_{\lambda,u}(t_1) = \mathcal{J}_\lambda(t_1 u) > 0.$$

Moreover, $\Phi_{\lambda,u}$ is strictly increasing on $[t_2, t_1]$ and strictly decreasing on (t_1, ∞) with $\Phi_{\lambda,u}(t) \to -\infty$ as $t \to \infty$. Hence, by the intermediate value theorem, there exists a unique $t^* \in (t_2, t_1)$ such that $\Phi_{\lambda,u}(t^*) = 0$. Consequently,

$$\mathcal{J}_\lambda(t_2 u) = \Phi_{\lambda,u}(t_2) = \inf_{0 \le t \le t^*} \Phi_{\lambda,u}(t) = \inf_{0 \le t \le t^*} \mathcal{J}_\lambda(tu)$$

and

$$\mathcal{J}_\lambda(t_1 u) = \Phi_{\lambda,u}(t_1) = \sup_{t \geq 0} \Phi_{\lambda,u}(t) = \sup_{t \geq 0} \mathcal{J}_\lambda(tu).$$

This completes the proof of (ii). □

4 Existence of Nontrivial Nonnegative Solutions

In this section, we prove Theorem 1.1 and show that problem (1.1) admits at least two nontrivial nonnegative solutions.

Proposition 4.1 *For every $\lambda \in (0, \tau)$, the functional $\mathcal{J}_\lambda$ admits a minimizer u_λ on $\mathcal{N}_\lambda^+(\Omega)$ such that $\mathcal{J}_\lambda(u_\lambda) = \Theta_\lambda^+ < 0$, and u_λ is a nontrivial solution of (1.1).*

Proof From Corollary 3.7, it follows that the functional $\mathcal{J}_\lambda$ is coercive and bounded from below on $\mathcal{N}_\lambda^+(\Omega)$. Thus, there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+(\Omega)$ such that

$$\lim_{n\to\infty} \mathcal{J}_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u).$$

Clearly, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$. Hence, there exists $u_\lambda \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } W^{1,\vec{p}(\cdot)}(\Omega).$$

Using Lemmas 2.6 and 2.7, we obtain

$$u_n \to u_\lambda \quad \text{in } L_{k(\cdot)}^{\alpha(\cdot)}(\Omega), \text{ in } L_{h(\cdot)}^{\beta(\cdot)}(\partial\Omega) \text{ and a.e. in } \overline{\Omega}.$$

By Proposition 2.2, we have

$$\lim_{n\to\infty} \int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x, \tag{4.1}$$

$$\lim_{n\to\infty} \int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} h(x)|u_\lambda|^{\beta(x)}\,\mathrm{d}\sigma. \tag{4.2}$$

Using (4.1), (4.2), and Fatou's lemma, we deduce

$$\lim_{n\to\infty} \int_\Omega \frac{k(x)}{\alpha(x)}|u_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega \frac{k(x)}{\alpha(x)}|u_\lambda|^{\alpha(x)}\,\mathrm{d}x, \tag{4.3}$$

$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{h(x)}{\beta(x)}|u_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} \frac{h(x)}{\beta(x)}|u_\lambda|^{\beta(x)}\,\mathrm{d}\sigma. \tag{4.4}$$

Next, we show that $u_\lambda \neq 0$ and $\int_\Omega k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x > 0$. We argue by contradiction. Suppose that this is not the case. Then

$$\lim_{n\to\infty}\int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x = 0. \tag{4.5}$$

Since $u_n \in \mathcal{N}_\lambda^+(\Omega)$, we have $\Phi'_{\lambda,u_n}(1) = 0$. Hence,

$$\begin{aligned}&\int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma\\ &= \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)}\right)\mathrm{d}x - \lambda\int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x.\end{aligned} \tag{4.6}$$

On the other hand, by the definition of $\mathcal{J}_\lambda$,

$$\begin{aligned}\mathcal{J}_\lambda(u_n) \geq \frac{1}{P_+^+}\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)}\right)\mathrm{d}x - \frac{\lambda}{\alpha^-}\int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x\\ - \frac{1}{\beta^-}\int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma.\end{aligned}$$

In view of (4.6), we obtain from the above inequality that

$$\begin{aligned}\mathcal{J}_\lambda(u_n) \geq \left(\frac{1}{P_+^+} - \frac{1}{\beta^-}\right)\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)}\right)\mathrm{d}x\\ - \lambda\left(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\right)\int_\Omega k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x.\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (4.5), we get

$$\lim_{n\to\infty}\mathcal{J}_\lambda(u_n) \geq 0. \tag{4.7}$$

However, by Lemma 3.8, we have

$$\lim_{n\to\infty}\mathcal{J}_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)}\mathcal{J}_\lambda(u) = \Theta_\lambda^+ < 0,$$

which contradicts (4.7). Hence, $u_\lambda \in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}$ and $\int_\Omega k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x > 0$.

Next, we show that $u_n \to u_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Suppose this is not the case. Since $u_n \rightharpoonup u_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ and convex lower semicontinuous functionals on reflexive

Banach spaces are weakly lower semicontinuous, we get

$$
\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u_\lambda}{\partial x_i} \right|^{p_i(x)} + |u_\lambda|^{p_i(x)} \right) \mathrm{d}x \\
&\quad < \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^N \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left(\left| \frac{\partial u_\lambda}{\partial x_i} \right|^{p_i(x)} + |u_\lambda|^{p_i(x)} \right) \mathrm{d}x \\
&\quad < \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x.
\end{aligned} \tag{4.8}
$$

From (4.3), (4.4) and (4.8), it follows that

$$
\begin{aligned}
\lim_{n\to\infty} &\mathcal{J}_\lambda(u_n) = \liminf_{n\to\infty} \mathcal{J}_\lambda(u_n) \\
&= \liminf_{n\to\infty} \Bigg[\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \\
&\quad - \lambda \int_\Omega \frac{k(x)}{\alpha(x)} |u_n|^{\alpha(x)} \, \mathrm{d}x - \int_{\partial\Omega} \frac{h(x)}{\beta(x)} |u_n|^{\beta(x)} \, \mathrm{d}\sigma \Bigg] \\
&\geq \liminf_{n\to\infty} \Bigg[\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \Bigg] \\
&\quad - \lambda \int_\Omega \frac{k(x)}{\alpha(x)} |u_\lambda|^{\alpha(x)} \, \mathrm{d}x - \int_{\partial\Omega} \frac{h(x)}{\beta(x)} |u_\lambda|^{\beta(x)} \, \mathrm{d}\sigma \\
&> \int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left(\left| \frac{\partial u_\lambda}{\partial x_i} \right|^{p_i(x)} + |u_\lambda|^{p_i(x)} \right) \mathrm{d}x - \lambda \int_\Omega \frac{k(x)}{\alpha(x)} |u_\lambda|^{\alpha(x)} \, \mathrm{d}x \\
&\quad - \int_{\partial\Omega} \frac{h(x)}{\beta(x)} |u_\lambda|^{\beta(x)} \, \mathrm{d}\sigma \\
&= \mathcal{J}_\lambda(u_\lambda).
\end{aligned} \tag{4.9}
$$

By Lemma 3.11, there exists a unique $t_2 > 0$ such that $t_2 u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$. Since $u_n \rightharpoonup u_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, we also have $t_2 u_n \rightharpoonup t_2 u_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Hence, repeating the previous arguments with $t_2 u_n$ and $t_2 u_\lambda$ along with the definition of $\Phi'_{\lambda,u}$, we have

$$
\begin{aligned}
&\liminf_{n\to\infty} \Phi'_{\lambda,u_n}(t_2) \\
&\quad = \liminf_{n\to\infty} \Bigg[\int_\Omega \sum_{i=1}^N t_2^{p_i(x)-1} \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda \int_\Omega t_2^{\alpha(x)-1} k(x)|u_n|^{\alpha(x)}\,\mathrm{d}x - \int_{\partial\Omega} t_2^{\beta(x)-1} h(x)|u_n|^{\beta(x)}\,\mathrm{d}\sigma \Bigg]\\
&\geq \liminf_{n\to\infty} \Bigg[\int_\Omega \sum_{i=1}^N t_2^{p_i(x)-1} \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + |u_n|^{p_i(x)} \right) \mathrm{d}x \Bigg]\\
&- \lambda \int_\Omega t_2^{\alpha(x)-1} k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x - \int_{\partial\Omega} t_2^{\beta(x)-1} h(x)|u_\lambda|^{\beta(x)}\,\mathrm{d}\sigma\\
&> \int_\Omega \sum_{i=1}^N t_2^{p_i(x)-1} \left(\left|\frac{\partial u_\lambda}{\partial x_i}\right|^{p_i(x)} + |u_\lambda|^{p_i(x)} \right) \mathrm{d}x - \lambda \int_\Omega t_2^{\alpha(x)-1} k(x)|u_\lambda|^{\alpha(x)}\,\mathrm{d}x\\
&- \int_{\partial\Omega} t_2^{\beta(x)-1} h(x)|u_\lambda|^{\beta(x)}\,\mathrm{d}\sigma\\
&= \Phi'_{\lambda,u_\lambda}(t_2) = \frac{1}{t_2}\langle \mathcal{J}'_\lambda(t_2u_\lambda), t_2u_\lambda\rangle = 0.
\end{aligned}
$$

We conclude that $\Phi'_{\lambda,u_n}(t_2) > 0$ for n sufficiently large. Since $u_n \in \mathcal{N}_\lambda^+(\Omega)$ for all $n \in \mathbb{N}$, it follows that $\Phi'_{\lambda,u_n}(1) = 0$ and $\Phi''_{\lambda,u_n}(1) > 0$. Thus, 1 is the point of minimum of Φ_{λ,u_n}, $\Phi'_{\lambda,u_n}(t) < 0$ for all $t \in (0,1)$ and $\Phi'_{\lambda,u_n}(t) > 0$ for all $t > 1$. It follows that $t_2 > 1$. Since $t_2u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$, the number t_2 is the point of minimum for Φ_{λ,u_λ} and therefore, Φ_{λ,u_λ} is strictly decreasing on $(0,t_2)$. From (4.9), one has

$$
\mathcal{J}_\lambda(t_2u_\lambda) \leq \mathcal{J}_\lambda(u_\lambda) < \lim_{n\to\infty} \mathcal{J}_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u),
$$

which contradicts the fact that $t_2u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$. Thus, $u_n \to u_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ and

$$
\mathcal{J}_\lambda(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u) < 0.
$$

Thus, u_λ is a minimizer of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+(\Omega)$. Since $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, Lemma 3.4 implies that u_λ is a critical point of $\mathcal{J}_\lambda$. Therefore, u_λ is a nontrivial solution of (1.1). This completes the proof. □

Proposition 4.2 *For every $\lambda \in (0,\tau)$, the functional $\mathcal{J}_\lambda$ admits a minimizer v_λ on $\mathcal{N}_\lambda^-(\Omega)$ such that $\mathcal{J}_\lambda(v_\lambda) = \Theta_\lambda^- > 0$, and v_λ is a nontrivial solution of* (1.1).

Proof Due to Corollary 3.7, $\mathcal{J}_\lambda$ is coercive and bounded from below on $\mathcal{N}_\lambda^-(\Omega)$. Therefore, there exists a minimizing sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^-(\Omega)$ such that

$$
\lim_{n\to\infty} \mathcal{J}_\lambda(v_n) = \inf_{v\in\mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(v).
$$

Moreover, the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$. It follows that we can find $v_\lambda \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$
v_n \rightharpoonup v_\lambda \quad \text{in } W^{1,\vec{p}(\cdot)}(\Omega).
$$

Using Lemmas 2.6 and 2.7, we obtain

$$v_n \to v_\lambda \quad \text{in } L^{\alpha(\cdot)}_{k(\cdot)}(\Omega), \text{ in } L^{\beta(\cdot)}_{h(\cdot)}(\partial\Omega), \text{ and a.e. in } \overline{\Omega}.$$

By Proposition 2.2, we infer

$$\lim_{n\to\infty} \int_\Omega k(x)|v_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega k(x)|v_\lambda|^{\alpha(x)}\,\mathrm{d}x, \tag{4.10}$$

$$\lim_{n\to\infty} \int_{\partial\Omega} h(x)|v_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma. \tag{4.11}$$

Consequently, from (4.10), (4.11) and Fatou's lemma, we deduce that

$$\lim_{n\to\infty} \int_\Omega \frac{k(x)}{\alpha(x)}|v_n|^{\alpha(x)}\,\mathrm{d}x = \int_\Omega \frac{k(x)}{\alpha(x)}|v_\lambda|^{\alpha(x)}\,\mathrm{d}x,$$
$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{h(x)}{\beta(x)}|v_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} \frac{h(x)}{\beta(x)}|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma.$$

We claim that $v_\lambda \neq 0$. In fact, we first show that $\int_{\partial\Omega} h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma > 0$. Assume, by contradiction, that

$$\lim_{n\to\infty} \int_{\partial\Omega} h(x)|v_n|^{\beta(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma = 0. \tag{4.12}$$

Recall that $v_n \in \mathcal{N}_\lambda^-(\Omega)$ and so $\Phi'_{\lambda,v_n}(1) = 0$. This implies that

$$\begin{aligned} &\lambda \int_\Omega k(x)|v_n|^{\alpha(x)}\,\mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)} + |v_n|^{p_i(x)} \right) \mathrm{d}x - \int_{\partial\Omega} h(x)|v_n|^{\beta(x)}\,\mathrm{d}\sigma. \end{aligned} \tag{4.13}$$

From the definition of $\mathcal{J}_\lambda$ and (4.13), we get

$$\begin{aligned} \mathcal{J}_\lambda(v_n) \le \left(\frac{1}{P_-^-} - \frac{1}{\alpha^+}\right) \int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)} + |v_n|^{p_i(x)} \right) \mathrm{d}x \\ - \left(\frac{1}{\beta^+} - \frac{1}{\alpha^+}\right) \int_{\partial\Omega} h(x)|v_n|^{\beta(x)}\,\mathrm{d}\sigma. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (4.12), we obtain

$$\lim_{n\to\infty} \mathcal{J}_\lambda(v_n) \le 0. \tag{4.14}$$

On the other hand, by Lemma 3.9, we have

$$0<\zeta\leq\lim_{n\to\infty}\mathcal{J}_\lambda(v_n)=\inf_{v\in\mathcal{N}_\lambda^-(\Omega)}\mathcal{J}_\lambda(v),$$

contradicting (4.14). Hence, $\int_{\partial\Omega}h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma>0$, and in particular $v_\lambda\neq 0$. Now, by Lemma 3.11, there exists a unique $t_1>0$ such that $t_1v_\lambda\in\mathcal{N}_\lambda^-(\Omega)$.

We next prove that $v_n\to v_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Assume, by contradiction, that this is not true. Since $v_n\rightharpoonup v_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, and since every convex lower semicontinuous functional on a reflexive Banach space is weakly lower semicontinuous, we obtain

$$\int_\Omega\sum_{i=1}^N\left(\left|\frac{\partial v_\lambda}{\partial x_i}\right|^{p_i(x)}+|v_\lambda|^{p_i(x)}\right)\mathrm{d}x<\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^N\left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}+|v_n|^{p_i(x)}\right)\mathrm{d}x$$

and

$$\begin{aligned}&\int_\Omega\sum_{i=1}^N\frac{1}{p_i(x)}\left(\left|\frac{\partial v_\lambda}{\partial x_i}\right|^{p_i(x)}+|v_\lambda|^{p_i(x)}\right)\mathrm{d}x\\&\quad<\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^N\frac{1}{p_i(x)}\left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}+|v_n|^{p_i(x)}\right)\mathrm{d}x.\end{aligned}$$

The same convergences remain valid if we replace v_n and v_λ by t_1v_n and t_1v_λ, respectively. Hence,

$$\begin{aligned}&\liminf_{n\to\infty}\mathcal{J}_\lambda(t_1v_n)\\&=\liminf_{n\to\infty}\Bigg[\int_\Omega\sum_{i=1}^N\frac{t_1^{p_i(x)}}{p_i(x)}\left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}+|v_n|^{p_i(x)}\right)\mathrm{d}x\\&\qquad-\lambda\int_\Omega\frac{t_1^{\alpha(x)}k(x)}{\alpha(x)}|v_n|^{\alpha(x)}\,\mathrm{d}x-\int_{\partial\Omega}\frac{t_1^{\beta(x)}h(x)}{\beta(x)}|v_n|^{\beta(x)}\,\mathrm{d}\sigma\Bigg]\\&\geq\liminf_{n\to\infty}\Bigg[\int_\Omega\sum_{i=1}^N\frac{t_1^{p_i(x)}}{p_i(x)}\left(\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}+|v_n|^{p_i(x)}\right)\mathrm{d}x\Bigg]\\&\qquad-\lambda\int_\Omega\frac{t_1^{\alpha(x)}k(x)}{\alpha(x)}|v_\lambda|^{\alpha(x)}\,\mathrm{d}x-\int_{\partial\Omega}\frac{t_1^{\beta(x)}h(x)}{\beta(x)}|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma\\&>\int_\Omega\sum_{i=1}^N\frac{t_1^{p_i(x)}}{p_i(x)}\left(\left|\frac{\partial v_\lambda}{\partial x_i}\right|^{p_i(x)}+|v_\lambda|^{p_i(x)}\right)\mathrm{d}x-\lambda\int_\Omega\frac{t_1^{\alpha(x)}k(x)}{\alpha(x)}|v_\lambda|^{\alpha(x)}\,\mathrm{d}x\\&\qquad-\int_{\partial\Omega}\frac{t_1^{\beta(x)}h(x)}{\beta(x)}|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma\\&=\mathcal{J}_\lambda(t_1v_\lambda).\end{aligned}\tag{4.15}$$

Similarly, from the definition of $\Phi'_{\lambda,u}$, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \Phi'_{\lambda,v_n}(t_1) \\
&= \liminf_{n\to\infty} \Bigg[\int_\Omega \sum_{i=1}^N t_1^{p_i(x)-1} \left(\left| \frac{\partial v_n}{\partial x_i} \right|^{p_i(x)} + |v_n|^{p_i(x)} \right) \mathrm{d}x \\
&\quad - \lambda \int_\Omega t_1^{\alpha(x)-1} k(x)|v_n|^{\alpha(x)}\,\mathrm{d}x - \int_{\partial\Omega} t_1^{\beta(x)-1} h(x)|v_n|^{\beta(x)}\,\mathrm{d}\sigma \Bigg] \\
&\geq \liminf_{n\to\infty} \left[\int_\Omega \sum_{i=1}^N t_1^{p_i(x)-1} \left(\left| \frac{\partial v_n}{\partial x_i} \right|^{p_i(x)} + |v_n|^{p_i(x)} \right) \mathrm{d}x \right] \\
&\quad - \lambda \int_\Omega t_1^{\alpha(x)-1} k(x)|v_\lambda|^{\alpha(x)}\,\mathrm{d}x - \int_{\partial\Omega} t_1^{\beta(x)-1} h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma \\
&> \int_\Omega \sum_{i=1}^N t_1^{p_i(x)-1} \left(\left| \frac{\partial v_\lambda}{\partial x_i} \right|^{p_i(x)} + |v_\lambda|^{p_i(x)} \right) \mathrm{d}x - \lambda \int_\Omega t_1^{\alpha(x)-1} k(x)|v_\lambda|^{\alpha(x)}\,\mathrm{d}x \\
&\quad - \int_{\partial\Omega} t_1^{\beta(x)-1} h(x)|v_\lambda|^{\beta(x)}\,\mathrm{d}\sigma \\
&= \Phi'_{\lambda,v_\lambda}(t_1) = \frac{1}{t_1}\langle \mathcal{J}'_\lambda(t_1 v_\lambda), t_1 v_\lambda \rangle = 0.
\end{aligned}$$

Therefore, for n sufficiently large, $\Phi'_{\lambda,v_n}(t_1) > 0$. Since $v_n \in \mathcal{N}_\lambda^-(\Omega)$ for all $n \in \mathbb{N}$, we have $\Phi'_{\lambda,v_n}(1) = 0$ and $\Phi''_{\lambda,v_n}(1) < 0$. Hence, 1 is the point of maximum of Φ_{λ,v_n}, $\Phi'_{\lambda,v_n}(t) > 0$ for all $t \in (0,1)$ and $\Phi'_{\lambda,v_n}(t) < 0$ for all $t > 1$. This shows that $t_1 < 1$. On the other hand, $t_1 v_\lambda \in \mathcal{N}_\lambda^-(\Omega)$, so t_1 is the unique global maximum point of Φ_{λ,v_λ}. Combining this with (4.15), we deduce that

$$\mathcal{J}_\lambda(t_1 v_\lambda) < \liminf_{n\to\infty} \mathcal{J}_\lambda(t_1 v_n) \leq \liminf_{n\to\infty} \mathcal{J}_\lambda(v_n) = \lim_{n\to\infty} \mathcal{J}_\lambda(v_n) = \inf_{v\in\mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(v),$$

which contradicts the fact that $t_1 v_\lambda \in \mathcal{N}_\lambda^-(\Omega)$. Therefore, $v_n \to v_\lambda$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Furthermore,

$$\mathcal{J}_\lambda(v_\lambda) = \inf_{v\in\mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(v) = \Theta_\lambda^- > 0.$$

Hence, v_λ is a minimizer of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^-(\Omega)$. Because of $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, by Lemma 3.4, we obtain v_λ is a critical point of $\mathcal{J}_\lambda$. It follows that v_λ is a nontrivial solution of (1.1). □

We are now in the position to state the proof of Theorem 1.1.

Proof of Theorem 1.1 By Proposition 4.1, u_λ is a nontrivial solution of (1.1) and

$$\mathcal{J}_\lambda(|u_\lambda|) = \mathcal{J}_\lambda(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u).$$

Hence, $u_* := |u_\lambda|$ is a minimizer of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+(\Omega)$. Since Lemma 3.5 implies that $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, it follows from Lemma 3.4 that u_* is a nontrivial nonnegative solution of (1.1).

Next, by Proposition 4.2, v_λ is a nontrivial solution of (1.1) and

$$\mathcal{J}_\lambda(|v_\lambda|) = \mathcal{J}_\lambda(v_\lambda) = \inf_{v \in \mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(v).$$

Therefore, $v_* := |v_\lambda|$ is a minimizer of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^-(\Omega)$. Again, since $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, Lemma 3.4 implies that v_* is a nontrivial nonnegative solution of (1.1).

Finally, since $\mathcal{N}_\lambda^+(\Omega) \cap \mathcal{N}_\lambda^-(\Omega) = \emptyset$, the two solutions u_* and v_* are distinct. Consequently, problem (1.1) admits at least two distinct nontrivial nonnegative solutions. This completes the proof. $\square$

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Declarations

Competing interests The authors declare no competing interests.

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