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# Elliptic problems with unbalanced growth, unbounded coefficient and intrinsic operator

Shengda Zeng*
*National Center for Applied Mathematics in Chongqing and School of Mathematical Sciences Chongqing Normal University, Chongqing 401331, P. R. China*
*zengshengda@163.com*

Yasi Lu
*National Center for Applied Mathematics in Chongqing Chongqing Normal University Chongqing 401331, P. R. China*
*yasilu507@163.com*

Patrick Winkert
*Technische Universität Berlin, Institut für Mathematik Straße des 17. Juni 136 10623 Berlin, Germany*
*winkert@math.tu-berlin.de*

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We investigate a variable exponent double phase problem involving an unbounded coefficient in the principal part and a multi-valued convection term defined via an intrinsic operator. Under suitable growth and coercivity conditions, truncation techniques and a surjectivity result for multivalued pseudomonotone operators yield bounded weak solutions with bounded boundary traces. Furthermore, we examine special cases of the double phase problem by selecting distinct intrinsic operators, including convolution-type operators. The analysis is motivated by [D. Motreanu and E. Tornatore, Elliptic equations with unbounded coefficient, convection term and intrinsic operator, *Math. Z.*

*Corresponding author.

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**308**(2) (2024) Paper No. 38, 13 pp.] and incorporates mixed boundary conditions and multivalued source terms.

*Keywords*: Bounded solution; elliptic inclusions; convection term; double phase problems; intrinsic operator; unbounded coefficient; variable exponents

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## 1. Introduction

Our starting point is the variable exponent double phase inclusion below, whose principal coefficient may be unbounded and whose convection term involves an intrinsic operator

$$\begin{aligned} -\operatorname{div}\left[h(u,\|u\|_{k(\cdot)},\|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\right] &\in f(x,u,Tu,\nabla u,\nabla(Tu)) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x,u,Tu) \quad \text{on } \Gamma_2. \end{aligned} \tag{P1}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded Lipschitz domain, and

$$\mathcal{K}(\nabla u) = |\nabla u|^{\vartheta(x)-2}\nabla u + \mu(x)|\nabla u|^{\kappa(x)-2}\nabla u, \quad \mathcal{K}(0) = 0.$$

The sets $\Gamma_1$ and $\Gamma_2$ form a partition of $\partial\Omega$ satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. Here, $\vartheta, \kappa, k \in C(\overline{\Omega})$ fulfill $1 < \vartheta(x) < N$, $\vartheta(x) \leq \kappa(x)$ and $1 < k(x) < \vartheta^*(x) = \frac{N\vartheta(x)}{N-\vartheta(x)}$ for all $x \in \overline{\Omega}$ while $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. Moreover, the intrinsic operator $T\colon V \to W^{1,\vartheta-}(\Omega)$ (see Remark 2.9 for $V$) is continuous, the coefficient $h\colon \mathbb{R} \times [0,\infty) \times [0,\infty) \to [a_0,+\infty)$ with $a_0 > 0$ is continuous and possibly unbounded and $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ is a multivalued function defined in the domain $\Omega$ and $f_{\Gamma_2}\colon \Gamma_2 \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ is a multivalued function defined on the boundary $\Gamma_2$. Furthermore,

$$\frac{\partial u}{\partial n} := h(u,\|u\|_{k(\cdot)},\|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\cdot\nu,$$

where $\nu$ represents the outward unit normal at $\Gamma_2$.

The operator governing the variable exponent mixed boundary problem (P1) has a double phase structure. Applications that have motivated research on this class of problems include image processing, transonic flow models, nonlinear versions of Derrick's problem, and the nonlinear theory of composite materials. We refer, for instance, to the works by Bahrouni–Rădulescu–Repovš [4], Benci–D'Avenia–Fortunato–Pisani [8], Charkaoui–Ben-Loghfyry–Zeng [12], Cherfils–Il'yasov [13] and Harjulehto–Hästö [24]. In its classical form, this operator reads

$$\operatorname{div}\left(|\nabla w|^{\vartheta-2}\nabla w + \mu(x)|\nabla w|^{\kappa-2}\nabla w\right),$$

with the associated growth functional

$$w \mapsto \int_\Omega \left(|\nabla w|^\vartheta + \mu(x)|\nabla w|^\kappa\right) \mathrm{d}x. \tag{1.1}$$

For a weight function $0 \le \mu(\cdot) \in L^\infty(\Omega)$, the integrand of (1.1) given by $\Phi(x,t) = t^\vartheta + \mu(x)t^\kappa$ exhibits unbalanced growth, that is

$$t^\vartheta \le \Phi(x,t) \le k_1\,(1+t^\kappa) \quad \text{for all } (x,t) \in \Omega \times [0,\infty),$$

with some constant $k_1 > 0$.

Functionals of the type (1.1) were first studied by Zhikov [47] and Marcellini [28, 29] in connection with nonlinear elasticity of composite materials. Such functionals also arise in duality theory and in the study of the Lavrentiev phenomenon; see Zhikov [48]. The qualitative behavior of (1.1) is strongly influenced by the modulating coefficient $\mu(\cdot)$. More precisely, on the set

$$\Omega_{>0} := \{x \in \Omega\colon \mu(x) > 0\},$$

the functional (1.1) exhibits $\kappa$-growth and $\kappa$-ellipticity in the gradient, while it reduces to $\vartheta$-growth and $\vartheta$-ellipticity on the complement where $\mu$ vanishes. The term "double phase" refers to these two elliptic regimes. The regularity of local minimizers of (1.1) was studied by Baroni–Colombo–Mingione [5, 6] and Colombo–Mingione [14, 15]. Other studies of regularity under unbalanced growth include those of Beck–Mingione [7], Byun–Ok–Song [9], Fuchs–Mingione [20], and Prasad–Tewary [41]. Boundedness of weak solutions for fractional double phase problems with logarithmic perturbation is established by Zeng–Lu–Rădulescu–Winkert [44]. The relevant Musielak–Orlicz Sobolev spaces and properties of double phase operators are treated by Arora–Crespo-Blanco–Winkert [2, 3], Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], Lu–Vetro–Zeng [27], and Vetro–Zeng [42]. For different choices of the right-hand side, existence has been studied by Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], Gasiński–Papageorgiou [22], Gasiński–Winkert [23], Liu–Dai [26], Papageorgiou–Rădulescu– Repovš [37, 38], Papageorgiou–Vetro–Vetro [39], Perera–Squassina [40], Zeng–Bai–Gasiński–Winkert [43] and Zeng–Rădulescu-Winkert [45, 46], see also the references therein.

The second feature of problem (P1) is the occurrence of an unbounded nonlocal coefficient

$$h\colon \mathbb{R} \times [0,\infty) \times [0,\infty) \to [a_0, +\infty), \quad a_0 > 0,$$

which is continuous but may be unbounded from above. Importantly, the function $h$ depends not only on the unknown function $u$, but also on its norms in the spaces $L^{k(\cdot)}(\Omega)$ with $1 < k(x) < \vartheta^*(x)$ for all $x \in \overline{\Omega}$, and in $W^{1,\Phi}(\Omega)$. The study of Dirichlet problems for the $p$-Laplacian with unbounded coefficients was initiated by Motreanu [32]. Later, Motreanu–Tornatore [35] investigated a Dirichlet problem driven by a degenerate $p$-Laplacian with an unbounded weight, where the reaction term involves convection. Further contributions to the theory of Dirichlet problems with unbounded coefficients can be found in Diblík–Galewski–Kossowski–Motreanu [17], Galewski–Motreanu [21], and Motreanu–Tornatore [36]. These works commonly rely on truncation methods as a key tool to address the challenges posed by unbounded weights.

Problem (P1) also involves the intrinsic operator

$$T\colon V \to W^{1,\vartheta-}(\Omega).$$

In (P1), the multivalued term $f$ describes convection in $\Omega$, whereas $f_{\Gamma_2}$ acts on the boundary. Intrinsic operators are of particular relevance due to their wide range of applications. Prominent examples include convolution, truncation, and the inverse of the $p$-Laplacian, see the paper by Motreanu–Tornatore [36]. Further studies of intrinsic operators in $p,q$-Laplacian equations include those by Medeiros–Motreanu [30] and Motreanu–Motreanu [33, 34].

We study the variable exponent double phase inclusion (P1) with an unbounded coefficient, a multivalued convection term and an intrinsic operator, together with its constant-exponent counterpart. It is worth noting that problem (P1) covers a wide range of special cases. For example, by choosing $1 < \vartheta(x) \equiv \vartheta$ and $1 < \kappa(x) \equiv \kappa$ for all $x \in \overline{\Omega}$, the corresponding constant exponent inclusion reads

$$\begin{cases} -\operatorname{div}\ \big[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{L}(\nabla u)\big] \in f(x, u, Tu, \nabla u, \nabla(Tu)) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} \in f_{\Gamma_2}(x, u, Tu) & \text{on } \Gamma_2, \end{cases}$$

where $\mathcal{L}(\nabla u) = |\nabla u|^{\vartheta-2}\nabla u + \mu(x)|\nabla u|^{\kappa-2}\nabla u$. When $|\Gamma_2| = 0$, (P1) becomes a variable exponent double phase inclusion subject to a Dirichlet condition

$$\begin{aligned} -\operatorname{div}\big[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\big] &\in f(x, u, Tu, \nabla u, \nabla(Tu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, if $\mu(x) = 0$ for a.e. $x \in \Omega$, the principal part of problem (P1) is an unbounded coefficient $\vartheta(\cdot)$-Laplacian, and the inclusion reads

$$\begin{aligned} -\operatorname{div}\Big(h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})|\nabla u|^{\vartheta(x)-2}\nabla u\Big) &\in f(x, u, Tu, \nabla u, \nabla(Tu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, Tu) \quad \text{on } \Gamma_2. \end{aligned}$$

Several choices are possible for the intrinsic operator $T$. For further details, see Sec. 4 and the work by Motreanu–Tornatore [36].

Establishing the existence of weak solutions to (P1) poses two main challenges. First, the unbounded coefficient $h$, which may be unbounded from above, precludes a direct treatment of (P1) by standard operator theory. Second, the convection term $f(x, u, Tu, \nabla u, \nabla(Tu))$, whose arguments include the gradient of $u$, generally prevents a direct variational treatment. Our strategy to overcome these obstacles follows the approach by Motreanu–Tornatore [36]. Section 3 introduces an auxiliary problem obtained by truncation. A surjectivity result for pseudomonotone multivalued operators then provides a solution. Uniform bounds for solutions of the truncated problem allow us to recover solutions of the original problem (P1). The

analysis is carried out in two distinct cases, reflecting the different requirements for variable versus constant exponents:

(i) In the variable exponent case, the coercivity estimates use growth at the minimum exponent $\vartheta_-$.
(ii) In the constant exponent case, a broader range of subcritical growth is admitted.

In both cases, the composition condition in (H5)(ii) or (H′5)(ii) is used for uniform boundedness. Detailed proofs are presented in Subsec. 3.1 (variable exponents) and Subsec. 3.2 (constant exponents).

Section 2 collects the Musielak–Orlicz space framework for the variable exponent double phase operator and the required definitions for multivalued operators. In Sec. 3, a surjectivity result yields solutions of an auxiliary truncated problem. Their uniform bounds then give weak solutions of (P1). Finally, in Sec. 4, we explore applications of problem (P1) by considering various choices of the intrinsic operator $T$.

## 2. Preliminaries

The Musielak–Orlicz space framework used below is based primarily on the works of Crespo-Blanco–Gasiński–Harjulehto–Winkert [16] and Lu–Vetro–Zeng [27], with emphasis on variable exponent double phase operators. Related multivalued double phase inclusions are studied by Cen–Costea [11]. We work on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, whose boundary $\partial\Omega$ is Lipschitz.

**Definition 2.1**

(i) A continuous, convex mapping $\varphi\colon [0,\infty) \to [0,\infty)$ is an $N$-function when $\varphi(t) = 0$ only for $t = 0$ and the following limits hold:

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{\varphi(t)}{t} = +\infty.$$

(ii) For a mapping $\varphi\colon \Omega \times [0,\infty) \to [0,\infty)$, membership in the generalized $N$-function class, written $\varphi \in N(\Omega)$, requires $\varphi(\cdot,t)$ to be measurable at each $t \geq 0$ and $\varphi(x,\cdot)$ to be an $N$-function for a.e. $x \in \Omega$.

**Definition 2.2.** For a function $\varphi$, the doubling condition $\varphi \in \Delta_2$ means that a constant $c_\varphi \geq 1$ can be chosen so that

$$\varphi(x,2t) \leq c_\varphi \varphi(x,t)$$

for a.e. $x \in \Omega$ and for all $t > 0$. Moreover, if $\varphi \in \Delta_2$, then

$$\varphi(x,t+s) \leq c_\varphi \varphi(x,t) + c_\varphi \varphi(x,s)$$

for a.e. $x \in \Omega$ and for all $t,s \geq 0$. For $\varphi,\psi \in N(\Omega)$, the relation $\varphi \prec \psi$ means that $\varphi$ is weaker than $\psi$: there are positive $c_1, c_2$ and a non-negative $h \in L^1(\Omega)$ for

which

$$\varphi(x,t) \le c_1\psi(x,c_2t) + h(x)$$

for a.e. $x \in \Omega$ and for all $t > 0$.

For $\varphi \in N(\Omega)$, its Fenchel conjugate is defined by

$$\varphi^*(x,t) = \sup_{s\ge 0}(st - \varphi(x,s)) \quad \text{for a.e. } x \in \Omega \text{ and for all } t > 0,$$

which satisfies Young's inequality

$$ts \le \varphi(x,t) + \varphi^*(x,s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s,t > 0.$$

Moreover, for $f \in L^{\varphi}(\Omega)$ and $g \in L^{\varphi^*}(\Omega)$, we have

$$\int_\Omega |f||g|\,\mathrm{d}x \le 2\|f\|_\varphi \|g\|_{\varphi^*}.$$

If $\varphi \in \Delta_2$ is convex with derivative $\varphi'$, then for some constants $c, \widetilde{c} > 1$,

$$c^{-1}\varphi(x,t) \le \varphi'(x,t)t \le c\varphi(x,t) \quad \text{and} \quad \varphi^*(x,\varphi'(x,t)) \le \widetilde{c}\varphi(x,t)$$

for a.e. $x \in \Omega$ and for all $t \ge 0$. To define the Musielak–Orlicz space $L^{\varphi}(\Omega)$ for $\varphi \in N(\Omega)$, we first introduce its modular

$$\rho_\varphi(u) = \int_\Omega \varphi(x,|u|)\,\mathrm{d}x,$$

and the set

$$L^{\varphi}(\Omega) := \{u \in M(\Omega)\colon \text{ there exists } \lambda > 0 \text{ such that } \rho_\varphi(\lambda u) < +\infty\}$$

whose norm is the Luxemburg norm

$$\|u\|_{\varphi,\Omega} := \inf\left\{\lambda > 0\colon \rho_\varphi\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Here $M(\Omega)$ denotes the space of Lebesgue measurable real-valued functions on $\Omega$, with functions identified when they agree almost everywhere. We suppress the domain in the norm notation, writing $\|u\|_\varphi$ for $\|u\|_{\varphi,\Omega}$. The associated Musielak–Orlicz Sobolev space consists of the functions

$$W^{1,\varphi}(\Omega) := \{u \in L^{\varphi}(\Omega)\colon |\nabla u| \in L^{\varphi}(\Omega)\}$$

with norm

$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi,$$

where $\|\nabla u\|_\varphi = \| \, |\nabla u| \, \|_\varphi$ and $\rho_\varphi(\nabla u) = \rho_\varphi(|\nabla u|)$. When $\varphi \in N(\Omega)$ is locally integrable (i.e. $\varphi(\cdot,t) \in L^1(\Omega)$ for all $t > 0$), completing $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$ yields $W_0^{1,\varphi}(\Omega)$. Thus $W_0^{1,\varphi}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,\varphi}(\Omega)}$. Musielak states the next two results in his monograph [31, Theorems 7.7 and 8.5].

**Proposition 2.3**

(i) *For $\varphi \in N(\Omega)$, the norm $\|\cdot\|_\varphi$ makes the Musielak–Orlicz space complete. Equivalently, $(L^{\varphi}(\Omega), \|\cdot\|_\varphi)$ is a Banach space.*

(ii) *Local integrability of* $\varphi, \psi \in N(\Omega)$ *together with* $\varphi \prec \psi$ *implies the continuous embedding* $L^{\psi}(\Omega) \hookrightarrow L^{\varphi}(\Omega)$.

The unit ball properties below are recorded in Musielak [31, Theorem 8.13] and Diening–Harjulehto–Hästö–Růžička [18, Lemma 2.1.14].

**Proposition 2.4.** *The following assertions are valid*:

(i) *Let* $\varphi \in N(\Omega) \cap \Delta_2$. *Then*

$$L^{\varphi}(\Omega) = \{u \in M(\Omega) \colon \rho_{\varphi}(u) < +\infty\}.$$

(ii) *Let* $\varphi \in N(\Omega) \cap \Delta_2$ *and* $u \in L^{\varphi}(\Omega)$. *Then* $\rho_{\varphi}(u) < 1$(respectively, $= 1; > 1$) *if and only if* $\|u\|_{\varphi} < 1$ (respectively, $= 1; > 1$).

For completeness and reflexivity of the Sobolev spaces, we use the following sufficient conditions from Musielak [31, Theorem 10.2] and Fan [19, Proposition 1.7 and 1.8].

**Theorem 2.5.** *Assume that* $\varphi \in N(\Omega)$ *is locally integrable and satisfies*

$$\inf_{x \in \Omega} \varphi(x, 1) > 0.$$

*The spaces* $W^{1,\varphi}(\Omega)$ *and* $W_0^{1,\varphi}(\Omega)$ *are then Banach spaces. If* $\varphi$ *also satisfies the* $\Delta_2$ *condition, these spaces are separable. Reflexivity of* $L^{\varphi}(\Omega)$ *also entails reflexivity of* $W^{1,\varphi}(\Omega)$ *and* $W_0^{1,\varphi}(\Omega)$.

For the particular choice $\varphi(x,t) = t^{\vartheta(x)}$, the conjugate takes the form

$$\varphi^*(x,t) = \vartheta(x)^{-\vartheta'(x)}(\vartheta(x) - 1)t^{\vartheta'(x)},$$

where $1 < \vartheta(\cdot) \in C(\overline{\Omega})$ and the relation $\frac{1}{\vartheta(x)} + \frac{1}{\vartheta'(x)} = 1$ holds for all $x \in \overline{\Omega}$. This choice of $\varphi$ yields the variable exponent Lebesgue space $L^{\vartheta(\cdot)}(\Omega)$.

For $\vartheta \in C(\overline{\Omega})$, we write its minimum and maximum as

$$\vartheta_- := \min_{x \in \overline{\Omega}} \vartheta(x) \quad \text{and} \quad \vartheta_+ := \max_{x \in \overline{\Omega}} \vartheta(x).$$

Analogous notation is used for $\kappa$. Suppose $1 < \vartheta(x) < N$ for all $x \in \overline{\Omega}$. The critical Sobolev exponents for $\Omega$ and $\partial\Omega$, respectively, are

$$\vartheta^*(x) := \frac{N\vartheta(x)}{N - \vartheta(x)} \quad \text{and} \quad \vartheta_*(x) := \frac{(N-1)\vartheta(x)}{N - \vartheta(x)} \quad \text{for all } x \in \overline{\Omega}.$$

We impose the following hypothesis:

(H0) Let $\vartheta, \kappa \in C(\overline{\Omega})$ such that $1 < \vartheta(x) < N$ and $\vartheta(x) \le \kappa(x) < \vartheta^*(x)$ for all $x \in \overline{\Omega}$. Let $0 \le \mu(\cdot) \in L^{\infty}(\Omega)$. The case $\mu \equiv 0$ is allowed.

The critical exponents corresponding to $\vartheta_-$ are denoted by

$$(\vartheta_-)^* := \frac{N\vartheta_-}{N - \vartheta_-} \quad \text{and} \quad (\vartheta_-)_* := \frac{(N-1)\vartheta_-}{N - \vartheta_-},$$

respectively. The double phase structure is described by the function $\Phi\colon \Omega \times [0,\infty) \to [0,\infty)$, namely

$$\Phi(x,t) := t^{\vartheta(x)} + \mu(x)t^{\kappa(x)} \quad \text{for all } (x,t) \in \Omega \times [0,\infty). \tag{2.1}$$

The energy density associated with $\mathcal{K}$ is given by

$$\Psi(x,t) := \frac{t^{\vartheta(x)}}{\vartheta(x)} + \mu(x)\frac{t^{\kappa(x)}}{\kappa(x)}.$$

For $\xi \neq 0$, $\mathcal{K}(\xi) = \partial_t \Psi(x,|\xi|)\xi/|\xi|$, and $\mathcal{K}(0) = 0$. Under (H0), the function $\Phi$ in (2.1) is a locally integrable N-function. The associated modular for $\Phi$ reads

$$\rho_\Phi(u) = \int_\Omega \Phi(x,|u|)\,\mathrm{d}x.$$

The resulting Musielak–Orlicz space takes the form

$$L^\Phi(\Omega) = \{u \in M(\Omega)\colon \rho_\Phi(u) < +\infty\},$$

and carries the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{\lambda > 0\colon \rho_\Phi\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Moreover, the Sobolev space $W^{1,\Phi}(\Omega)$ is defined as

$$W^{1,\Phi}(\Omega) := \{u \in L^\Phi(\Omega)\colon |\nabla u| \in L^\Phi(\Omega)\},$$

while $W_0^{1,\Phi}(\Omega)$ is obtained by completing $C_0^\infty(\Omega)$ in the norm of $W^{1,\Phi}(\Omega)$. On both $W_0^{1,\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ we use the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|\nabla u\|_\Phi.$$

Under hypothesis (H0), the Luxemburg norm $\|\cdot\|_\Phi$ makes $L^\Phi(\Omega)$ a Banach space that is both separable and reflexive. These properties also hold for $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$.

An equivalent norm on $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{\hat{\rho}_\Phi} := \inf\left\{\lambda > 0\colon \hat{\rho}_\Phi\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

with $\hat{\rho}_\Phi$ denoting the modular

$$\hat{\rho}_\Phi(u) = \int_\Omega \left(|\nabla u|^{\vartheta(x)} + \mu(x)|\nabla u|^{\kappa(x)}\right)\mathrm{d}x + \int_\Omega |u|^{\vartheta(x)}\,\mathrm{d}x,$$

for $u \in W^{1,\Phi}(\Omega)$, see Amoroso–Crespo-Blanco–Pucci–Winkert [1, Proposition 3.1].

The norm-modular estimates below are given by Amoroso–Crespo-Blanco–Pucci–Winkert [1, Proposition 3.2].

**Proposition 2.6.** *Let hypothesis* (H0) *be satisfied and* $u \in W^{1,\Phi}(\Omega)$.

(i) *If* $\|u\|_{\hat{\rho}_\Phi} < 1$, *then* $\|u\|_{\hat{\rho}_\Phi}^{\kappa_+} \leq \hat{\rho}_\Phi(u) \leq \|u\|_{\hat{\rho}_\Phi}^{\vartheta_-}$.
(ii) *If* $\|u\|_{\hat{\rho}_\Phi} > 1$, *then* $\|u\|_{\hat{\rho}_\Phi}^{\vartheta_-} \leq \hat{\rho}_\Phi(u) \leq \|u\|_{\hat{\rho}_\Phi}^{\kappa_+}$.

We write $X \hookrightarrow Y$ when the embedding of $X$ into $Y$ is continuous, and $X \hookrightarrow\hookrightarrow Y$ when the embedding of $X$ into $Y$ is compact. A function $h\colon \overline{\Omega} \to \mathbb{R}$ in $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ is log-Hölder continuous if there is a constant $C > 0$ satisfying

$$|h(x) - h(y)| \leq \frac{C}{|\log|x-y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } 0 < |x-y| < \frac{1}{2}.$$

For a bounded domain $\Omega \subset \mathbb{R}^N$ and $\gamma > N$, it holds

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

The embeddings listed below are established in Proposition 2.16 of Crespo-Blanco–Gasiński–Harjulehto–Winkert [16].

**Proposition 2.7.** *Let hypothesis* (H0) *be satisfied. Then the following hold*:

(i) $L^{\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq \vartheta(x)$ *for all* $x \in \Omega$.

(ii) *Let* $\vartheta \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, *then* $W^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq \vartheta^*(x)$ *for all* $x \in \overline{\Omega}$. *Under* (H0), *if* $1 \leq r(x) < \vartheta^*(x)$ *for all* $x \in \overline{\Omega}$, *both embeddings are compact without the additional log-Hölder assumption.*

(iii) *Let* $\vartheta \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma > N$, *then* $W^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *and* $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq \vartheta_*(x)$ *for all* $x \in \overline{\Omega}$.

(iv) $W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega), W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < \vartheta_*(x)$ *for all* $x \in \overline{\Omega}$.

**Proposition 2.8.** *Let hypothesis* (H0) *be satisfied. Then the following hold*:

(i) $W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{\Phi}(\Omega)$.

(ii) *A constant* $C > 0$, *independent of* $u$, *can be chosen such that*

$$\|u\|_{\Phi} \leq C\|\nabla u\|_{\Phi} \tag{2.2}$$

*for all* $u \in W_0^{1,\Phi}(\Omega)$.

**Remark 2.9.** Following the norm-equivalence approach of Cen–Costea [11], we show that $\|\nabla u\|_{\Phi}$ is an equivalent norm on the closed mixed-boundary space

$$V := \{u \in W^{1,\Phi}(\Omega)\colon \operatorname{Tr} u = 0 \text{ a.e. on } \Gamma_1\}.$$

Take $P(u) := \int_{\Gamma_1} |\operatorname{Tr} u| \, d\sigma$. The trace estimate implies $P(u) \leq C\|u\|_{1,\Phi}$, and $P(c) = |c||\Gamma_1|$ for constants $c$. It remains to prove that, for some $C > 0$,

$$\|u\|_{1,\Phi} \leq C\,(\|\nabla u\|_{\Phi} + P(u)) \quad \text{for all } u \in W^{1,\Phi}(\Omega).$$

If this estimate failed, homogeneity would provide a sequence $u_n \in W^{1,\Phi}(\Omega)$ with

$$\|u_n\|_{1,\Phi} = 1, \quad \|\nabla u_n\|_{\Phi} + P(u_n) \to 0.$$

The reflexivity of $W^{1,\Phi}(\Omega)$ and its compact embedding into $L^{\Phi}(\Omega)$ in Proposition 2.8(i) yield, after passage to a subsequence, $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ and $u_n \to u$

in $L^\Phi(\Omega)$. Since $\nabla u_n \to 0$ in $L^\Phi$, we have $\nabla u = 0$, so $u$ is constant on the domain $\Omega$. It follows that $u_n \to u$ in $W^{1,\Phi}(\Omega)$, and continuity of $P$ gives $P(u) = 0$. Thus $u = 0$ because $|\Gamma_1| > 0$, contradicting $\|u_n\|_{1,\Phi} = 1$. Consequently, $\|\nabla u\|_\Phi + P(u)$ is equivalent to $\|u\|_{1,\Phi}$ on $W^{1,\Phi}(\Omega)$. On $V$, $P(u) = 0$, so (2.2) holds and $\|\nabla u\|_\Phi$ is an equivalent norm. We retain $\|u\| := \|u\|_{\hat\rho_\Phi}$ on $V$, including in the arguments of $h$. Choose $D \geq 1$ such that $\|u\| \leq D\|\nabla u\|_\Phi$ on $V$ and set $a_* := a_0 D^{-\vartheta_-}$. For $\|u\| \geq D$, we have $\|\nabla u\|_\Phi \geq \|u\|/D \geq 1$, and the norm-modular inequalities give

$$\int_\Omega \Phi(x, |\nabla u|)\,\mathrm{d}x \geq \|\nabla u\|_\Phi^{\vartheta_-} \geq D^{-\vartheta_-}\|u\|^{\vartheta_-}.$$

Multiplying by $a_0$ yields

$$a_0 \int_\Omega \Phi(x, |\nabla u|)\,\mathrm{d}x \geq a_*\|u\|^{\vartheta_-}.$$

Write $C_r, C_{r,\Gamma_2}$ for the embedding constants of $V$ (with norm $\|\cdot\|$) into $L^r(\Omega), L^r(\Gamma_2)$ whenever the relevant embedding holds, and choose $G_{\vartheta_-}$ so that $\|\nabla u\|_{\vartheta_-} \leq G_{\vartheta_-}\|u\|$. The output $Tu$ need not have zero trace on $\Gamma_1$.

Next, we recall standard definitions for multivalued operators.

**Definition 2.10.** Consider a real reflexive Banach space $X$, its dual $X^*$, and the duality pairing $\langle\cdot,\cdot\rangle$. For a multivalued operator $B\colon X \to 2^{X^*}$, we use the following terminology:

(i) pseudomonotone if and only if
   (a) For each $v \in X$, the value $B(v)$ is a nonempty, closed, convex and bounded set.
   (b) On every finite-dimensional subspace of $X$, the restriction of $B$ is upper semicontinuous with respect to the weak topology of $X^*$.
   (c) For any sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$ with $v_n \rightharpoonup v$ in $X$ and $v_n^* \in B(v_n)$ with $\limsup\langle v_n^*, v_n - v\rangle \leq 0$, there exists $v^*(w) \in B(v)$ satisfying
   $$\liminf\langle v_n^*, v_n - w\rangle \geq \langle v^*(w), v - w\rangle$$
   for all $w \in X$,

(ii) generalized pseudomonotone if and only if $\{v_n\}_{n\in\mathbb{N}} \subset X$, $\{v_n^*\}_{n\in\mathbb{N}} \subset X^*$ with $v_n^* \in B(v_n)$ such that $v_n \rightharpoonup v$ in $X$, $v_n^* \rightharpoonup v^*$ in $X^*$ and
   $$\limsup\langle v_n^*, v_n - v\rangle \leq 0$$
   imply $v^* \in B(v)$ and
   $$\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle,$$

(iii) coercive if and only if for $v \in D(B)$ such that $\|v\|_X \to \infty$, it holds that
   $$\frac{\inf\{\langle v^*, v\rangle\colon v^* \in B(v)\}}{\|v\|_X} \to +\infty.$$

The following surjective theorem is essential for our existence results, see Carl–Le [10].

**Theorem 2.11.** *Suppose $X$ is a real reflexive Banach space and $T\colon X \to 2^{X^*}$ is bounded and pseudomonotone. Assume the following coercivity condition: a real-valued function $c\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{r\to+\infty} c(r) = +\infty$ and a point $v_0 \in X$ can be chosen so that*

$$\langle v^*, v - v_0\rangle \geq c(\|v\|_X)\|v\|_X \quad \text{for all } (v, v^*) \in \mathrm{Gr}\ (T).$$

*Its range is then the whole space, that is,* range $(T) = X^*$, *so $T$ is surjective.*

## 3. Existence of Uniformly Bounded Weak Solutions

To obtain uniformly bounded weak solutions of (P1), we truncate the unbounded coefficient $h$ to define an auxiliary problem, and then establish its solvability via a surjectivity theorem for multivalued pseudomonotone operators. Uniform estimates independent of the truncation parameters allow us to choose those parameters so that an auxiliary solution also satisfies the original problem.

First, we focus on the problem involving variable exponents and present the corresponding assumptions for the function $h$, the multivalued functions $f\colon \Omega \times \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ and $f_{\Gamma_2}\colon \Gamma_2\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ as well as the intrinsic operator $T\colon V \to W^{1,\vartheta_-}(\Omega)$.

(H1) The function $h\colon \mathbb{R}\times[0,\infty)\times[0,\infty) \to [a_0,\infty)$ is continuous, where $a_0$ is a positive constant.

(H2) The functions $f\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ and $f_{\Gamma_2}\colon \Gamma_2\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ are graph measurable with $f(x,\cdot,\cdot,\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ being upper semicontinuous for a.e. $x\in\Omega$ and $f_{\Gamma_2}(x,\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R}\to 2^{\mathbb{R}}\backslash\{\emptyset\}$ being upper semicontinuous for a.e. $x\in\Gamma_2$. Both maps have nonempty closed convex values.

(H3) The growth of the interior term is controlled by constants $a_i \geq 0$ for $i = 1,2,3,4$ and a function $\sigma_1 \in L^{\frac{\vartheta_-}{\vartheta_- - 1}}(\Omega)$ through the estimate

$$\begin{aligned}&\sup\{|\eta|\colon \eta \in f(x,s_1,s_2,\xi_1,\xi_2)\}\\ &\leq \sigma_1(x) + a_1|s_1|^{\vartheta_- - 1} + a_2|s_2|^{\vartheta_- - 1} + a_3|\xi_1|^{\vartheta_- - 1} + a_4|\xi_2|^{\vartheta_- - 1},\end{aligned}$$

for a.e. $x\in\Omega$, for all $s_1, s_2 \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

(H4) For the boundary term, there are constants $b_i \geq 0$ for $i = 1,2$ and a function $\sigma_2 \in L^{\frac{\vartheta_-}{\vartheta_- - 1}}(\Gamma_2)$ for which

$$\sup\{|\xi|\colon \xi \in f_{\Gamma_2}(x,s_1,s_2)\} \leq \sigma_2(x) + b_1|s_1|^{\vartheta_- - 1} + b_2|s_2|^{\vartheta_- - 1},$$

for a.e. $x\in\Gamma_2$ and for all $s_1, s_2 \in \mathbb{R}$.

(H5) (i) There are constants $0 \leq \Lambda < a_*$ and $K \geq 0$ such that

$$\int_\Omega \eta u \,\mathrm{d}x - \int_{\Gamma_2} \zeta u \,\mathrm{d}\sigma \leq \Lambda \|u\|^{\vartheta_-} + K\|u\| \tag{3.1}$$

for every $u \in V$ and all measurable selections $\eta(x) \in f(x, u, Tu, \nabla u, \nabla(Tu))$ and $\zeta(x) \in f_{\Gamma_2}(x, u, Tu)$ almost everywhere. The norm and $a_*$ are fixed in Remark 2.9.

(ii) There exist exponents

$$\vartheta_- \leq d < (\vartheta_-)^*, \quad \vartheta_- \leq e < (\vartheta_-)_*,$$
$$\frac{N}{\vartheta_-} < \gamma_1 < \infty, \quad \frac{N-1}{\vartheta_- - 1} < \gamma_2 < \infty$$

such that, for every $M > 0$, there exists $L_M > 0$ with the following property. For each $u \in V$ with $\|u\| \leq M$, there are non-negative functions $\omega_{u,1} \in L^{\gamma_1}(\Omega)$ and $\omega_{u,2} \in L^{\gamma_2}(\Gamma_2)$ satisfying

$$\|\omega_{u,1}\|_{\gamma_1} + \|\omega_{u,2}\|_{\gamma_2,\Gamma_2} \leq L_M$$

and

$$\sup_{\eta \in f(x,u,Tu,\nabla u,\nabla(Tu))} |\eta|$$
$$\leq \omega_{u,1}(x) + L_M \left( |u|^{d-1} + |\nabla u|^{\vartheta_-(d-1)/d} \right) \quad \text{a.e. in } \Omega,$$

as well as

$$\sup_{\zeta \in f_{\Gamma_2}(x,u,Tu)} |\zeta| \leq \omega_{u,2}(x) + L_M |u|^{e-1} \quad \text{a.e. on } \Gamma_2.$$

Hypothesis (H5)(i) provides the energy estimate. Hypothesis (H5)(ii) is used in Lemma 3.7 to obtain uniform bounds independent of the truncation parameters.

**Remark 3.1.** Under (H3)–(H4), a sufficient condition for (H5)(i) is that $T$ satisfies the following estimates with constants $c_i \geq 0$ for $i = 1, 2, 3, 4$:

$$\|Tu\|_{\vartheta_-}^{\vartheta_- - 1} \leq c_1 \|u\|_{\hat{\rho}_\Phi}^{\vartheta_- - 1} + c_3,$$
$$\|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- - 1} \leq c_2 \|u\|_{\hat{\rho}_\Phi}^{\vartheta_- - 1} + c_3,$$
$$\|Tu\|_{\vartheta_-,\Gamma_2}^{\vartheta_- - 1} \leq c_4 \|u\|_{\hat{\rho}_\Phi}^{\vartheta_- - 1} + c_3,$$

for all $u \in V$, together with

$$\Lambda := a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 c_1 C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 c_2 C_{\vartheta_-}$$
$$+ b_1 C_{\vartheta_-,\Gamma_2}^{\vartheta_-} + b_2 c_4 C_{\vartheta_-,\Gamma_2} < a_*, \tag{3.2}$$

Indeed, Hölder's inequality and these estimates give

$$\begin{aligned}\int_\Omega \eta u\,\mathrm{d}x - \int_{\Gamma_2} \zeta u\,\mathrm{d}\sigma &\le \int_\Omega |\eta||u|\,\mathrm{d}x + \int_{\Gamma_2} |\zeta||u|\,\mathrm{d}\sigma \\ &\le \left(\|\sigma_1\|_{\vartheta_-'} + a_2\|Tu\|_{\vartheta_-}^{\vartheta_- -1} + a_4\|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- -1}\right)\|u\|_{\vartheta_-} \\ &\quad + a_1\|u\|_{\vartheta_-}^{\vartheta_-} + a_3\|\nabla u\|_{\vartheta_-}^{\vartheta_- -1}\|u\|_{\vartheta_-} \\ &\quad + \left(\|\sigma_2\|_{\vartheta_-',\Gamma_2} + b_2\|Tu\|_{\vartheta_-,\Gamma_2}^{\vartheta_- -1}\right)\|u\|_{\vartheta_-,\Gamma_2} + b_1\|u\|_{\vartheta_-,\Gamma_2}^{\vartheta_-} \\ &\le \Lambda\|u\|^{\vartheta_-} + K\|u\|,\end{aligned}$$

where

$$K = C_{\vartheta_-}[\|\sigma_1\|_{\vartheta_-'} + (a_2+a_4)c_3] + C_{\vartheta_-,\Gamma_2}[\|\sigma_2\|_{\vartheta_-',\Gamma_2} + b_2c_3].$$

Thus (3.1) holds with constants independent of $u$.

If, in addition, $\vartheta_- > N-1$, the same norm estimates imply (H5)(ii). Indeed,

$$\vartheta_-' > \frac{N}{\vartheta_-} \quad \text{and} \quad \vartheta_-' > \frac{N-1}{\vartheta_- - 1}.$$

Take $d = e = \vartheta_-$ and $\gamma_1 = \gamma_2 = \vartheta_-'$, and set

$$\omega_{u,1} = \sigma_1 + a_2|Tu|^{\vartheta_- -1} + a_4|\nabla(Tu)|^{\vartheta_- -1},$$

$$\omega_{u,2} = \sigma_2 + b_2|Tu|^{\vartheta_- -1}.$$

Since $(\vartheta_- - 1)\vartheta_-' = \vartheta_-$, the displayed norm estimates give, for $\|u\| \le M$,

$$\begin{aligned}\|\omega_{u,1}\|_{\vartheta_-'} + \|\omega_{u,2}\|_{\vartheta_-',\Gamma_2} &\le \|\sigma_1\|_{\vartheta_-'} + \|\sigma_2\|_{\vartheta_-',\Gamma_2} \\ &\quad + (a_2c_1 + a_4c_2 + b_2c_4)M^{\vartheta_- -1} + (a_2+a_4+b_2)c_3.\end{aligned}$$

The remaining terms in (H3)–(H4) have exactly the powers required in (H5)(ii). Choose $L_M$ larger than the displayed norm bound and $a_1, a_3, b_1$.

We use the following weak formulation of (P1).

**Definition 3.2.** We say that a function $u \in V$ is a weak solution of problem (P1), if there exist fixed measurable selections $\eta \in f(x,u,Tu,\nabla u,\nabla(Tu))$ a.e. in $\Omega$ and $\zeta \in f_{\Gamma_2}(x,u,Tu)$ a.e. on $\Gamma_2$, such that, for every $v \in V$, $h(u,\|u\|_{k(\cdot)},\|u\|_{\hat\rho_\Phi})\mathcal{K}(\nabla u)\cdot\nabla v \in L^1(\Omega)$, $\eta v \in L^1(\Omega)$, $\zeta v \in L^1(\Gamma_2)$, and the following identity holds:

$$\begin{aligned}&\int_\Omega h(u,\|u\|_{k(\cdot)},\|u\|_{\hat\rho_\Phi})\left(|\nabla u|^{\vartheta(x)-2}\nabla u + \mu(x)|\nabla u|^{\kappa(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x \\ &\quad = \int_\Omega \eta v\,\mathrm{d}x - \int_{\Gamma_2}\zeta v\,\mathrm{d}\sigma. \qquad (3.3)\end{aligned}$$

The main existence statements for weak solutions of (P1) are given next.

**Theorem 3.3.** *Let hypotheses* (H0)–(H5) *be satisfied. Then problem* (P1) *possesses a uniformly bounded weak solution $u \in V$. Its trace on $\Gamma_2$ is also bounded.*

Next, we consider the particular case when the exponents are constants, that is, $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$. In this scenario, the growth conditions for the multivalued functions $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ and $f_{\Gamma_2}\colon \Gamma_2 \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ are weaker as seen below:

(H′3) There exist constants $\tilde{a}_i \geq 0$ for $i = 1, 2, 3, 4$, $\tilde{r}_1 \in (1, \tilde{\vartheta}^*)$ and $\tilde{\sigma}_1 \in L^{\tilde{r}_1'}(\Omega)$ satisfying

$$\begin{aligned} &\sup\{|\eta|\colon \eta \in f(x, s_1, s_2, \xi_1, \xi_2)\} \\ &\quad \leq \tilde{\sigma}_1(x) + \tilde{a}_1 |s_1|^{\tilde{\vartheta}-1} + \tilde{a}_2 |s_2|^{\tilde{r}_1 - 1} + \tilde{a}_3 |\xi_1|^{\tilde{\vartheta}-1} + \tilde{a}_4 |\xi_2|^{\frac{\tilde{\vartheta}}{\tilde{r}_1'}}, \end{aligned}$$

for a.e. $x \in \Omega$, for all $s_1, s_2 \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

(H′4) For the boundary term, choose constants $\tilde{b}_i \geq 0$ for $i = 1, 2$, $\tilde{r}_2 \in (1, \tilde{\vartheta}_*)$ and a function $\tilde{\sigma}_2 \in L^{\tilde{r}_2'}(\Gamma_2)$ so that

$$\sup\{|\xi|\colon \xi \in f_{\Gamma_2}(x, s_1, s_2)\} \leq \tilde{\sigma}_2(x) + \tilde{b}_1 |s_1|^{\tilde{\vartheta}-1} + \tilde{b}_2 |s_2|^{\tilde{r}_2 - 1},$$

for a.e. $x \in \Gamma_2$ and for all $s_1, s_2 \in \mathbb{R}$.

(H′5) (i) There are constants $0 \leq \widetilde{\Lambda} < a_*$ and $\widetilde{K} \geq 0$ such that

$$\int_\Omega \eta u \,\mathrm{d}x - \int_{\Gamma_2} \zeta u \,\mathrm{d}\sigma \leq \widetilde{\Lambda} \|u\|^{\tilde{\vartheta}} + \widetilde{K} \|u\| \tag{3.4}$$

for every $u \in V$ and all measurable selections specified in (H5)(i).

(ii) Hypothesis (H5)(ii) holds with $\vartheta_- = \tilde{\vartheta}$.

**Remark 3.4.** Under (H′3)–(H′4), a sufficient condition for (H′5)(i) is that $T$ satisfies the following estimates with constants $\tilde{c}_i \geq 0$ for $i = 1, 2, 3, 4$:

$$\begin{aligned} \|Tu\|_{\tilde{\vartheta}^*}^{\tilde{r}_1 - 1} &\leq \tilde{c}_1 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + \tilde{c}_3, \\ \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\frac{\tilde{\vartheta}}{\tilde{r}_1'}} &\leq \tilde{c}_2 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + \tilde{c}_3, \\ \|Tu\|_{\tilde{\vartheta}_*, \Gamma_2}^{\tilde{r}_2 - 1} &\leq \tilde{c}_4 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + \tilde{c}_3, \end{aligned}$$

for all $u \in V$, together with

$$\begin{aligned} \widetilde{\Lambda} &:= \tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 \tilde{c}_1 C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{a}_3 G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}} + \tilde{a}_4 \tilde{c}_2 C_{\tilde{r}_1} \\ &\quad + \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 \tilde{c}_4 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2} < a_*. \end{aligned} \tag{3.5}$$

The terms involving $T$ are estimated by Hölder's inequality as follows:

$$\begin{aligned} \int_\Omega |Tu|^{\tilde{r}_1 - 1} |u| \,\mathrm{d}x &\leq \|Tu\|_{\tilde{\vartheta}^*}^{\tilde{r}_1 - 1} \|u\|_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}}, \\ \int_\Omega |\nabla(Tu)|^{\tilde{\vartheta}/\tilde{r}_1'} |u| \,\mathrm{d}x &\leq \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}_1'} \|u\|_{\tilde{r}_1}, \\ \int_{\Gamma_2} |Tu|^{\tilde{r}_2 - 1} |u| \,\mathrm{d}\sigma &\leq \|Tu\|_{\tilde{\vartheta}_*, \Gamma_2}^{\tilde{r}_2 - 1} \|u\|_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2}. \end{aligned}$$

Here the first and third pairs of Hölder exponents follow from

$$\frac{\tilde{r}_1 - 1}{\tilde{\vartheta}^*} + \frac{\tilde{\vartheta}^* - \tilde{r}_1 + 1}{\tilde{\vartheta}^*} = 1, \quad \frac{\tilde{r}_2 - 1}{\tilde{\vartheta}_*} + \frac{\tilde{\vartheta}_* - \tilde{r}_2 + 1}{\tilde{\vartheta}_*} = 1.$$

By substituting the norm bounds for $T$ and estimating the remaining terms as in Remark 3.1, we obtain (3.4) with the displayed $\widetilde{\Lambda}$ and

$$\widetilde{K} = \|\tilde{\sigma}_1\|_{\tilde{r}_1'} C_{\tilde{r}_1} + \tilde{c}_3(\tilde{a}_2 C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{a}_4 C_{\tilde{r}_1}) + \|\tilde{\sigma}_2\|_{\tilde{r}_2', \Gamma_2} C_{\tilde{r}_2, \Gamma_2} + \tilde{b}_2 \tilde{c}_3 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2}.$$

When $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta} > N - 1$, these estimates also imply (H′5)(ii). The critical domain and trace norm estimates control the corresponding $L^{\tilde{\vartheta}}$ norms on these finite-measure sets, so the argument at the end of Remark 3.1 applies.

We have the following existence result.

**Theorem 3.5.** *Suppose $\vartheta(x) \equiv \tilde{\vartheta} > 1$, $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and assume* (H0)–(H2) *together with* (H′3) –(H′5). *Problem* (P1) *then has a weak solution $u \in V$ that is uniformly bounded in $\Omega$, with bounded trace on $\Gamma_2$.*

### 3.1. *Variable exponents*

We now give the proof of Theorem 3.3.

**Lemma 3.6.** *Assume* (H0)–(H5). *Every weak solution $u \in V$ of* (P1), *or of its auxiliary problem with $h_R$, obeys $\|u\|_{\hat{\rho}_\Phi} \le M_1$ for a constant $M_1 \ge 1$ depending on the problem data and the lower bound $a_0$ of $h$.*

**Proof.** Let $u$ be a solution of (P1). Choosing $v = u$ as a test function in (3.3) yields

$$\begin{aligned} &\int_\Omega h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left( |\nabla u|^{\vartheta(x)} + \mu(x) |\nabla u|^{\kappa(x)} \right) \mathrm{d}x \\ &\quad = \int_\Omega \eta u \, \mathrm{d}x - \int_{\Gamma_2} \zeta u \, \mathrm{d}\sigma. \end{aligned} \tag{3.6}$$

Assume $\|u\|_{\hat{\rho}_\Phi} \ge D$. By (H1), Proposition 2.6 and Remark 2.9, we obtain

$$\begin{aligned} &\int_\Omega h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left( |\nabla u|^{\vartheta(x)} + \mu(x) |\nabla u|^{\kappa(x)} \right) \mathrm{d}x \\ &\quad \ge a_0 \int_\Omega \Phi(x, |\nabla u|) \, \mathrm{d}x \ge a_* \|u\|_{\hat{\rho}_\Phi}^{\vartheta_-}. \end{aligned} \tag{3.7}$$

By (H5)(i), (3.6) and (3.7), we obtain

$$(a_* - \Lambda) \|u\|^{\vartheta_-} \le K \|u\|.$$

Hence, we have

$$\|u\|_{\hat{\rho}_\Phi} \le \left( \frac{K}{a_* - \Lambda} \right)^{1/(\vartheta_- - 1)} =: M \quad \text{whenever } \|u\|_{\hat{\rho}_\Phi} \ge D.$$

Taking $M_1 = \max\{D, 1, M\}$ shows the assertion. The estimate uses only the lower bound $a_0$, and therefore also holds for $h_R$, independently of $R$. □

**Lemma 3.7.** *Under* (H0), (H1) *and* (H5)(ii), *let* $u \in V$ *satisfy the weak formulation of the original or an auxiliary problem, with a principal coefficient bounded below by* $a_0$, *and assume* $\|u\|_{\hat{\rho}_\Phi} \le M$. *A constant* $M_2(M) > 0$, *independent of the upper bound of that coefficient, satisfies* $\|u\|_\infty + \|u\|_{\infty,\Gamma_2} \le M_2(M)$.

**Proof.** We use a Moser iteration, as in Motreanu–Tornatore [36]. Put $u^+ = \max\{u, 0\}$ and $u_\ell = \min\{u^+, \ell\}$ for $\ell > 1$. For $\alpha \ge 0$, take

$$v = u^+ u_\ell^{\alpha\vartheta_-}, \quad z = u^+ u_\ell^{\alpha}.$$

Both functions belong to $V$, since the corresponding scalar truncation maps are Lipschitz and vanish at zero. Their gradients satisfy

$$\nabla v = (1 + \alpha\vartheta_- \mathbf{1}_{\{0<u^+<\ell\}}) u_\ell^{\alpha\vartheta_-} \nabla u^+,$$
$$\nabla z = (1 + \alpha \mathbf{1}_{\{0<u^+<\ell\}}) u_\ell^{\alpha} \nabla u^+.$$

Denote the principal coefficient by $H_u \ge a_0$. Since $|\xi|^{\vartheta_-} \le 1 + |\xi|^{\vartheta(x)}$, testing the weak formulation with $v$ and discarding the non-negative extra gradient terms yields

$$a_0 \int_\Omega u_\ell^{\alpha\vartheta_-} |\nabla u^+|^{\vartheta_-} \,\mathrm{d}x \le a_0 \int_\Omega u_\ell^{\alpha\vartheta_-} \,\mathrm{d}x + \int_\Omega |\eta| u^+ u_\ell^{\alpha\vartheta_-} \,\mathrm{d}x + \int_{\Gamma_2} |\zeta| u^+ u_\ell^{\alpha\vartheta_-} \,\mathrm{d}\sigma.$$

Taking (H5)(ii) into account, apply Young's inequality with conjugate exponents $d/(d-1)$ and $d$ to

$$\left(|\nabla u^+|^{\vartheta_-} u_\ell^{\alpha\vartheta_-}\right)^{(d-1)/d} \left(u^+ u_\ell^{\alpha\vartheta_-/d}\right).$$

After choosing the Young parameter in terms of $a_0$ and $L_M$, we obtain

$$L_M |\nabla u^+|^{\vartheta_- - (d-1)/d} u^+ u_\ell^{\alpha\vartheta_-} \le \frac{a_0}{2} |\nabla u^+|^{\vartheta_-} u_\ell^{\alpha\vartheta_-} + C_M (u^+)^d u_\ell^{\alpha\vartheta_-}.$$

Moreover, $|\nabla z| \le (\alpha+1) u_\ell^\alpha |\nabla u^+|$, $u_\ell^{\alpha\vartheta_-} \le 1 + z^{\vartheta_-}$ and $u^+ u_\ell^{\alpha\vartheta_-} \le 1 + z^{\vartheta_-}$. Absorbing the gradient term, we obtain

$$\|\nabla z\|_{\vartheta_-}^{\vartheta_-} \le C_M (\alpha+1)^{\vartheta_-} \left[1 + \int_\Omega \left(1 + \omega_{u,1} + |u|^{d-\vartheta_-}\right) z^{\vartheta_-} \,\mathrm{d}x + \int_{\Gamma_2} \left(\omega_{u,2} + |u|^{e-\vartheta_-}\right) z^{\vartheta_-} \,\mathrm{d}\sigma\right]. \tag{3.8}$$

Here and below $C_M \ge 1$ may be enlarged. The bounds on $\|\omega_{u,1}\|_{\gamma_1}$ and $\|\omega_{u,2}\|_{\gamma_2,\Gamma_2}$ in (H5)(ii) ensure that $C_M$ is independent of the choice of $u$ with $\|u\| \le M$, of $\alpha$

and $\ell$, and of the upper bound of $H_u$. Set

$$r_\Omega = \max\left\{\vartheta_-, \vartheta_-\gamma_1', \frac{\vartheta_-(\vartheta_-)^*}{(\vartheta_-)^* - d + \vartheta_-}\right\} < (\vartheta_-)^*,$$

$$r_\Gamma = \max\left\{\vartheta_-, \vartheta_-\gamma_2', \frac{\vartheta_-(\vartheta_-)_*}{(\vartheta_-)_* - e + \vartheta_-}\right\} < (\vartheta_-)_*.$$

The strict inequalities follow from (H5)(ii). Hölder's inequality yields the interior estimates

$$\int_\Omega \omega_{u,1} z^{\vartheta_-}\,\mathrm{d}x \le \|\omega_{u,1}\|_{\gamma_1}\|z\|^{\vartheta_-}_{\vartheta_-\gamma_1'},$$

$$\int_\Omega |u|^{d-\vartheta_-} z^{\vartheta_-}\,\mathrm{d}x \le \|u\|^{d-\vartheta_-}_{(\vartheta_-)^*}\|z\|^{\vartheta_-}_{\frac{\vartheta_-(\vartheta_-)^*}{(\vartheta_-)^*-d+\vartheta_-}} \quad \text{if } d > \vartheta_-.$$

The second estimate uses conjugate exponents $(\vartheta_-)^*/(d-\vartheta_-)$ and $(\vartheta_-)^*/((\vartheta_-)^* - d+\vartheta_-)$. If $d = \vartheta_-$, the weight is 1, and the integral is $\|z\|^{\vartheta_-}_{\vartheta_-}$. The same calculations on $\Gamma_2$ use $\gamma_2$, $e$ and $(\vartheta_-)_*$. The norm bound on $u$ and the Sobolev and trace embeddings therefore turn (3.8) into

$$\|\nabla z\|^{\vartheta_-}_{\vartheta_-} \le C_M(\alpha+1)^{\vartheta_-}\left(1 + \|z\|^{\vartheta_-}_{r_\Omega} + \|z\|^{\vartheta_-}_{r_\Gamma,\Gamma_2}\right).$$

Choose $0 < \theta_\Omega, \theta_\Gamma < 1$ such that

$$\frac{1}{r_\Omega} = \frac{1-\theta_\Omega}{\vartheta_-} + \frac{\theta_\Omega}{(\vartheta_-)^*},$$

$$\frac{1}{r_\Gamma} = \frac{1-\theta_\Gamma}{\vartheta_-} + \frac{\theta_\Gamma}{(\vartheta_-)_*}.$$

Interpolation and the Sobolev and trace inequalities on the partial zero-trace space give

$$\|z\|^{\vartheta_-}_{r_\Omega} \le C\|z\|^{\vartheta_-(1-\theta_\Omega)}_{\vartheta_-}\|\nabla z\|^{\vartheta_-\theta_\Omega}_{\vartheta_-},$$

$$\|z\|^{\vartheta_-}_{r_\Gamma,\Gamma_2} \le C\|z\|^{\vartheta_-(1-\theta_\Gamma)}_{\vartheta_-,\Gamma_2}\|\nabla z\|^{\vartheta_-\theta_\Gamma}_{\vartheta_-}.$$

Young's inequality with exponents $1/\theta_\Omega$, $1/(1-\theta_\Omega)$, and their boundary counterparts implies that, for every $0 < \varepsilon \le 1$,

$$\|z\|^{\vartheta_-}_{r_\Omega} + \|z\|^{\vartheta_-}_{r_\Gamma,\Gamma_2} \le \varepsilon\|\nabla z\|^{\vartheta_-}_{\vartheta_-} + C\varepsilon^{-b}\left(\|z\|^{\vartheta_-}_{\vartheta_-} + \|z\|^{\vartheta_-}_{\vartheta_-,\Gamma_2}\right),$$

where $b = \max\{\theta_\Omega/(1-\theta_\Omega), \theta_\Gamma/(1-\theta_\Gamma)\} > 0$. Take $\varepsilon = [2C_M(\alpha+1)^{\vartheta_-}]^{-1}$. The coefficient of the gradient term on the right is then $1/2$, so absorption yields, with $\beta = \vartheta_-(1+b)$,

$$\|\nabla z\|^{\vartheta_-}_{\vartheta_-} \le C_M(\alpha+1)^{\beta}\left(1 + \|z\|^{\vartheta_-}_{\vartheta_-} + \|z\|^{\vartheta_-}_{\vartheta_-,\Gamma_2}\right).$$

For $s \ge 1$, write

$$\|w\|_{s,\Omega\cup\Gamma_2} := \left(\int_\Omega |w|^s\,\mathrm{d}x + \int_{\Gamma_2} |w|^s\,\mathrm{d}\sigma\right)^{1/s}.$$

Since $(\vartheta_-)_* < (\vartheta_-)^*$, the same Sobolev and trace inequalities control $\|z\|_{(\vartheta_-)_*,\Omega\cup\Gamma_2}$ by $\|\nabla z\|_{\vartheta_-}$. Letting $\ell \to \infty$ and using Fatou's lemma therefore yields

$$\begin{aligned}&\max\left\{1, \|u^+\|_{(\alpha+1)(\vartheta_-)_*,\Omega\cup\Gamma_2}\right\}\\ &\quad\le \left[C_M(\alpha+1)^\beta\right]^{1/[\vartheta_-(\alpha+1)]} \max\left\{1, \|u^+\|_{\vartheta_-(\alpha+1),\Omega\cup\Gamma_2}\right\},\end{aligned} \tag{3.9}$$

whenever the norm on the right is finite. Set $\chi = (\vartheta_-)_*/\vartheta_- > 1$ and take $\alpha_j + 1 = \chi^j$ for $j = 0, 1, \ldots$. The initial norm is bounded by $C_0 M$, and repeated application of (3.9) bounds all norms with exponents $\vartheta_-\chi^j$ by $P_M \max\{1, C_0 M\}$, where

$$\begin{aligned}\log P_M &= \sum_{j=0}^{\infty} \frac{\log C_M + \beta j \log\chi}{\vartheta_-\chi^j}\\ &= \frac{\chi \log C_M}{\vartheta_-(\chi-1)} + \frac{\beta\chi\log\chi}{\vartheta_-(\chi-1)^2} < \infty.\end{aligned}$$

Since $\mathrm{d}x + \mathrm{d}\sigma$ on $\Omega\cup\Gamma_2$ is a finite measure, letting $j \to \infty$ proves

$$\max\{\|u^+\|_\infty, \|u^+\|_{\infty,\Gamma_2}\} \le P_M \max\{1, C_0 M\}.$$

Testing with $v = -u^- \min\{u^-, \ell\}^{\alpha\vartheta_-}$, where $u^- = \max\{-u, 0\}$, leads to the same estimates for $u^-$. This proves the asserted bound $M_2(M)$, independently of the truncation parameters. □

Since the coefficient $h$ may be unbounded, the standard operator theory cannot be applied directly to problem (P1). We therefore replace it with the truncated function $h_R\colon \mathbb{R}\times[0,\infty)\times[0,\infty)\to\mathbb{R}$ defined by

$$h_R(t,s,l) = \begin{cases} h(R_1,R_2,R_3) & \text{if } t > R_1, s > R_2, l > R_3,\\ h(R_1,R_2,l) & \text{if } t > R_1, s > R_2, 0 \le l \le R_3,\\ h(R_1,s,R_3) & \text{if } t > R_1, 0 \le s \le R_2, l > R_3,\\ h(R_1,s,l) & \text{if } t > R_1, 0 \le s \le R_2, 0 \le l \le R_3,\\ h(t,R_2,R_3) & \text{if } -R_1 \le t \le R_1, s > R_2, l > R_3,\\ h(t,R_2,l) & \text{if } -R_1 \le t \le R_1, s > R_2, 0 \le l \le R_3,\\ h(t,s,R_3) & \text{if } -R_1 \le t \le R_1, 0 \le s \le R_2, l > R_3,\\ h(t,s,l) & \text{if } -R_1 \le t \le R_1, 0 \le s \le R_2, 0 \le l \le R_3,\\ h(-R_1,R_2,R_3) & \text{if } t < -R_1, s > R_2, l > R_3,\\ h(-R_1,R_2,l) & \text{if } t < -R_1, s > R_2, 0 \le l \le R_3,\\ h(-R_1,s,R_3) & \text{if } t < -R_1, 0 \le s \le R_2, l > R_3,\\ h(-R_1,s,l) & \text{if } t < -R_1, 0 \le s \le R_2, 0 \le l \le R_3,\end{cases} \tag{3.10}$$

for parameters $R_1, R_2, R_3 > 0$. Clearly, $h_R$ is bounded and continuous, with $a_0 \leq h_R \leq M_R$ for some $M_R > 0$. The corresponding auxiliary inclusion reads

$$\begin{aligned} -\operatorname{div}\left[h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\right] &\in f(x, u, Tu, \nabla u, \nabla(Tu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\cdot\nu &\in f_{\Gamma_2}(x, u, Tu) \quad \text{on } \Gamma_2. \end{aligned} \tag{P2}$$

The operator associated with (P2), $\mathcal{J}_R\colon V \to V^*$, acts through

$$\begin{aligned} &\langle \mathcal{J}_R(u), v\rangle \\ &\quad = \int_\Omega h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\vartheta(x)-2}\nabla u + \mu(x)|\nabla u|^{\kappa(x)-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x, \end{aligned}$$

for all $u, v \in V$.

**Definition 3.8.** We say that a function $\tilde{u} \in V$ is a weak solution of problem (P2), if $h_R(\tilde{u}, \|\tilde{u}\|_{k(\cdot)}, \|\tilde{u}\|_{\hat{\rho}_\Phi})(|\nabla\tilde{u}|^{\vartheta(x)-2}\nabla\tilde{u} + \mu(x)|\nabla\tilde{u}|^{\kappa(x)-2}\nabla\tilde{u})\cdot\nabla v \in L^1(\Omega)$ and there exist fixed measurable selections $\eta \in f(x, \tilde{u}, T\tilde{u}, \nabla\tilde{u}, \nabla(T\tilde{u}))$ as well as $\zeta \in f_{\Gamma_2}(x, \tilde{u}, T\tilde{u})$ with $\eta v \in L^1(\Omega)$ and $\zeta v \in L^1(\Gamma_2)$, such that

$$\begin{aligned} &\int_\Omega h_R(\tilde{u}, \|\tilde{u}\|_{k(\cdot)}, \|\tilde{u}\|_{\hat{\rho}_\Phi}) \left(|\nabla\tilde{u}|^{\vartheta(x)-2}\nabla\tilde{u} + \mu(x)|\nabla\tilde{u}|^{\kappa(x)-2}\nabla\tilde{u}\right)\cdot\nabla v \,\mathrm{d}x \\ &\quad = \int_\Omega \eta v \,\mathrm{d}x - \int_{\Gamma_2} \zeta v \,\mathrm{d}\sigma, \end{aligned}$$

for all $v \in V$. The pointwise memberships hold almost everywhere on the corresponding sets.

The operator $\mathcal{J}_R$ fulfills the following useful properties.

**Proposition 3.9.** *Under* (H0) *and* (H1), $\mathcal{J}_R\colon V \to V^*$ *is a bounded and continuous operator. It also has the* $(\mathrm{S}_+)$*-property, i.e.*

$$u_n \rightharpoonup u \text{ in } V \quad \text{and} \quad \limsup_{n\to\infty}\langle \mathcal{J}_R(u_n), u_n - u\rangle \leq 0 \;\Rightarrow\; u_n \to u \text{ in } V.$$

**Proof.** We use the basic double phase operator $\mathcal{A}\colon V \to V^*$, namely

$$\begin{aligned} \langle \mathcal{A}(u), v\rangle &= \int_\Omega \partial_t\Psi(x, |\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla v \,\mathrm{d}x \\ &= \int_\Omega \left(|\nabla u|^{\vartheta(x)-2}\nabla u + \mu(x)|\nabla u|^{\kappa(x)-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x, \quad u, v \in V. \end{aligned}$$

The double phase estimates of Crespo-Blanco–Gasiński–Harjulehto–Winkert [16] show that $\mathcal{A}$ is bounded, continuous and of type $(\mathrm{S}_+)$. Their gradient estimates apply on $V$ by the norm equivalence in Remark 2.9, and remain valid when the two exponents coincide or $\mu$ vanishes. The strict monotonicity of the first power flux also implies

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle > 0 \quad \text{for } u \neq v.$$

Indeed, equality would imply $\nabla u = \nabla v$ almost everywhere. Their difference is constant, and its zero trace on $\Gamma_1$ forces that constant to vanish.

Continuity of $h_R$ provides the finite constant

$$M_R = \max_{(t,s,l)\in[-R_1,R_1]\times[0,R_2]\times[0,R_3]} h(t,s,l).$$

For $H_u = h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})$, we have $a_0 \le H_u \le M_R$. The flux map $\xi \mapsto \mathcal{K}(\xi)$ is continuous from $L^\Phi(\Omega;\mathbb{R}^N)$ to $L^{\Phi^*}(\Omega;\mathbb{R}^N)$ and is bounded on bounded sets. Hölder's inequality gives

$$|\langle \mathcal{J}_R(u), v\rangle| \le 2M_R \|\mathcal{K}(\nabla u)\|_{\Phi^*} \|\nabla v\|_\Phi.$$

Together with boundedness of $\mathcal{A}$, this estimate bounds $\mathcal{J}_R$.

For continuity, assume $u_n \to u$ in $V$, and write $H_n = h_R(u_n, \|u_n\|_{k(\cdot)}, \|u_n\|_{\hat{\rho}_\Phi})$ and $H = H_u$. The subcritical embedding in Proposition 2.7(ii) and convergence in $V$ imply $\|u_n\|_{k(\cdot)} \to \|u\|_{k(\cdot)}$ and $\|u_n\|_{\hat{\rho}_\Phi} \to \|u\|_{\hat{\rho}_\Phi}$. Every subsequence has a further subsequence with $u_n(x) \to u(x)$ almost everywhere, and continuity of $h_R$ implies $H_n(x) \to H(x)$ there. The power bounds in (H0) imply the $\Delta_2$ condition for $\Phi^*$, and the fixed flux belongs to its Orlicz heart. Since $|H_n - H| \le 2M_R$, for every $\lambda > 0$ the integrand below tends to zero almost everywhere and is dominated by the integrable function $\Phi^*(x, 2M_R|\mathcal{K}(\nabla u)|/\lambda)$. Thus

$$\rho_{\Phi^*}\left(\frac{(H_n - H)\mathcal{K}(\nabla u)}{\lambda}\right) = \int_\Omega \Phi^*\left(x, \frac{|H_n - H|\,|\mathcal{K}(\nabla u)|}{\lambda}\right) \mathrm{d}x \to 0.$$

For each $\lambda > 0$ this modular is eventually at most 1, so the Luxemburg norm is eventually at most $\lambda$. Since $\lambda$ is arbitrary, we obtain

$$\|(H_n - H)\mathcal{K}(\nabla u)\|_{\Phi^*} \to 0. \tag{3.11}$$

The subsequence principle extends this convergence for the whole sequence. For $v \in V$, Hölder's inequality now yields the estimate

$$\begin{aligned}
&|\langle \mathcal{J}_R(u_n) - \mathcal{J}_R(u), v\rangle| \\
&\le \left|\int_\Omega H_n[\mathcal{K}(\nabla u_n) - \mathcal{K}(\nabla u)] \cdot \nabla v \,\mathrm{d}x\right| + \left|\int_\Omega (H_n - H)\mathcal{K}(\nabla u)\cdot \nabla v \,\mathrm{d}x\right| \\
&\le 2\left(M_R\|\mathcal{K}(\nabla u_n) - \mathcal{K}(\nabla u)\|_{\Phi^*} + \|(H_n - H)\mathcal{K}(\nabla u)\|_{\Phi^*}\right)\|\nabla v\|_\Phi.
\end{aligned}$$

Flux continuity and (3.11) show that $\mathcal{J}_R(u_n) \to \mathcal{J}_R(u)$ in $V^*$.

To verify the $(S_+)$-property, assume $u_n \rightharpoonup u$ in $V$ and the stated limsup condition. The compact embedding $V \hookrightarrow\hookrightarrow L^\Phi(\Omega)$ in Proposition 2.8(i) implies $u_n \to u$ in $L^\Phi(\Omega)$. From any subsequence we may extract a further subsequence such that

$$u_n(x) \to u(x) \text{ a.e.}, \quad \|u_n\|_{k(\cdot)} \to s, \quad \|u_n\|_{\hat{\rho}_\Phi} \to l.$$

Set $\overline{H}(x) = h_R(u(x), s, l)$. As above,

$$\|(H_n - \overline{H})\mathcal{K}(\nabla u)\|_{\Phi^*} \to 0.$$

Weak convergence of the gradients and Hölder's inequality therefore imply

$$\begin{aligned} I_n :&= \int_\Omega H_n \mathcal{K}(\nabla u)\cdot\nabla(u_n-u)\,\mathrm{d}x \\ &= \int_\Omega \overline{H}\mathcal{K}(\nabla u)\cdot\nabla(u_n-u)\,\mathrm{d}x + \int_\Omega (H_n-\overline{H})\mathcal{K}(\nabla u)\cdot\nabla(u_n-u)\,\mathrm{d}x \to 0. \end{aligned}$$

Pointwise monotonicity of the flux, together with $H_n \geq a_0$, yields

$$\begin{aligned} 0 &\leq a_0\langle \mathcal{A}(u_n)-\mathcal{A}(u), u_n-u\rangle \\ &\leq \int_\Omega H_n[\mathcal{K}(\nabla u_n)-\mathcal{K}(\nabla u)]\cdot\nabla(u_n-u)\,\mathrm{d}x \\ &= \langle \mathcal{J}_R(u_n), u_n-u\rangle - I_n. \end{aligned}$$

It follows that $\limsup_n \langle \mathcal{A}(u_n), u_n-u\rangle \leq 0$, since $\langle \mathcal{A}(u), u_n-u\rangle \to 0$. The $(\mathrm{S}_+)$-property of $\mathcal{A}$ yields strong convergence in $V$ along this further subsequence. Applying the subsequence principle to the arbitrary initial subsequence proves $u_n \to u$ in $V$. □

Define the selection sets

$$\begin{aligned} \mathcal{S}_f(u) &:= \{\eta \in L^{\vartheta_-{}'}(\Omega)\colon \eta(x) \in f(x,u,Tu,\nabla u,\nabla(Tu)) \text{ a.e.}\}, \\ \mathcal{S}_\Gamma(u) &:= \{\zeta \in L^{\vartheta_-{}'}(\Gamma_2)\colon \zeta(x) \in f_{\Gamma_2}(x,u,Tu) \text{ a.e.}\}. \end{aligned}$$

Each selection defines a functional on $V$ by integration against the test function. The corresponding multivalued operators $B_f, B_{f_{\Gamma_2}} : V \to 2^{V^*}$ are defined by

$$\begin{aligned} B_f(u) &= \left\{ v \mapsto \int_\Omega \eta v\,\mathrm{d}x \colon \eta \in \mathcal{S}_f(u)\right\}, \\ B_{f_{\Gamma_2}}(u) &= \left\{ v \mapsto \int_{\Gamma_2} \zeta v\,\mathrm{d}\sigma \colon \zeta \in \mathcal{S}_\Gamma(u)\right\}. \end{aligned}$$

We write $B_\eta$ and $B_\zeta$ for the corresponding linear functionals. Hölder's inequality and the embeddings fixed in Remark 2.9 give

$$\begin{aligned} |B_\eta(v)| &\leq C_{\vartheta_-}\|\eta\|_{\vartheta'_-}\|v\|, \\ |B_\zeta(v)| &\leq C_{\vartheta_-,\Gamma_2}\|\zeta\|_{\vartheta'_-,\Gamma_2}\|v\|. \end{aligned}$$

Thus these functionals belong to $V^*$, and $B_f(u)$ and $B_{f_{\Gamma_2}}(u)$ collect all functionals obtained from the selections at $u$.

By (H2)–(H4), the scalar values are nonempty and compact almost everywhere. Graph measurability and the measurable selection theorem ensure that $\mathcal{S}_f(u)$ and $\mathcal{S}_\Gamma(u)$ are nonempty. Their $L^{\vartheta'_-}$ integrability follows from (H3)–(H4) for fixed $u$.

Convexity of the pointwise values implies convexity of the selection sets. If a sequence of selections converges in $L^{\vartheta'_-}$, an almost everywhere convergent subsequence and pointwise closedness show that its limit remains a selection. The sets are

therefore norm closed, convex and bounded, hence weakly compact in the reflexive spaces $L^{\vartheta'_-}(\Omega)$ and $L^{\vartheta'_-}(\Gamma_2)$.

On the ball $\|u\| \leq M$, (H5)(ii) and the continuous embeddings into $L^d(\Omega)$ and $L^e(\Gamma_2)$ imply

$$\|\eta\|_{\min\{\gamma_1,d'\}} + \|\zeta\|_{\min\{\gamma_2,e'\},\Gamma_2} \leq C_M, \quad \eta \in \mathcal{S}_f(u), \quad \zeta \in \mathcal{S}_\Gamma(u).$$

Indeed, $(d-1)d' = d$ and $\vartheta_-(d-1)d'/d = \vartheta_-$, so

$$\big\||u|^{d-1}\big\|_{d'} = \|u\|_d^{d-1}, \quad \Big\||\nabla u|^{\vartheta_-(d-1)/d}\Big\|_{d'} = \|\nabla u\|_{\vartheta_-}^{\vartheta_-(d-1)/d}.$$

Similarly, $\||u|^{e-1}\|_{e',\Gamma_2} = \|u\|_{e,\Gamma_2}^{e-1}$. The growth bounds and the finite-measure embeddings therefore yield the displayed uniform estimate. Since $d, \gamma_1' < (\vartheta_-)^*$ and $e, \gamma_2' < (\vartheta_-)_*$, the associated dual embeddings

$$V \hookrightarrow\hookrightarrow L^{\max\{d,\gamma_1'\}}(\Omega), \quad V \hookrightarrow\hookrightarrow L^{\max\{e,\gamma_2'\}}(\Gamma_2)$$

are compact.

**Proposition 3.10.** *Under* (H0)–(H5), $\mathcal{F}_R := \mathcal{J}_R - B_f + B_{f_{\Gamma_2}} : V \to 2^{V^*}$ *is bounded, coercive and pseudomonotone.*

**Proof.** We first verify the closed-graph property of the selection maps. Suppose $u_n \to u$ in $V$, $\eta_n \in \mathcal{S}_f(u_n)$, $\zeta_n \in \mathcal{S}_\Gamma(u_n)$, and

$$\eta_n \rightharpoonup \eta \quad \text{in } L^{\min\{\gamma_1,d'\}}(\Omega), \quad \zeta_n \rightharpoonup \zeta \quad \text{in } L^{\min\{\gamma_2,e'\}}(\Gamma_2).$$

Continuity of $T$ implies $Tu_n \to Tu$ in $W^{1,\vartheta_-}(\Omega)$. After extraction, all interior arguments converge almost everywhere, as do the traces of $u_n$ and $Tu_n$ on $\Gamma_2$. Set $F_u(x) = f(x, u, Tu, \nabla u, \nabla(Tu))$. Upper semicontinuity and compactness of $F_u(x)$ imply

$$\operatorname{dist}(\eta_n(x), F_u(x)) \to 0 \quad \text{for a.e. } x \in \Omega.$$

By Mazur's lemma, there are finite convex combinations $\widehat{\eta}_n = \sum_{j=n}^{k_n} \alpha_{n,j}\eta_j$, where $\alpha_{n,j} \geq 0$ and $\sum_{j=n}^{k_n} \alpha_{n,j} = 1$, such that $\widehat{\eta}_n \to \eta$ in $L^{\min\{\gamma_1,d'\}}(\Omega)$. A further subsequence converges almost everywhere. Convexity of $F_u(x)$ yields

$$\operatorname{dist}(\widehat{\eta}_n(x), F_u(x)) \leq \sup_{j \geq n} \operatorname{dist}(\eta_j(x), F_u(x)) \to 0.$$

Pointwise closedness therefore implies $\eta(x) \in F_u(x)$ almost everywhere. The growth bound (H3) at the limiting input shows that $\eta \in L^{\vartheta'_-}(\Omega)$, so $\eta \in \mathcal{S}_f(u)$. The same argument on $\Gamma_2$, using (H4), proves $\zeta \in \mathcal{S}_\Gamma(u)$.

The preceding selection bounds and the bound for $\mathcal{J}_R$ imply boundedness. Each value is nonempty, convex and weakly compact, hence closed. For $u^* = \mathcal{J}_R(u) - B_\eta + B_\zeta \in \mathcal{F}_R(u)$ and $\|u\| \geq D$, the calculation in Lemma 3.6 yields

$$\langle u^*, u\rangle \geq (a_* - \Lambda)\|u\|^{\vartheta_-} - K\|u\|.$$

Dividing by $\|u\|$ shows that the lower bound tends to $+\infty$ as $\|u\| \to \infty$. For $\|u\| < D$, boundedness on this ball gives $\langle u^*, u\rangle \geq -C_D\|u\|$. Combining the two estimates verifies the coercivity condition in Theorem 2.11 with $v_0 = 0$.

Let $u_n \rightharpoonup u$, choose $u_n^* = \mathcal{J}_R(u_n) - B_{\eta_n} + B_{\zeta_n} \in \mathcal{F}_R(u_n)$, and suppose $\limsup_n \langle u_n^*, u_n - u\rangle \leq 0$. If $Q = \max\{1, \sup_n \|u_n\|_{\hat{\rho}_\Phi}\} < \infty$, (H5)(ii) and Hölder's inequality give a constant $C_Q > 0$ such that

$$\int_\Omega |\eta_n||u_n - u|\,\mathrm{d}x \leq C_Q\left(\|u_n - u\|_d + \|u_n - u\|_{\gamma_1'}\right),$$

$$\int_{\Gamma_2} |\zeta_n||u_n - u|\,\mathrm{d}\sigma \leq C_Q\left(\|u_n - u\|_{e,\Gamma_2} + \|u_n - u\|_{\gamma_2',\Gamma_2}\right).$$

The compact embeddings just established give

$$\int_\Omega \eta_n(u_n - u)\,\mathrm{d}x - \int_{\Gamma_2} \zeta_n(u_n - u)\,\mathrm{d}\sigma \to 0.$$

Consequently $\limsup_n \langle \mathcal{J}_R(u_n), u_n - u\rangle \leq 0$, and Proposition 3.9 implies $u_n \to u$ in $V$. Fix $w \in V$ and pass to a subsequence attaining $\liminf_n \langle u_n^*, u_n - w\rangle$. Extract selections $\eta_n \rightharpoonup \eta$ in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $\zeta_n \rightharpoonup \zeta$ in $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. The closed-graph property just proved implies $\eta \in \mathcal{S}_f(u)$, $\zeta \in \mathcal{S}_\Gamma(u)$. By continuity of $\mathcal{J}_R$ and strong convergence of the inputs,

$$\liminf_n \langle u_n^*, u_n - w\rangle = \langle \mathcal{J}_R(u) - B_\eta + B_\zeta, u - w\rangle.$$

This verifies condition (c) of pseudomonotonicity. Condition (a) was established above. To verify condition (b), let $E \subset V$ be a finite-dimensional subspace, $u \in E$, and let $\mathcal{O} \subset V^*$ be weakly open with $\mathcal{F}_R(u) \subset \mathcal{O}$. If $\mathcal{F}_R|_E$ were not upper semicontinuous at $u$, there would exist sequences satisfying

$$u_n \in E, \quad \|u_n - u\| < \frac{1}{n}, \quad u_n^* \in \mathcal{F}_R(u_n) \setminus \mathcal{O}.$$

The inputs lie in a norm ball, so boundedness of $\mathcal{F}_R$ and reflexivity of $V^*$ imply, after extraction, $u_n^* \rightharpoonup w^*$ in $V^*$. Write $u_n^* = \mathcal{J}_R(u_n) - B_{\eta_n} + B_{\zeta_n}$. The uniform selection bounds yield a further subsequence with $\eta_n \rightharpoonup \eta$ in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $\zeta_n \rightharpoonup \zeta$ in $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. Since $u_n \to u$ in $V$, the closed-graph property implies $\eta \in \mathcal{S}_f(u)$ and $\zeta \in \mathcal{S}_\Gamma(u)$. Continuity of $\mathcal{J}_R$ and the dual embeddings established above show, for every $v \in V$, that

$$\langle w^*, v\rangle = \lim_{n\to\infty} \langle u_n^*, v\rangle = \langle \mathcal{J}_R(u), v\rangle - \int_\Omega \eta v\,\mathrm{d}x + \int_{\Gamma_2} \zeta v\,\mathrm{d}\sigma.$$

Thus $w^* \in \mathcal{F}_R(u) \subset \mathcal{O}$. Weak convergence and weak openness of $\mathcal{O}$ imply $u_n^* \in \mathcal{O}$ for all sufficiently large $n$, contradicting the choice of $u_n^*$. Consequently, $\mathcal{F}_R|_E$ is upper semicontinuous from $E$ with its norm topology to $V^*$ with its weak topology. This verifies condition (b). □

**Theorem 3.11.** *Under* (H0)–(H5), (P2) *has a weak solution* $u \in V$, *and every such solution satisfies* $\|u\| \leq M_1$ *and* $\|u\|_\infty + \|u\|_{\infty,\Gamma_2} \leq M_2(M_1)$, *independently of* $R$.

**Proof.** Proposition 3.10 and Theorem 2.11 yield a solution of $0 \in \mathcal{F}_R(u)$. The two estimates follow from Lemmas 3.6 and 3.7, applied to the auxiliary coefficient $h_R \geq a_0$. □

We can now complete the proof of Theorem 3.3.

**Proof of Theorem 3.3.** Choose $R_3 \geq M_1$, $R_2 \geq C_k M_1$ and $R_1 \geq M_2(M_1)$, where $C_k$ is the continuous embedding constant into $L^{k(\cdot)}(\Omega)$. Solve (P2) for these parameters. The estimates in Theorem 3.11 imply that all three truncations in (3.10) are inactive at this solution, so $h_R = h$ a.e. in its weak formulation. Thus it solves (P1), with the asserted bounds. □

### 3.2. *Constant exponents*

We turn to the proof of Theorem 3.5.

**Lemma 3.12.** *Suppose $\vartheta(x) \equiv \tilde{\vartheta} > 1$, $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and assume* (H0)–(H1) *together with* (H′3)–(H′5). *Every weak solution $u \in V$ of* (P1), *or of its auxiliary problem with $h_R$, obeys $\|u\|_{\hat{\rho}_\Phi} \leq \widetilde{M}_1$ for a constant $\widetilde{M}_1 \geq 1$ depending on the problem data and the lower bound $a_0$ of $h$.*

**Proof.** Take a weak solution $u \in V$ of (P1). By choosing $v = u$ in (3.3) we arrive at

$$\int_\Omega h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left( |\nabla u|^{\tilde{\vartheta}} + \mu(x) |\nabla u|^{\tilde{\kappa}} \right) \mathrm{d}x = \int_\Omega \eta u \, \mathrm{d}x - \int_{\Gamma_2} \zeta u \, \mathrm{d}\sigma. \quad (3.12)$$

If $\|u\|_{\hat{\rho}_\Phi} \geq D$, hypotheses (H1), Proposition 2.6 and Remark 2.9 give

$$\int_\Omega h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left( |\nabla u|^{\tilde{\vartheta}} + \mu(x) |\nabla u|^{\tilde{\kappa}} \right) \mathrm{d}x \geq a_* \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}}. \quad (3.13)$$

By (H′5)(i), (3.12) and (3.13), we obtain

$$(a_* - \widetilde{\Lambda}) \|u\|^{\tilde{\vartheta}} \leq \widetilde{K} \|u\|.$$

Hence,

$$\tilde{M} = \left( \frac{\widetilde{K}}{a_* - \widetilde{\Lambda}} \right)^{1/(\tilde{\vartheta}-1)}.$$

From this we conclude that $\|u\|_{\hat{\rho}_\Phi} \leq \tilde{M}$ whenever $\|u\|_{\hat{\rho}_\Phi} \geq D$. Setting $\widetilde{M}_1 = \max\{D, 1, \tilde{M}\}$ completes the proof. As in Lemma 3.6, this estimate is also valid for the auxiliary coefficient $h_R$, independently of its truncation parameters. □

**Lemma 3.13.** *Under the assumptions of Lemma* 3.12, *every such solution satisfies $\|u\|_\infty + \|u\|_{\infty,\Gamma_2} \leq M_2(\widetilde{M}_1)$, independently of the truncation parameters.*

**Proof.** The assertion follows from Lemma 3.7 with $M = \widetilde{M}_1$, since (H′5)(ii) supplies (H5)(ii) and its proof uses the $W^{1,\tilde{\vartheta}}$ embeddings and the lower bound $a_0$. □

**Proposition 3.14.** *Under* (H0)–(H2) *and* (H′3)–(H′5), $\mathcal{F}_R\colon V \to 2^{V^*}$ *is bounded, coercive and pseudomonotone.*

**Proof.** The selection sets are defined as above, now in $L^{d_0'}(\Omega)$ and $L^{e_0'}(\Gamma_2)$, where $d_0 = \max\{\tilde{\vartheta}, \tilde{r}_1\}$ and $e_0 = \max\{\tilde{\vartheta}, \tilde{r}_2\}$. For fixed $u$, the growth estimates imply bounded selections in these spaces: the $u$ and $\nabla u$ terms have $L^{\tilde{\vartheta}'}$ bounds. The $Tu$ and $\nabla(Tu)$ terms have $L^{\tilde{r}_1'}$ bounds, and the boundary terms have $L^{\tilde{\vartheta}'}$ or $L^{\tilde{r}_2'}$ bounds. On finite-measure sets these bounds imply the asserted $L^{d_0'}$, $L^{e_0'}$ bounds. On each norm ball, (H′5)(ii) also bounds the selections uniformly in $L^{\min\{\gamma_1,d'\}}(\Omega)$ and $L^{\min\{\gamma_2,e'\}}(\Gamma_2)$. The corresponding dual embeddings and boundedness of $\mathcal{J}_R$ therefore imply boundedness of $\mathcal{F}_R$. As above, (H2) and these growth bounds ensure nonempty, convex and weakly compact values, and the Mazur argument in Proposition 3.10 establishes their closed graph for strong convergence of inputs in $V$ and weak convergence of selections in $L^{\min\{\gamma_1,d'\}}(\Omega)$ and $L^{\min\{\gamma_2,e'\}}(\Gamma_2)$. The estimate in Lemma 3.12 implies $\langle u^*, u\rangle \geq (a_* - \widetilde{\Lambda})\|u\|^{\tilde{\vartheta}} - \widetilde{K}\|u\|$ for $u^* \in \mathcal{F}_R(u)$ and $\|u\| \geq D$, proving coercivity. For a weakly convergent input sequence in the pseudomonotonicity condition, (H′5)(ii) yields uniform bounds in $L^{\min\{\gamma_1,d'\}}(\Omega)$ and $L^{\min\{\gamma_2,e'\}}(\Gamma_2)$. The compact embeddings into $L^{\max\{d,\gamma_1'\}}(\Omega)$ and $L^{\max\{e,\gamma_2'\}}(\Gamma_2)$ imply that both source pairings tend to zero, exactly as in Proposition 3.10. Proposition 3.9 then implies strong convergence of the inputs in $V$. Passing to weak limits of the selections verifies condition (c) as in Proposition 3.10. For condition (b), apply the weak-open-set contradiction from that proposition to a norm-convergent sequence in a finite-dimensional input space. The same uniform bounds and closed-graph argument identify the weak limit with an element of the value at the limiting input. □

**Proof of Theorem 3.5.** Proposition 3.14 and Theorem 2.11 give an auxiliary solution for every $R$. Lemmas 3.12 and 3.13 provide bounds independent of $R$. Choose $R_3 \geq \widetilde{M}_1$, $R_2 \geq C_k\widetilde{M}_1$ and $R_1 \geq M_2(\widetilde{M}_1)$. All truncations are then inactive, and the auxiliary solution solves the original problem. This completes the constant-exponent argument. □

## 4. Applications

Following Motreanu–Tornatore [36], we examine several choices of the intrinsic operator $T\colon V \to W^{1,\vartheta-}(\Omega)$ and their applications. We verify the sufficient norm conditions in Remarks 3.1 and 3.4 for each choice. We also verify (H5)(ii) from the concrete assumptions in each application. In the sufficient conditions below, $\gamma_1, \gamma_2$ satisfy the inequalities in (H5)(ii). All domain and trace embedding constants are those fixed in Remark 2.9, unless the full space $W^{1,\vartheta-}(\Omega)$ is specified.

Choosing the convolution operator $Tu = (\rho * Eu)|_\Omega$, where $\rho \in L^1(\mathbb{R}^N)$ and $E\colon W^{1,\vartheta_-}(\Omega) \to W^{1,\vartheta_-}(\mathbb{R}^N)$ is a bounded linear extension operator, gives the inclusion

$$\begin{aligned} -\operatorname{div}\ \big[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\big] &\in f(x, u, \rho * Eu, \nabla u, \nabla(\rho * Eu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, \rho * Eu) \quad \text{on } \Gamma_2. \end{aligned} \tag{4.1}$$

Fix $E_0 > 0$ so that $\|Eu\|_{W^{1,\vartheta_-}(\mathbb{R}^N)} \le E_0\|u\|_{\hat{\rho}_\Phi}$, and put $L = E_0\|\rho\|_{L^1(\mathbb{R}^N)}$. For the output of this operator, let $S_r, S_{r,\Gamma_2}$ denote the embedding constants of the full space $W^{1,\vartheta_-}(\Omega)$ into $L^r(\Omega)$ and $L^r(\Gamma_2)$.

**Theorem 4.1.** *Assume* (H0)–(H4) *and either* $\vartheta_- > N - 1$, *or* $\rho \in L^{\vartheta'_-}(\mathbb{R}^N)$, $\sigma_1 \in L^{\gamma_1}(\Omega)$ *and* $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$. *Suppose also that*

$$\begin{aligned} &a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 L^{\vartheta_- - 1} C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 L^{\vartheta_- - 1} C_{\vartheta_-} \\ &+ b_1 C_{\vartheta_-,\Gamma_2}^{\vartheta_-} + b_2 (S_{\vartheta_-,\Gamma_2} L)^{\vartheta_- - 1} C_{\vartheta_-,\Gamma_2} < a_*. \end{aligned}$$

*Under these conditions, problem* (4.1) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace on* $\Gamma_2$ *is also bounded.*

**Proof.** To apply Theorem 3.3, we verify the sufficient conditions in Remark 3.1. The extension estimate reads

$$\|Eu\|_{W^{1,\vartheta_-}(\mathbb{R}^N)} \le E_0\|u\|_{\hat{\rho}_\Phi} \quad \text{for all } u \in V.$$

Young's convolution inequality, including its derivative version, gives

$$\begin{aligned} \|Tu\|_{\vartheta_-,\Omega} &\le \|\rho * Eu\|_{\vartheta_-,\mathbb{R}^N} \le \|\rho\|_{1,\mathbb{R}^N}\|Eu\|_{\vartheta_-,\mathbb{R}^N} \le L\|u\|_{\hat{\rho}_\Phi}, \\ \|\nabla(Tu)\|_{\vartheta_-,\Omega} &\le \|\rho * \nabla(Eu)\|_{\vartheta_-,\mathbb{R}^N} \le \|\rho\|_{1,\mathbb{R}^N}\|\nabla(Eu)\|_{\vartheta_-,\mathbb{R}^N} \le L\|u\|_{\hat{\rho}_\Phi}. \end{aligned} \tag{4.2}$$

Using the full $W^{1,\vartheta_-}$ norm as the sum of the function and gradient norms, the same argument also yields

$$\|Tu\|_{W^{1,\vartheta_-}(\Omega)} \le \|\rho\|_{1,\mathbb{R}^N}\|Eu\|_{W^{1,\vartheta_-}(\mathbb{R}^N)} \le L\|u\|_{\hat{\rho}_\Phi}.$$

For the boundary estimate, the trace embedding of the full space implies

$$\|Tu\|_{\vartheta_-,\Gamma_2} \le S_{\vartheta_-,\Gamma_2}\|Tu\|_{W^{1,\vartheta_-}(\Omega)} \le S_{\vartheta_-,\Gamma_2} L\|u\|_{\hat{\rho}_\Phi}.$$

Thus the norm estimates in Remark 3.1 hold with

$$c_1 = L^{\vartheta_- - 1}, \quad c_2 = L^{\vartheta_- - 1}, \quad c_3 = 0, \quad c_4 = (S_{\vartheta_-,\Gamma_2} L)^{\vartheta_- - 1}.$$

The extension and convolution maps are continuous, so $T\colon V \to W^{1,\vartheta_-}(\Omega)$ is continuous. The displayed smallness inequality is exactly (3.2) with these constants, so Remark 3.1 implies (H5)(i).

If $\vartheta_- > N-1$, Remark 3.1 also verifies (H5)(ii). In the other case, the hypotheses on $\rho, \sigma_1, \sigma_2$ apply. Hölder's inequality and the derivative identity in (4.2) yield

$$\|Tu\|_{\infty,\Omega} + \|\nabla(Tu)\|_{\infty,\Omega} \le \|\rho\|_{\vartheta_-{}',\mathbb{R}^N} \|Eu\|_{W^{1,\vartheta_-}(\mathbb{R}^N)} \le E_0 \|\rho\|_{\vartheta_-{}',\mathbb{R}^N} \|u\|_{\hat\rho_\Phi}.$$

The convolution has bounded continuous representatives, and the same bound applies to its trace. Hence (H3)–(H4) imply (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i + C_M$ for $i = 1, 2$ on every norm ball. Theorem 3.3 therefore applies. □

In particular, a smooth compactly supported kernel and bounded source functions $\sigma_1, \sigma_2$ satisfy the second set of sufficient conditions.

**Theorem 4.2.** *Let $\vartheta(x) \equiv \tilde\vartheta > 1$ and $\kappa(x) \equiv \tilde\kappa > 1$ for all $x \in \overline{\Omega}$, and set $\tilde r_1 = \tilde r_2 = \tilde\vartheta$. Assume* (H0) –(H2), (H′3)–(H′4) *and either $\tilde\vartheta > N-1$, or $\rho \in L^{\tilde\vartheta'}(\mathbb{R}^N)$, $\tilde\sigma_1 \in L^{\gamma_1}(\Omega)$ and $\tilde\sigma_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$\tilde a_1 C_{\tilde\vartheta}^{\tilde\vartheta} + \tilde a_2 (S_{\tilde\vartheta^*} L)^{\tilde\vartheta - 1} C_{\frac{\tilde\vartheta^*}{\tilde\vartheta^* - \tilde r_1 + 1}} + \tilde a_3 G_{\tilde\vartheta}^{\tilde\vartheta - 1} C_{\tilde\vartheta} + \tilde a_4 L^{\tilde\vartheta - 1} C_{\tilde\vartheta}$$

$$+ \tilde b_1 C_{\tilde\vartheta,\Gamma_2}^{\tilde\vartheta} + \tilde b_2 (S_{\tilde\vartheta_*,\Gamma_2} L)^{\tilde\vartheta - 1} C_{\frac{\tilde\vartheta_*}{\tilde\vartheta_* - \tilde r_2 + 1},\Gamma_2} < a_*.$$

*Under these conditions, problem* (4.1) *has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.*

**Proof.** Theorem 3.5 reduces the proof to checking the sufficient conditions in Remark 3.4. For constant exponents, (4.2) yields

$$\|\nabla(Tu)\|_{\tilde\vartheta}^{\tilde\vartheta/\tilde r_1'} = \|\nabla(Tu)\|_{\tilde\vartheta}^{\tilde\vartheta - 1} \le L^{\tilde\vartheta - 1} \|u\|_{\hat\rho_\Phi}^{\tilde\vartheta - 1}.$$

The critical domain embedding of the full space yields

$$\|Tu\|_{\tilde\vartheta^*}^{\tilde r_1 - 1} = \|Tu\|_{\tilde\vartheta^*}^{\tilde\vartheta - 1} \le S_{\tilde\vartheta^*}^{\tilde\vartheta - 1} \|Tu\|_{W^{1,\tilde\vartheta}(\Omega)}^{\tilde\vartheta - 1} \le (S_{\tilde\vartheta^*} L)^{\tilde\vartheta - 1} \|u\|_{\hat\rho_\Phi}^{\tilde\vartheta - 1}.$$

For the boundary estimate, the critical trace embedding implies

$$\|Tu\|_{\tilde\vartheta_*,\Gamma_2}^{\tilde r_2 - 1} = \|Tu\|_{\tilde\vartheta_*,\Gamma_2}^{\tilde\vartheta - 1} \le S_{\tilde\vartheta_*,\Gamma_2}^{\tilde\vartheta - 1} \|Tu\|_{W^{1,\tilde\vartheta}(\Omega)}^{\tilde\vartheta - 1} \le (S_{\tilde\vartheta_*,\Gamma_2} L)^{\tilde\vartheta - 1} \|u\|_{\hat\rho_\Phi}^{\tilde\vartheta - 1}.$$

Thus the norm estimates in Remark 3.4 hold with $\tilde c_1 = (S_{\tilde\vartheta^*} L)^{\tilde\vartheta - 1}$, $\tilde c_2 = L^{\tilde\vartheta - 1}$, $\tilde c_3 = 0$ and $\tilde c_4 = (S_{\tilde\vartheta_*,\Gamma_2} L)^{\tilde\vartheta - 1}$. The stated inequality (3.5) verifies (H′5)(i) by Remark 3.4. If $\tilde\vartheta > N-1$, the same remark verifies (H′5)(ii). In the other case, the preceding convolution bound and $\tilde r_1 = \tilde r_2 = \tilde\vartheta$ verify (H′5)(ii) with $d = e = \tilde\vartheta$ and $\omega_{u,i} = \tilde\sigma_i + C_M$. Thus Theorem 3.5 applies. □

Next, we consider the continuous map $Tu = u^\pm$ with $u^+ = \max\{0, u\}$ and $u^- = -\min\{0, u\}$ in problem (P1). For this choice, problem (P1) becomes

$$\begin{aligned} -\operatorname{div}\left[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat\rho_\Phi}) \mathcal{K}(\nabla u)\right] &\in f(x, u, u^\pm, \nabla u, \nabla(u^\pm)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, u^\pm) \quad \text{on } \Gamma_2. \end{aligned} \tag{4.3}$$

The next existence statements apply to (4.3). We give the proof for $Tu = u^-$. The argument for $Tu = u^+$ is analogous.

**Theorem 4.3.** *Assume* (H0)–(H4), $\sigma_1 \in L^{\gamma_1}(\Omega)$, $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$, *and the inequality*

$$(a_1 + a_2)C_{\vartheta_-}^{\vartheta_-} + (a_3 + a_4)G_{\vartheta_-}^{\vartheta_- - 1}C_{\vartheta_-} + (b_1 + b_2)C_{\vartheta_-,\Gamma_2}^{\vartheta_-} < a_*.$$

*Under these conditions, problem* (4.3) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace is also bounded.*

**Proof.** We verify the norm estimates in Remark 3.1. For all $u \in V$, we get

$$\begin{aligned}
\|Tu\|_{\vartheta_-} &= \|u^-\|_{\vartheta_-} \le \|u\|_{\vartheta_-} \le C_{\vartheta_-}\|u\|_{\hat{\rho}_\Phi},\\
\|\nabla(Tu)\|_{\vartheta_-} &= \|\nabla u^-\|_{\vartheta_-} \le \|\nabla u\|_{\vartheta_-} \le G_{\vartheta_-}\|u\|_{\hat{\rho}_\Phi},\\
\|Tu\|_{\vartheta_-,\Gamma_2} &= \|u^-\|_{\vartheta_-,\Gamma_2} \le \|u\|_{\vartheta_-,\Gamma_2} \le C_{\vartheta_-,\Gamma_2}\|u\|_{\hat{\rho}_\Phi}.
\end{aligned}$$

Therefore, the norm estimates in Remark 3.1 hold with

$$c_1 = C_{\vartheta_-}^{\vartheta_- - 1}, \quad c_2 = G_{\vartheta_-}^{\vartheta_- - 1}, \quad c_3 = 0, \quad c_4 = C_{\vartheta_-,\Gamma_2}^{\vartheta_- - 1}.$$

The truncation map is continuous in $W^{1,\vartheta_-}(\Omega)$ and preserves the zero trace on $\Gamma_1$. To check (H5)(ii), the pointwise bounds $|u^-| \le |u|$ and $|\nabla u^-| \le |\nabla u|$ give

$$\sup_{\eta \in f(x,u,u^-,\nabla u,\nabla u^-)} |\eta| \le \sigma_1(x) + (a_1 + a_2)|u|^{\vartheta_- - 1} + (a_3 + a_4)|\nabla u|^{\vartheta_- - 1},$$

$$\sup_{\zeta \in f_{\Gamma_2}(x,u,u^-)} |\zeta| \le \sigma_2(x) + (b_1 + b_2)|u|^{\vartheta_- - 1}.$$

These are (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i$ for $i = 1, 2$. The smallness condition is precisely (3.2) for the displayed constants. Remark 3.1 verifies (H5)(i), and Theorem 3.3 proves the assertion. □

**Theorem 4.4.** *Let* $\vartheta(x) \equiv \tilde{\vartheta} > 1$ *and* $\kappa(x) \equiv \tilde{\kappa} > 1$ *for all* $x \in \overline{\Omega}$, *and set* $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. *Assume* (H0)–(H2), (H′3)–(H′4), $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$, $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$, *and the inequality*

$$\begin{aligned}
&\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 C_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + (\tilde{a}_3 + \tilde{a}_4) G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}}\\
&+\tilde{b}_1 C_{\tilde{\vartheta},\Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 C_{\tilde{\vartheta}^*,\Gamma_2}^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_2 + 1},\Gamma_2} < a_*.
\end{aligned}$$

*Under these conditions, problem* (4.3) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace is also bounded.*

**Proof.** Again, the proof reduces to checking the sufficient conditions in Remark 3.4.

For all $u \in V$, one can estimate that

$$\|Tu\|_{\tilde{\vartheta}_*}^{\tilde{r}_1-1} = \|u^-\|_{\tilde{\vartheta}_*}^{\tilde{\vartheta}-1} \le C_{\tilde{\vartheta}_*}^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1},$$

$$\|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}_1'} = \|\nabla u^-\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \le G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1},$$

$$\|Tu\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{r}_2-1} = \|u^-\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1} \le C_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1}.$$

Thus, we only need to choose $\tilde{c}_1 = C_{\tilde{\vartheta}_*}^{\tilde{\vartheta}-1}$, $\tilde{c}_2 = G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1}$, $\tilde{c}_3 = 0$ and $\tilde{c}_4 = C_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1}$. The same pointwise estimates verify (H′5)(ii), because $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Remark 3.4 and (3.5) give (H′5)(i), and Theorem 3.5 applies. □

Furthermore, let $Tu = (-\Delta_{\vartheta_-})^{-1}\ell(u)$ with $\ell\colon \mathbb{R} \to \mathbb{R}$ continuous and satisfying

$$|\ell(t)| \le d_1|t|^{\vartheta_- - 1} + d_2, \quad d_1, d_2 > 0.$$

Here $Tu$ is the unique solution in $W_0^{1,\vartheta_-}(\Omega)$ of $-\Delta_{\vartheta_-}(Tu) = \ell(u)$ with zero Dirichlet data on the whole boundary. This specifies the inverse operator, whose values are only required to belong to $W^{1,\vartheta_-}(\Omega)$. We consider the problem

$$\begin{aligned}
&-\operatorname{div}\left[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u)\right] \\
&\quad \in f(x, u, (-\Delta_{\vartheta_-})^{-1}\ell(u), \nabla u, \nabla((-\Delta_{\vartheta_-})^{-1}\ell(u))) \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \Gamma_1, \\
&-\frac{\partial u}{\partial n} \in f_{\Gamma_2}(x, u, (-\Delta_{\vartheta_-})^{-1}\ell(u)) \quad \text{on } \Gamma_2.
\end{aligned} \tag{4.4}$$

The existence statements for (4.4) are as follows. Let $P_{\vartheta_-}, P_{(\vartheta_-)^*}$ denote the Dirichlet Poincaré and Sobolev constants, and put

$$A_1 = P_{\vartheta_-} d_1 C_{\vartheta_-}^{\vartheta_- - 1}, \quad A_0 = P_{\vartheta_-} d_2 |\Omega|^{1/\vartheta_-{}'}.$$

**Theorem 4.5.** *Assume* (H0)–(H4) *and either* $\vartheta_- > N-1$, *or* $\Omega$ *has* $C^2$ *boundary,* $\ell$ *is bounded,* $\sigma_1 \in L^{\gamma_1}(\Omega)$ *and* $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$. *Suppose also that*

$$a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 P_{\vartheta_-}^{\vartheta_- - 1} A_1 C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 A_1 C_{\vartheta_-} + b_1 C_{\vartheta_-,\Gamma_2}^{\vartheta_-} < a_*.$$

*Under these conditions, problem* (4.4) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace is also bounded.*

**Proof.** Since for a given $u \in V$, $Tu$ solves $-\Delta_{\vartheta_-}(Tu) = \ell(u)$ in $\Omega$, testing with $Tu$ we get

$$\begin{aligned}
\|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_-} &= \int_\Omega |\nabla(Tu)|^{\vartheta_-}\,\mathrm{d}x = \int_\Omega \ell(u)Tu\,\mathrm{d}x \\
&\le \left(d_1\|u\|_{\vartheta_-}^{\vartheta_- - 1} + d_2|\Omega|^{1/\vartheta_-{}'}\right)\|Tu\|_{\vartheta_-} \\
&\le \left(d_1 C_{\vartheta_-}^{\vartheta_- - 1}\|u\|_{\hat{\rho}_\Phi}^{\vartheta_- - 1} + d_2|\Omega|^{1/\vartheta_-{}'}\right) P_{\vartheta_-}\|\nabla(Tu)\|_{\vartheta_-}.
\end{aligned}$$

If the gradient norm is nonzero, divide by it. The resulting inequality is also true when it is zero. Thus

$$\|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- -1} \le A_1 \|u\|_{\hat{\rho}_\Phi}^{\vartheta_- -1} + A_0.$$

The Dirichlet Poincaré inequality then implies

$$\|Tu\|_{\vartheta_-}^{\vartheta_- -1} \le P_{\vartheta_-}^{\vartheta_- -1} \|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- -1} \le P_{\vartheta_-}^{\vartheta_- -1} A_1 \|u\|_{\hat{\rho}_\Phi}^{\vartheta_- -1} + P_{\vartheta_-}^{\vartheta_- -1} A_0.$$

The trace of $Tu$ vanishes on the whole boundary. Consequently the norm estimates in Remark 3.1 hold with

$$c_1 = P_{\vartheta_-}^{\vartheta_- -1} A_1, \quad c_2 = A_1, \quad c_3 = \max\{1, P_{\vartheta_-}^{\vartheta_- -1}\} A_0, \quad c_4 = 0.$$

The continuity and growth assumptions on $\ell$ imply continuity of $u \mapsto \ell(u)$ from $L^{\vartheta_-}$ to $L^{\vartheta_-'}$. To verify continuity of the Dirichlet inverse, let $F_n \to F$ in $L^{\vartheta_-'}$ and let $w_n, w \in W_0^{1,\vartheta_-}(\Omega)$ solve $-\Delta_{\vartheta_-} w_n = F_n$ and $-\Delta_{\vartheta_-} w = F$. Testing with $w_n$ and using the Dirichlet Poincaré inequality gives

$$\|\nabla w_n\|_{\vartheta_-}^{\vartheta_- -1} \le P_{\vartheta_-} \|F_n\|_{\vartheta_-'},$$

where the estimate is also valid if the gradient vanishes. Hence a subsequence converges weakly to some $w_0$ in $W_0^{1,\vartheta_-}(\Omega)$. Writing $A_{\vartheta_-}$ for the Dirichlet $\vartheta_-$-Laplacian, we have

$$\langle A_{\vartheta_-}(w_n), w_n - w_0 \rangle = \int_\Omega (F_n - F)(w_n - w_0)\,\mathrm{d}x + \int_\Omega F(w_n - w_0)\,\mathrm{d}x \to 0.$$

The $(S_+)$ property of $A_{\vartheta_-}$ implies $w_n \to w_0$ strongly in $W_0^{1,\vartheta_-}(\Omega)$. Passing to the equation gives $-\Delta_{\vartheta_-} w_0 = F$, and uniqueness implies $w_0 = w$. The subsequence principle yields $w_n \to w$ in $W_0^{1,\vartheta_-}(\Omega)$ for the full sequence, proving continuity of the inverse. Hence $T\colon V \to W^{1,\vartheta_-}(\Omega)$ is continuous. The smallness inequality is (3.2) for these constants, so Remark 3.1 implies (H5)(i).

If $\vartheta_- > N - 1$, the verified norm estimates imply (H5)(ii) by Remark 3.1. In the other case, $\Omega$ has $C^2$ boundary and $\ell$ is bounded. Put $B = \|\ell\|_\infty$ and choose a ball $B_R(x_0)$ containing $\overline{\Omega}$. Its $\vartheta_-$-Laplacian torsion function is

$$\tau(x) = \frac{R^{\vartheta_-'} - |x - x_0|^{\vartheta_-'}}{\vartheta_-' N^{1/(\vartheta_- - 1)}}, \quad -\Delta_{\vartheta_-} \tau = 1 \quad \text{in } B_R(x_0).$$

Since $|\ell(u)| \le B$, weak comparison with $\pm B^{1/(\vartheta_- - 1)}\tau$ gives

$$|Tu| \le B^{1/(\vartheta_- - 1)}\tau \quad \text{in } \Omega, \quad \|Tu\|_\infty \le C(\Omega, \vartheta_-) B^{1/(\vartheta_- - 1)}.$$

If $B = 0$, uniqueness gives $Tu = 0$. Combining this solution bound with the global Dirichlet boundary regularity estimate for bounded forcing yields

$$\|Tu\|_{W^{1,\infty}(\Omega)} \le C(\Omega, \vartheta_-, \|\ell\|_\infty),$$

see Lieberman [25]. Substituting these bounds in (H3)–(H4) proves (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i + C_M$ for $i = 1, 2$. Theorem 3.3 therefore applies. □

For example, $\ell(t) = \tanh t$ and bounded $\sigma_1, \sigma_2$ satisfy the bounded-forcing assumptions on a $C^2$ domain.

Note that if $\vartheta(x) = \tilde{\vartheta}$ for all $x \in \overline{\Omega}$, then $\vartheta_- = \tilde{\vartheta}$.

**Theorem 4.6.** *Let $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and set $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Assume* (H0)–(H2), (H′3)–(H′4) *and either $\tilde{\vartheta} > N - 1$, or $\Omega$ has $C^2$ boundary, $\ell$ is bounded, $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$ and $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1 C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{a}_3 G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}} + \tilde{a}_4 A_1 C_{\tilde{\vartheta}} + \tilde{b}_1 C_{\tilde{\vartheta},\Gamma_2}^{\tilde{\vartheta}} < a_*.$$

*Under these conditions, problem* (4.4) *has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.*

**Proof.** The argument used for Theorem 4.5 yields the following estimates for every $u \in V$:

$$\begin{aligned}
\|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} &\le A_1 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + A_0, \\
\|Tu\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} &\le P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \le P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_0, \\
\|Tu\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1} &= 0.
\end{aligned}$$

Therefore the norm estimates in Remark 3.4 hold with $\tilde{c}_1 = P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1$, $\tilde{c}_2 = A_1$, $\tilde{c}_3 = \max\{1, P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1}\} A_0$ and $\tilde{c}_4 = 0$. The assumed inequality (3.5) verifies (H′5)(i) by Remark 3.4. If $\tilde{\vartheta} > N - 1$, that remark also verifies (H′5)(ii). In the other case, the bounded-forcing regularity argument just proved, together with $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$, verifies (H′5)(ii) with $d = e = \tilde{\vartheta}$ and $\omega_{u,i} = \tilde{\sigma}_i + C_M$. Theorem 3.5 applies in both cases. □

Finally, for $Tu = gu$ with $g \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ we consider the problem

$$\begin{aligned}
-\operatorname{div}\ \big[h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u)\big] &\in f(x, u, gu) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_1, \\
-\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, gu) \quad \text{on } \Gamma_2.
\end{aligned} \tag{4.5}$$

The source $f(x, u, gu)$ is independent of the gradient arguments. Thus $a_3 = a_4 = 0$ and $\tilde{a}_3 = \tilde{a}_4 = 0$ in the corresponding growth bounds. Write $G = \|g\|_\infty$, $G_\Gamma = \|g\|_{\infty,\Gamma_2}$ and $G_1 = \|\nabla g\|_N$. Since $1/\vartheta_- = 1/(\vartheta_-)^* + 1/N$, Hölder's inequality controls the product $u\nabla g$ in $L^{\vartheta_-}$. The trace of $g$ is bounded by its essential supremum in $\Omega$.

**Theorem 4.7.** *Assume* (H0)–(H4), *$\sigma_1 \in L^{\gamma_1}(\Omega)$, $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$, and the inequality*

$$a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 (G C_{\vartheta_-})^{\vartheta_- - 1} C_{\vartheta_-} + b_1 C_{\vartheta_-,\Gamma_2}^{\vartheta_-} + b_2 (G_\Gamma C_{\vartheta_-,\Gamma_2})^{\vartheta_- - 1} C_{\vartheta_-,\Gamma_2} < a_*.$$

*Under these conditions, problem* (4.5) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace is also bounded.*

**Proof.** For $u \in V$, we calculate that

$$\begin{aligned}
\|Tu\|_{\vartheta_-} &= \|gu\|_{\vartheta_-} \le \|g\|_\infty \|u\|_{\vartheta_-} \le GC_{\vartheta_-}\|u\|_{\hat{\rho}_\Phi},\\
\|\nabla(Tu)\|_{\vartheta_-} &= \|g\nabla u + u\nabla g\|_{\vartheta_-} \le G\|\nabla u\|_{\vartheta_-} + G_1\|u\|_{(\vartheta_-)^*}\\
&\le (GG_{\vartheta_-} + G_1 C_{(\vartheta_-)^*})\|u\|_{\hat{\rho}_\Phi},\\
\|Tu\|_{\vartheta_-,\Gamma_2} &= \|(g)(u)\|_{\vartheta_-,\Gamma_2} \le G_\Gamma \|u\|_{\vartheta_-,\Gamma_2} \le G_\Gamma C_{\vartheta_-,\Gamma_2}\|u\|_{\hat{\rho}_\Phi}.
\end{aligned}$$

Then, the norm estimates in Remark 3.1 hold with

$$\begin{aligned}
&c_1 = (GC_{\vartheta_-})^{\vartheta_- - 1}, \quad c_2 = (GG_{\vartheta_-} + G_1 C_{(\vartheta_-)^*})^{\vartheta_- - 1},\\
&c_3 = 0, \quad c_4 = (G_\Gamma C_{\vartheta_-,\Gamma_2})^{\vartheta_- - 1}.
\end{aligned}$$

The product rule also proves continuity of the multiplication map into $W^{1,\vartheta_-}(\Omega)$. Finally, the pointwise relations $|gu| \le G|u|$ and the boundary relation $|gu| \le G_\Gamma|u|$ give (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i$ for $i = 1, 2$. Remark 3.1 and (3.2) give (H5)(i), so Theorem 3.3 applies. □

**Theorem 4.8.** *Let* $\vartheta(x) \equiv \tilde{\vartheta} > 1$ *and* $\kappa(x) \equiv \tilde{\kappa} > 1$ *for all* $x \in \overline{\Omega}$, *and set* $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. *Assume* (H0) –(H2), (H′3)–(H′4), $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$, $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$, *and the inequality*

$$\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{b}_1 C_{\tilde{\vartheta},\Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 (G_\Gamma C_{\tilde{\vartheta}_*,\Gamma_2})^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1},\Gamma_2} < a_*.$$

*Under these conditions, problem* (4.5) *has a weak solution* $u \in V$ *that is uniformly bounded. Its trace is also bounded.*

**Proof.** For $u \in V$, we calculate that

$$\begin{aligned}
\|Tu\|_{\tilde{\vartheta}^*}^{\tilde{r}_1 - 1} &= \|gu\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} \le G^{\tilde{\vartheta}-1}\|u\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} \le (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1},\\
\|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}_1'} &= \|\nabla(gu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \le (GG_{\tilde{\vartheta}} + G_1 C_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1},\\
\|Tu\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{r}_2 - 1} &= \|(gu)\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1}\\
&\le G_\Gamma^{\tilde{\vartheta}-1}\|u\|_{\tilde{\vartheta}_*,\Gamma_2}^{\tilde{\vartheta}-1}\\
&\le (G_\Gamma C_{\tilde{\vartheta}_*,\Gamma_2})^{\tilde{\vartheta}-1}\|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1}.
\end{aligned}$$

Then, the norm estimates in Remark 3.4 hold with $\tilde{c}_1 = (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}$, $\tilde{c}_2 = (GG_{\tilde{\vartheta}} + G_1 C_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}$, $\tilde{c}_3 = 0$ and $\tilde{c}_4 = (G_\Gamma C_{\tilde{\vartheta}_*,\Gamma_2})^{\tilde{\vartheta}-1}$. As before, the absence of gradient arguments in $f$, the pointwise product bounds and the stated integrability of the source functions imply (H′5)(ii) with $d = e = \tilde{\vartheta}$. The inequality (3.5) verifies (H′5)(i) by Remark 3.4. Theorem 3.5 concludes the proof. □

## ORCID

Shengda Zeng https://orcid.org/0000-0003-1818-842X

Yasi Lu https://orcid.org/0009-0002-1103-1809

Patrick Winkert https://orcid.org/0000-0003-0320-7026

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