EXISTENCE OF SOLUTIONS FOR RESONANT DOUBLE PHASE PROBLEMS WITH MIXED BOUNDARY VALUE CONDITIONS

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Abstract. We study a double phase problem with mixed boundary value conditions with reaction terms that resonate at the first eigenvalue of the related eigenvalue problem. Based on the maximum principle and homological local linking, we are going to prove the existence of at least two bounded nontrivial solutions for this problem.

1. Introduction

In this paper, we study the following double phase problems with mixed boundary conditions

$$
\begin{aligned}
A(u)+|u|^{p-2}u+a(x)|u|^{q-2}u&=f(x,u) && \text{in } \Omega,\\
u&=0 && \text{on } \sigma,\\
\left(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu&=g(x,u) && \text{on } \Gamma,
\end{aligned}
\tag{1.1}
$$

where

$$A(u):=-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\right)$$

is the double phase operator, Ω is a bounded domain of $\mathbb{R}^N$, $N\geq 2$, with a C^1 boundary $\partial\Omega$ such that $\partial\Omega=\sigma\cup\Gamma$ and $\sigma\cap\Gamma=\emptyset$, $\nu(x)$ denotes the outer unit normal of Ω at $x\in\Gamma$,

$$1<p<N,\quad p<q<p_*=\frac{(N-1)p}{N-p}\quad\text{and}\quad 0<a(\cdot)\in L^\infty(\Omega). \tag{1.2}$$

Clearly, $q<p_*$ implies $q<p^*=\frac{Np}{N-p}$. The nonlinearities f and g satisfy the following hypotheses:

(H) $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ and $g\colon\Gamma\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions such that the following hold:

(i) There exist constants $C_1,C_2>0$ such that

$$
\begin{aligned}
|f(x,t)|&\leq C_1\left(1+|t|^{r_1-1}\right) && \text{for a.a. } x\in\Omega,\\
|g(x,t)|&\leq C_2\left(1+|t|^{r_2-1}\right) && \text{for a.a. } x\in\Gamma,
\end{aligned}
$$

for all $t\in\mathbb{R}$, where $q<r_1<p^*$ and $q<r_2<p_*$, respectively.

(ii)

$$\lim_{t\to\pm\infty}\frac{qF(x,t)}{|t|^q}\leq\lambda_1(q)\quad\text{uniformly for a.a. } x\in\Omega,$$

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$$\lim_{t\to\pm\infty} \frac{qG(x,t)}{|t|^q} \leq \lambda_1(q) \quad \text{uniformly for a.a. } x \in \Gamma,$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ and $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$;

(iii)

$$\lim_{|t|\to+\infty} [f(x,t)t - qF(x,t)] = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

$$\lim_{|t|\to+\infty} [g(x,t)t - qG(x,t)] = +\infty \quad \text{uniformly for a.a. } x \in \Gamma;$$

(iv) There exist $\delta > 0$, $\theta > \tilde{\lambda}_1(p)$ and $0 < \tilde{\lambda} < \tilde{\lambda}_2(p)$ such that

$$\theta\,|t|^p \leq pF(x,t) \leq \tilde{\lambda}\,|t|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } |t| \leq \delta,$$

$$\theta\,|t|^p \leq pG(x,t) \leq \tilde{\lambda}\,|t|^p \quad \text{for a.a. } x \in \Gamma \text{ and for all } |t| \leq \delta.$$

where $\lambda_1(q)$ stands for the first eigenvalue of the weighted q-Laplace mixed boundary condition problem while $\tilde{\lambda}_1(p)$ and $\tilde{\lambda}_2(p)$ represent the first and the second eigenvalues of the p-Laplace mixed boundary condition problem, respectively, see Section 2 for more details.

The solutions of problem (1.1) are understood in the weak sense, that is, $u \in X$ is a solution of (1.1) if

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x \\
&\quad + \int_\Omega \left(|u|^{p-2}u + a(x)|u|^{q-2}u\right)v\,\mathrm{d}x \\
&= \int_\Omega f(x,u)v\,\mathrm{d}x + \int_\Gamma g(x,u)v\,\mathrm{d}S,
\end{aligned}$$

is satisfied for all $v \in X$, where $X = \{u \in W^{1,\mathcal{H}}(\Omega) : u|_\sigma = 0\}$ is a closed subspace of $W^{1,\mathcal{H}}(\Omega)$, which will be defined in Section 2.

The differential operator

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right), \quad u \in W^{1,\mathcal{H}}(\Omega) \tag{1.3}$$

involved in problem (1.1) is the so-called double phase operator. The integral form of it is denoted by

$$\int_\Omega \left(|\nabla u|^p + a(x)|\nabla u|^q\right)\mathrm{d}x, \tag{1.4}$$

which was first introduced by Zhikov [40] to describe the phenomenon that hardening properties of strongly anisotropic materials drastically change with the point in the domain. The function $a(\cdot)$ was used as an aid to regulating the mixture between two different materials, with power hardening of rates p and q, respectively, see for instance the works of Zhikov [40–42]. The energy density of (1.4) exhibits ellipticity in the gradient of order q on the points x where $a(x)$ is positive and of order p on the points x where $a(x)$ vanishes. This is the reason why we call (1.3) as the double phase operator. Both theoretical and applications aspects of functionals of type (1.4) have been intensively studied by many researchers, see for example, Baroni-Colombo-Mingione [2–4], Baroni-Kuusi-Mingione [5], Byun-Oh [6], Colombo-Mingione [8, 9], De Filippis-Mingione [11], De Filippis-Palatucci [12], Gasiński-Winkert [14,15], Liu-Dai [18–20], Liu-Dai-Papageorgiou-Winkert [21], Liu-Winkert [22], Marcellini [24, 25], Ok [27, 28], Papageorgiou-Rădulescu-Repovš

[30,31], Perera-Squassina [36], Ragusa-Tachikawa [37], Zeng-Bai-Gasiński-Winkert [38,39] and the references therein.

The purpose of this paper is to study the multiplicity of solutions for problem (1.1). There are two main characteristics of this problem: one is that the reaction terms resonate at the corresponding eigenvalues; the other one is the appearance of nonlinear boundary conditions and mixed boundary conditions.

The main result in this paper is the following theorem.

Theorem 1.1. *Let hypotheses* (1.2) *and* (H) *be satisfied, then problem* (1.1) *has at least two nontrivial solutions* $u_1, u_2 \in X \cap L^\infty(\Omega)$.

Theorem 1.1 is related to the recent results obtained in Liu-Zeng-Gasiński-Kim [23], Papageorgiou-Rădulescu-Repovš [31] and Papageorgiou-Rădulescu-Zhang [33]. Papageorgiou-Rădulescu-Repovš [31] investigated the existence of multiple solutions to a double phase Robin problem when resonating at the first eigenvalue of the weighted p-Laplace Robin problem, applying the local linking of the Morse theory to derive the existence of at least two bounded solutions. Papageorgiou-Rădulescu-Zhang [33] considered the existence of multiple solutions to the Dirichlet double phase problem when resonating at the first eigenvalue of the weighted p-Laplace Dirichlet equation, using variational methods together with Morse theory to yield the existence of at least two bounded nontrivial solutions. Liu-Zeng-Gasiński-Kim [23] studied a nonlinear complementarity problem (NCP) with a double phase differential operator and a generalized multivalued boundary condition. By using the Moreau-Yosida approximation method, the regularization problem corresponding to NCP was introduced, and finally, the properties of the solution set of NCP were obtained. Inspired by the above papers, we are going to study the resonant double phase equations under mixed boundary conditions given in (1.1) in the present paper. The reaction terms resonate at the first eigenvalue of the weighted q-Laplace equation with the mixed boundary, which is different from Papageorgiou-Rădulescu-Repovš [31] and Papageorgiou-Rădulescu-Zhang [33]. The mixed boundary conditions are divided into two parts, one is the Dirichlet boundary condition and the other is the nonlinear boundary condition, which is different from Liu-Zeng-Gasiński-Kim [23]. These differences bring new challenges. In order to overcome these difficulties, we need to require more elaborate calculations to get the compactness condition and the homological local linking.

The proof of Theorem 1.1 is based on variational methods and Morse theoretic aspects, especially the homological local linking. First, by the hypotheses (H)(i) and (H)(iii), we show that the corresponding energy functional J of (1.1) satisfies the Cerami condition. Second, by (H)(ii) and (H)(iii), we prove that J is coercive, and then by the Weierstrass-Tonelli theorem, it is concluded that there exists $u_1 \neq 0$ such that $J'(u_1) = 0$. Finally, in order to obtain the second solution u_2, we verify that J has a local $(1,1)$-linking at 0 by hypothesis (H)(ii). In addition, we study the eigenvalue problem of the weighted q-Laplace equation with mixed boundary conditions.

The rest of this paper is organized as follows. In Section 2 we recall some main variational tools and introduce the Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ including some of its properties. We also present some properties of the weighted q-Laplace equation with mixed boundary conditions and the related first eigenvalue and its eigenfunction. The proof of the Theorem 1.1 is then given in Section 3.

2. Preliminaries

In this section, we first recall the main properties on the theory of Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, respectively. We refer to Colasuonno-Squassina [7], Crespo-Blanco-Gasiński-Harjulehto-Winkert [10], Harjulehto-Hästö [16] and Musielak [26] for the main results in this direction.

Suppose (1.2) and let $\mathcal{H}\colon \overline{\Omega}\times[0,\infty)\to[0,\infty)$ be the function defined by

$$\mathcal{H}(x,t)=t^p+a(x)t^q.$$

Then, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega)=\left\{u \;\Big|\; u\colon\Omega\to\mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}}(u)<+\infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}}=\inf\left\{\tau>0\ \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right)\le 1\right\},$$

where the modular function $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u):=\int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x=\int_\Omega\left(|u|^p+a(x)|u|^q\right)\mathrm{d}x.$$

We know that the space $L^{\mathcal{H}}(\Omega)$ is a reflexive Banach space. Moreover, we define the weighted Lebesgue space $L^q_a(\Omega)$

$$L^q_a(\Omega)=\left\{u \;\Big|\; u\colon\Omega\to\mathbb{R} \text{ is measurable and } \int_\Omega a(x)|u|^q\,\mathrm{d}x<+\infty\right\},$$

which is endowed with the seminorm

$$\|u\|_{q,a}=\left(\int_\Omega a(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

It is not easy to check the validity of the following continuous embeddings

$$L^q(\Omega)\hookrightarrow L^{\mathcal{H}}(\Omega)\hookrightarrow L^q_a(\Omega)\cap L^p(\Omega).$$

The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega)=\left\{u\in L^{\mathcal{H}}(\Omega)\,:\,|\nabla u|\in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}}=\|\nabla u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}}=\|\,|\nabla u|\,\|_{\mathcal{H}}$.

Similarly, we define

$$W^{1,\mathcal{K}}(\Omega)=\left\{u\in L^q_a(\Omega):|\nabla u|\in L^q_a(\Omega)\right\},$$

which is endowed with the norm

$$\|u\|_{1,\mathcal{K}}=\|\nabla u\|_{q,\mu}+\|u\|_{q,\mu}.$$

We know that $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{K}}(\Omega)$ are reflexive Banach spaces. Moreover, we have the following embedding results, see for example Crespo-Blanco-Gasiński-Harjulehto-Winkert [10, Proposition 2.16] or Gasiński-Winkert [15, Proposition 2.2].

Proposition 2.1. *Let* (1.2) *be satisfied and let*

$$p^* := \frac{Np}{N-p} \quad \text{and} \quad p_* := \frac{(N-1)p}{N-p}.$$

Then the following embedding hold:

(i) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1, p^*]$ *and compact for all* $r \in [1, p^*)$;

(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is continuous for all* $r \in [1, p_*]$ *and compact for all* $r \in [1, p_*)$;

(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,\mathcal{K}}(\Omega)$ *is continuous.*

Let

$$\varrho(u) = \int_\Omega \Big(|\nabla u|^p + a(x)|\nabla u|^q + |u|^p + a(x)|u|^q\Big)\,\mathrm{d}x. \tag{2.1}$$

The norm $\|\cdot\|$ and the modular function ϱ are related as follows, see Liu-Dai [18, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [10, Proposition 2.14].

Proposition 2.2. *Let* (1.2) *be satisfied, let* $u \in W^{1,\mathcal{H}}(\Omega)$ *and let* ϱ *be defined by* (2.1). *Then the following hold:*

(i) *If* $u \neq 0$, *then* $\|u\| = \lambda$ *if and only if* $\varrho(\frac{u}{\lambda}) = 1$;

(ii) $\|u\| < 1$ *(resp.* > 1, $= 1$*) if and only if* $\varrho(u) < 1$ *(resp.* > 1, $= 1$*)*;

(iii) *If* $\|u\| < 1$, *then* $\|u\|^q \leq \varrho(u) \leq \|u\|^p$;

(iv) *If* $\|u\| > 1$, *then* $\|u\|^p \leq \varrho(u) \leq \|y\|^q$;

(v) $\|u\| \to 0$ *if and only if* $\varrho(u) \to 0$;

(vi) $\|u\| \to +\infty$ *if and only if* $\varrho(u) \to +\infty$.

Let $L\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear operator given by

$$\begin{aligned}\langle L(u), v\rangle_{\mathcal{H}} &= \int_\Omega \big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big)\cdot\nabla v\,\mathrm{d}x \\ &\quad + \int_\Omega \big(|u|^{p-2}u + a(x)|u|^{q-2}u\big)v\,\mathrm{d}x\end{aligned} \tag{2.2}$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$. Here, $\langle\cdot\,,\cdot\rangle_{\mathcal{H}}$ stands for the the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$. The operator $L\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ has the following properties, see Liu-Dai [18] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [10, Proposition 3.5].

Proposition 2.3. *The operator* L *defined by* (2.2) *is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone (hence maximal monotone) and it is of type* (S_+).

Next, we recall some definitions and tools that will be used in this paper.

Definition 2.4. *Let* X *be a real Banach space and let* X^* *be its dual space. We say that* $J \in C^1(X)$ *satisfies the Cerami-condition (C-condition for short), if for any* $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ *such that* $\{J(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *is bounded and* $(1 + \|u_n\|)J'(u_n) \to 0$ *in* X^*, *admits a strongly convergent subsequence.*

The following result can be found in Ambrosetti-Malchiodi [1, Theorem 5.5].

Proposition 2.5. *Suppose that X is a reflexive Banach space. If $J\colon X\to\mathbb{R}$ is coercive and sequentially weakly lower semi-continuous on X, then I is bounded from below on X and has a minimum in X.*

Let X be a Banach space, $J\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$. We introduce the following sets

$$\begin{aligned} J^c &= \{u\in X\,:\,J(u)\le c\},\\ K_J &= \{u\in X\,:\,J'(u)=0\},\\ K_J^c &= \{u\in K_J\,:\,J(u)=c\}, \end{aligned}$$

where K_J is the set of all critical points of J.

Consider a topological pair (A,B) such that $B\subseteq A\subseteq X$. For every $k\in\mathbb{N}_0$ we denote by $H_k(A,B)$ the k^{th}-relative singular homology group with integer coefficients for the pair (A,B). If $u\in K_J^c$ is isolated, the critical groups of J at u are defined by

$$C_k(J,u)=H_k(J^c\cap U,J^c\cap U\setminus\{u\})\quad\text{for all } k\ge 0,$$

with U being a neighborhood of u such that $K_J\cap J^c\cap U=\{u\}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood U.

If $J\in C^1(X,\mathbb{R})$ satisfies the C-condition (see Definition 2.4), $\inf J(K_J)>-\infty$ and $c<\inf J(K_J)$, then the critical groups of J at infinity are defined by

$$C_k(J,\infty)=H_k(X,J^c)\quad\text{for all } k\ge 0.$$

Taking Corollary 5.3.12 of Papageorgiou-Rădulescu-Repovš [32] into account, this definition is independent of the choice of the level $c<\inf J(K_J)$.

We use the local (m,n)-linking method to prove the existence of a solution of problem (1.1). The following definition is originally due to Perera [35] (see also Papageorgiou-Rădulescu-Repovš [32, Definition 6.6.13]).

Definition 2.6. *Let X be a Banach space, $J\in C^1(X,\mathbb{R})$, and 0 an isolated critical point of J with $J(0)=0$. Let $m,n\in\mathbb{N}$. Suppose there exist a neighborhood U of 0 and nonempty sets $E_0\subseteq E\subseteq U$, $D\subseteq X$ such that $E_0\cap D=\emptyset$ and*

(a) $J^0\cap U\cap K_J=\{0\}$;
(b) $\dim\operatorname{im} i_*-\dim\operatorname{im} j_*\ge n$, *where*

$$i_*\colon H_{m-1}(E_0)\to H_{m-1}(X\setminus D)\ \textit{and}\ j_*\colon H_{m-1}(E_0)\to H_{m-1}(E)$$

are the homomorphisms induced by the inclusion map $i\colon E_0\to X\setminus D$ and $j:E_0\to E$;
(c) $J|_E\le 0<J|_{U\cap D\setminus\{0\}}$.

Then we say that J has a "local (m,n)-linking" near the origin.

A very helpful result is the following corollary, see Papageorgiou-Rădulescu-Repovš [32, Corollary 6.7.10].

Proposition 2.7. *If X is a Banach space, $J\in C^1(X)$ is bounded below and satisfies the C-condition, J has a local (m,n)-linking at 0 with $m,n\in\mathbb{N}$ and 0 is not a global minimizer of J, then J has at least three critical points.*

Next, we want to study an appropriate eigenvalue problem following the ideas of Papageorgiou-Rădulescu-Repovš [31] and Li-Liu-Cheng [17]. We consider the following weighted q-Laplacian eigenvalue problem with mixed boundary conditions

$$\begin{aligned} -\operatorname{div}\left(a(x)|\nabla u|^{q-2}\nabla u\right) + a(x)|u|^{q-2}u &= \lambda|u|^{q-2}u \qquad && \text{in} \quad \Omega,\\ u &= 0 && \text{on} \quad \sigma,\\ \left(a(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu &= \lambda|u|^{q-2}u && \text{on} \quad \Gamma, \end{aligned} \tag{2.3}$$

where q and $a(\cdot)$ satisfy the hypothesis (1.2). The first eigenvalue $\lambda_1(q) > 0$ of (2.3) has the following variational characterization

$$\lambda_1(q) = \inf\left\{ \frac{\displaystyle\int_\Omega a(x)\left(|\nabla u|^q + |u|^q\right)\mathrm{d}x}{\displaystyle\int_\Omega |u|^q\,\mathrm{d}x + \int_\Gamma |u|^q\,\mathrm{d}S} \,:\, u \in W^{1,\mathcal{K}}(\Omega)\setminus\{0\} \right\}. \tag{2.4}$$

The corresponding eigenfunction $u_1 \in W^{1,\mathcal{K}}(\Omega)$ to the first eigenvalue $\lambda_1 > 0$ satisfies $u_1 \in L^\infty(\Omega)$ and $u_1(x) > 0$ for a.a. $x \in \Omega$ which can be shown similar to Proposition 3 of Papageorgiou-Rădulescu-Zhang [33].

Furthermore, let $\tilde{\lambda}_1(p)$ be the first eigenvalue of the following p-Laplacian mixed boundary value problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= \lambda|u|^{p-2}u \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \sigma,\\ \left(|\nabla u|^{p-2}\nabla u\right)\cdot\nu &= \lambda|u|^{p-2}u && \text{on } \Gamma. \end{aligned} \tag{2.5}$$

Based on the results of Li-Liu-Cheng [17] we know that the first eigenvalue $\tilde{\lambda}_1(p)$ of (2.5) is positive, simple and isolate. Let $\tilde{u}_1$ be the positive eigenfunction associated with $\tilde{\lambda}_1(p)$, then $\tilde{u}_1 \in L^\infty(\Omega)$. Moreover, the second eigenvalue $\tilde{\lambda}_2(p)$ of (2.5) can be written as

$$\tilde{\lambda}_2(p) = \inf\left\{\tilde{\lambda}(p) \,:\, \tilde{\lambda}(p) \text{ is an eigenvalue of (2.5) with } \tilde{\lambda}(p) > \tilde{\lambda}_1(p)\right\},$$

see Li-Liu-Cheng [17, Proposition 5.2].

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. Recall that $X = \{u \in W^{1,\mathcal{H}}(\Omega) : u|_\sigma = 0\}$ and let $\|u\| = \|u\|_{1,\mathcal{H}}$ for all $u \in X$ be the norm of X. The corresponding energy functional $J\colon X \to \mathbb{R}$ related to problem (1.1) is given by

$$\begin{aligned} J(u) = {} & \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right)\mathrm{d}x + \frac{1}{q}\int_\Omega a(x)\left(|\nabla u|^q + |u|^q\right)\mathrm{d}x\\ & - \int_\Omega F(x,u)\,\mathrm{d}x - \int_\Gamma G(x,u)\,\mathrm{d}S. \end{aligned}$$

Under our assumptions, it is standard to check that $J\colon X \to \mathbb{R}$ is well-defined and of class C^1 and the solutions of problem (1.1) are the critical points of $J\colon X \to \mathbb{R}$. First, we will show that $J\colon X \to \mathbb{R}$ satisfies the C-condition.

Proposition 3.1. *Let hypotheses (1.2) and (H) be satisfied, then the energy functional $J\colon X \to \mathbb{R}$ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ be a sequence such that

$$|J(u_n)| \leq M_1 \quad \text{for some} \quad M_1 > 0 \tag{3.1}$$

and

$$(1+\|u_n\|)J'(u_n) \to 0 \quad \text{in } X^* \quad \text{for } n \to \infty. \tag{3.2}$$

By (3.2) and (2.2), we have

$$\begin{aligned} |\langle J'(u_n), v\rangle| &= \left| \langle L(u_n), v\rangle_{\mathcal{H}} - \int_\Omega f(x,u_n)v \,\mathrm{d}x - \int_\Gamma g(x,u_n)v \,\mathrm{d}S \right| \\ &\leq \frac{\varepsilon_n \|v\|}{1+\|u_n\|} \end{aligned} \tag{3.3}$$

for all $v \in X$ with $\varepsilon_n \to 0^+$, which implies that

$$-\frac{\varepsilon_n\|v\|}{1+\|u_n\|} \leq \langle J'(u_n), v\rangle \leq \frac{\varepsilon_n\|v\|}{1+\|u_n\|}. \tag{3.4}$$

Taking $v = u_n$ in (3.3), it follows from (3.3) and (3.4) that,

$$\begin{aligned} &-\int_\Omega \left[(|\nabla u_n|^p + |u_n|^p) + a(x)(|\nabla u_n|^q + |u_n|^q)\right] \mathrm{d}x \\ &+ \int_\Omega f(x,u_n)u_n \,\mathrm{d}x + \int_\Gamma g(x,u_n)u_n \,\mathrm{d}S \ \leq \varepsilon_n. \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Moreover, by (3.1) we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{q}{p}(|\nabla u_n|^p + |u_n|^p) + a(x)(|\nabla u_n|^q + |u_n|^q)\right) \mathrm{d}x \\ &- \int_\Omega qF(x,u_n)\,\mathrm{d}x - \int_\Gamma qG(x,u_n)\,\mathrm{d}S \leq qM_1. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6) we have

$$\begin{aligned} &\left(\frac{q}{p}-1\right)\int_\Omega (|\nabla u_n|^p + |u_n|^p)\,\mathrm{d}x + \int_\Omega (f(x,u_n)u_n - qF(x,u_n))\,\mathrm{d}x \\ &+ \int_\Gamma (g(x,u_n)u_n - qG(x,u_n))\,\mathrm{d}S \leq M_2, \end{aligned} \tag{3.7}$$

for some $M_2 > 0$. Since $p < q$, we get in particular that

$$\int_\Omega (f(x,u_n)u_n - qF(x,u_n))\,\mathrm{d}x + \int_\Gamma (g(x,u_n)u_n - qG(x,u_n))\,\mathrm{d}S \leq M_2. \tag{3.8}$$

Claim: $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is bounded.

Suppose that $\|u_n\| \to \infty$. We take $v_n = \frac{u_n}{\|u_n\|}$ which implies that $\|v_n\| = 1$. Then we may assume that

$$v_n \rightharpoonup v \quad \text{in } X \quad \text{and} \quad v_n \to v \quad \text{in } L^{r_1}(\Omega) \text{ and } L^{r_2}(\partial\Omega)$$

for some $v \in X$, see Proposition 2.1 (ii), (iv).

Suppose $v = 0$. Let $\mu \geq 1$ and put $\tilde{v}_n = (q\mu)^{\frac{1}{q}} v_n$ for all $n \in \mathbb{N}$. So we have $\tilde{v}_n \to 0$ in $L^{r_1}(\Omega)$ and $L^{r_2}(\partial\Omega)$, which implies that

$$\int_\Omega F(x,\tilde{v}_n)\,\mathrm{d}x + \int_\Gamma G(x,\tilde{v}_n)\,\mathrm{d}S \to 0 \quad \text{as } n \to \infty.$$

Thus, for all $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$\int_\Omega F(x, \tilde{v}_n)\,\mathrm{d}x + \int_\Gamma G(x, \tilde{v}_n)\,\mathrm{d}S < \varepsilon. \tag{3.9}$$

for all $n \geq n_0$. Now we choose $t_n \in [0,1]$ such that

$$J(t_n u_n) = \max\{J(tu_n) \,:\, 0 \leq t \leq 1\} \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Recalling $\|u_n\| \to \infty$ as $n \to \infty$, we can find $n_1 \in \mathbb{N}$ such that

$$0 < \frac{(q\mu)^{\frac{1}{q}}}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_1. \tag{3.11}$$

Taking $\varepsilon = \frac{1}{2}\min\left\{\frac{q^{\frac{p}{q}}}{p}, 1\right\}\mu^{\frac{p}{q}}$ in (3.9), we conclude from (3.9), (3.10) and (3.11) that

$$\begin{aligned}
J(t_n u_n) &\geq J\left(\frac{(q\mu)^{\frac{1}{q}}}{\|u_n\|}u_n\right) = J(\tilde{v}_n) \\
&= \frac{1}{p}q^{\frac{p}{q}}\mu^{\frac{p}{q}}\left(\|\nabla v_n\|_p^p + \|v_n\|_p^p\right) + \mu\left(\|\nabla v_n\|_{a,q}^q + \|v_n\|_{a,q}^q\right) \\
&\quad - \int_\Omega F(x, \tilde{v}_n)\,\mathrm{d}x - \int_\Gamma G(x, \tilde{v}_n)\,\mathrm{d}S \\
&\geq \min\left\{\frac{q^{\frac{p}{q}}}{p}, 1\right\}\mu^{\frac{p}{q}}\varrho(v_n) - \int_\Omega F(x, \tilde{v}_n)\,\mathrm{d}x - \int_\Gamma G(x, \tilde{v}_n)\,\mathrm{d}S \\
&\geq \frac{1}{2}\min\left\{\frac{q^{\frac{p}{q}}}{p}, 1\right\}\mu^{\frac{p}{q}},
\end{aligned}$$

for all $n \geq \max\{n_1, n_0\}$. Since $\mu \geq 1$ is arbitrary, we obtain

$$J(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.12}$$

Recall that (3.1) implies $J(u_n) \leq M_1$ for all $n \in \mathbb{N}$. Obviously $J(0) = 0$. Hence there exists $n_2 \geq \mathbb{N}$ such that

$$t_n \in (0,1) \quad \text{for all } n \geq n_2. \tag{3.13}$$

It follows from (3.10) and (3.13) by using the chain rule that

$$\begin{aligned}
0 &= t_n \frac{\mathrm{d}}{\mathrm{d}t}J(tu_n)|_{t=t_n} = \langle J'(t_n u_n), t_n u_n\rangle \\
&= \|\nabla(t_n u_n)\|_p^p + \|t_n u_n\|_p^p + \|\nabla(t_n u_n)\|_{a,q}^q + \|t_n u_n\|_{a,q}^q \\
&\quad - \int_\Omega f(x, t_n u_n)t_n u_n\,\mathrm{d}x - \int_\Gamma g(x, t_n u_n)t_n u_n\,\mathrm{d}S
\end{aligned}$$

for all $n \geq n_2$, which can be equivalently written as

$$\begin{aligned}
&\|\nabla(t_n u_n)\|_p^p + \|t_n u_n\|_p^p + \|\nabla(t_n u_n)\|_{a,q}^q + \|t_n u_n\|_{a,q}^q \\
&= \int_\Omega f(x, t_n u_n)t_n u_n\,\mathrm{d}x + \int_\Gamma g(x, t_n u_n)t_n u_n\,\mathrm{d}S
\end{aligned} \tag{3.14}$$

for all $n \geq n_2$. Hence, from (3.14) we have

$$qJ(t_n u_n) = \frac{q}{p}\left(\|\nabla(t_n u_n)\|_p^p + \|t_n u_n\|_p^p\right) + \|\nabla(t_n u_n)\|_{a,q}^q + \|t_n u_n\|_{a,q}^q$$

$$
\begin{aligned}
&- \int_\Omega qF(x,t_nu_n)\,\mathrm{d}x - \int_\Gamma qG(x,t_nu_n)\,\mathrm{d}S\\
&= \left(\frac{q}{p}-1\right)\left(\|\nabla (t_nu_n)\|_p^p + \|t_nu_n\|_p^p\right)\\
&\quad + \int_\Omega \left(f(x,t_nu_n)u_n - qF(x,t_nu_n)\right)\mathrm{d}x\\
&\quad + \int_\Gamma \left(g(x,t_nu_n)u_n - qG(x,t_nu_n)\right)\mathrm{d}S
\end{aligned}
$$

for all $n \geq n_2$. It follows from (3.7) that

$$
q\varphi(t_nu_n) \leq M_2 \quad \text{for all } n \geq n_2.
$$

which contradicts (3.12).

Suppose now $v \not\equiv 0$. Let $\hat{\Omega} = \Omega \cup \Gamma$ and define

$$
\hat{\Omega}_+ = \left\{x \in \hat{\Omega} \,:\, v(x) > 0\right\} \quad \text{and} \quad \hat{\Omega}_- = \left\{x \in \hat{\Omega} \,:\, v(x) < 0\right\}.
$$

Then at least one of these measurable sets has a positive Lebesgue measure on $\mathbb{R}^N$. Note that

$$
u_n(x) \to +\infty \quad \text{for a.a.}\, x \in \hat{\Omega}_+ \quad \text{and} \quad u_n(x) \to -\infty \quad \text{for a.a.}\, x \in \hat{\Omega}_-.
$$

Let $\hat{\Omega}_1 = \hat{\Omega}_+ \cup \hat{\Omega}_-$ and let $|\cdot|$ be the Lebesgue measure on $\mathbb{R}^N$. Then, $|\hat{\Omega}_1| > 0$. By (H)(i) and (H)(iii), we have

$$
\begin{aligned}
f(x,y)y - qF(x,y) &\geq c_1 \quad \text{for a.a.}\, x \in \Omega,\\
g(x,y)y - qG(x,y) &\geq c_2 \quad \text{for a.a.}\, x \in \Gamma,
\end{aligned}
$$

for all $y \in \mathbb{R}$ and for some $c_1, c_2 > 0$. Using this, we obtain

$$
\begin{aligned}
&\int_\Omega \left(f(x,u_n)u_n - qF(x,u_n)\right)\mathrm{d}x + \int_\Gamma \left(g(x,u_n)u_n - qG(x,u_n)\right)\mathrm{d}S\\
&= \int_{\hat{\Omega}_1} \left(f(x,u_n)u_n - qF(x,u_n) + g(x,u_n)u_n - qG(x,u_n)\right)\mathrm{d}x\\
&\quad + \int_{\Omega\setminus\hat{\Omega}_1} \left(f(x,u_n)u_n - qF(x,u_n) + g(x,u_n)u_n - qG(x,u_n)\right)\mathrm{d}x\\
&\geq \int_{\hat{\Omega}_1} \left(f(x,u_n)u_n - qF(x,u_n) + g(x,u_n)u_n - qG(x,u_n)\right)\mathrm{d}x + c_3|\Omega \setminus \hat{\Omega}_1|
\end{aligned}
$$

for some $c_3 = \min\{c_1, c_2\} > 0$. From (H)(iii) it follows that

$$
\int_\Omega \left(f(x,u_n)u_n - qF(x,u_n)\right)\mathrm{d}x + \int_\Gamma \left(g(x,u_n)u_n - qG(x,u_n)\right)\mathrm{d}S \to +\infty,
$$

which contradicts (3.8). Therefore, $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is bounded. This proves the claim.

From the boundedness of the sequence, we can find a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$
u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad u_n \to u \quad \text{in } L^{r_1}(\Omega) \text{ and in } L^{r_2}(\partial\Omega).
$$

We choose $v = u_n - u$ in (3.3) and obtain using the convergence properties above that

$$
\lim_{n\to\infty} \langle L(u_n), u_n - u\rangle_{\mathcal{H}} = 0.
$$

Therefore, it follows that $u_n \to u$ in X since L is a mapping of type (S_+), see Proposition 2.3. □

Next, we prove that $J\colon X \to \mathbb{R}$ is coercive.

Proposition 3.2. *Let hypotheses* (1.2) *and* (H) *be satisfied, then the energy functional* $J\colon X \to \mathbb{R}$ *is coercive.*

Proof. Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(x,t)}{|t|^q}\right) &= \frac{f(x,t)|t|^q - q|t|^{q-2}tF(x,t)}{|t|^{2q}}\\ &= \frac{|t|^{q-2}t\big(f(x,t)t - qF(x,t)\big)}{|t|^{2q}}.\end{aligned}$$

From hypothesis (H)(iii), for any $\delta > 0$, there exists $M_\delta > 0$ such that

$$f(x,t)t - qF(x,t) \geq \delta \quad \text{for a.a. } x \in \Omega \text{ and for all } |t| \geq M_\delta.$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(x,t)}{|t|^q}\right) \begin{cases} \geq \dfrac{\delta}{t^{q+1}}, & \text{if } t \geq M_\delta,\\ \leq -\dfrac{\delta}{|t|^{q+1}}, & \text{if } t \leq -M_\delta.\end{cases}$$

Integrating this inequality, we obtain

$$\frac{F(x,t)}{|t|^q} - \frac{F(x,u)}{|u|^q} \geq -\frac{\delta}{q}\left(\frac{1}{|t|^q} - \frac{1}{|u|^q}\right) \tag{3.15}$$

for a.a. $x \in \Omega$ and for all $|t| \geq |u| \geq M_\delta$. By hypothesis (H)(ii), for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$F(x,t) \leq \frac{1}{q}\left(\lambda_1(q) + \varepsilon\right)|t|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } |t| \geq M_\varepsilon.$$

Using this inequality in (3.15) and letting $|t| \to \infty$, we obtain

$$\frac{1}{q}(\lambda_1(q) + \varepsilon) - \frac{F(x,u)}{|u|^q} \geq \frac{\delta}{q}\frac{1}{|u|^q},$$

that is,

$$(\lambda_1(q) + \varepsilon)|u|^q - qF(x,u) \geq \delta$$

for a.a. $x \in \Omega$ and for all $|u| \geq M = \max\{M_\delta, M_\varepsilon\}$. Letting $\varepsilon \to 0$, we obtain

$$\lambda_1(q)|u|^q - qF(x,u) \geq \delta. \tag{3.16}$$

Similar arguments apply to $G(\cdot,\cdot)$, that is, we can show

$$\lambda_1(q)|u|^q - qG(x,u) \geq \delta \tag{3.17}$$

for a.a. $x \in \Gamma$ and for all $|u| \geq M$.

Now, we claim that $J\colon X \to \mathbb{R}$ is coercive. Indeed, for any $u \in X$, it follows from $X \subset W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,\mathcal{K}}$, (2.4) and (3.16) as well as (3.17) that

$$\begin{aligned}J(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\,\mathrm{d}x + \frac{1}{q}\int_\Omega a(x)(|\nabla u|^q + |u|^q)\,\mathrm{d}x\\ - \int_\Omega F(x,u)\,\mathrm{d}x - \int_\Gamma G(x,u)\,\mathrm{d}S\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{p}\int_\Omega (|\nabla u|^p+|u|^p)\,\mathrm{d}x+\frac{1}{q}\lambda_1(q)\left(\int_\Omega |u|^q\,\mathrm{d}x+\int_\Gamma |u|^q\,\mathrm{d}S\right)\\
&\quad -\int_\Omega F(x,u)\,\mathrm{d}x-\int_\Gamma G(x,u)\,\mathrm{d}S\\
&=\frac{1}{p}\int_\Omega (|\nabla u|^p+|u|^p)\,\mathrm{d}x+\frac{1}{q}\int_\Omega \left(\lambda_1(q)\,|u|^q-qF(x,u)\right)\,\mathrm{d}x\\
&\quad +\frac{1}{q}\int_\Gamma \left(\lambda_1(q)\,|u|^q-qG(x,u)\right)\,\mathrm{d}S\\
&\geq \frac{\delta}{q}\left(|\Omega|+|\Gamma|\right)
\end{aligned}
$$

for a.a. $x\in\Omega\cup\Gamma$ and for all $|u|\geq M$, which implies that $J\colon X\to\mathbb{R}$ is coercive since δ is arbitrary and $|\Omega|,|\Gamma|>0$. □

Finally, we will prove that $J\colon X\to\mathbb{R}$ has a local $(1,1)$-linking at 0.

Proposition 3.3. *Let hypotheses* (1.2) *and* (H) *be satisfied, then the energy functional $J\colon X\to\mathbb{R}$ has a local $(1,1)$-linking at 0.*

Proof. Let V denote the space spanned by $\tilde{u}_1(p)$ and let

$$W=\left\{u\in X\ :\ \int_\Omega |\tilde{u}_1|^{p-1}u\,\mathrm{d}x+\int_\Gamma |\tilde{u}_1|^{p-1}u\,\mathrm{d}S=0\right\}.$$

We claim that

$$X=V\oplus W. \tag{3.18}$$

Indeed, for any $u\in X$, writing $u=\alpha\tilde{u}_1+w$ where $w\in X$ and

$$\alpha=\tilde{\lambda}_1(p)\frac{\int_\Omega |\tilde{u}_1|^{p-1}u\,\mathrm{d}x+\int_\Gamma |\tilde{u}_1|^{p-1}u\,\mathrm{d}S}{\int_\Omega |\nabla\tilde{u}_1|^p\,\mathrm{d}x}.$$

Recall that

$$\tilde{\lambda}_1(p)=\frac{\int_\Omega |\nabla\tilde{u}_1|^p\,\mathrm{d}x}{\int_\Omega |\tilde{u}_1|^p\,\mathrm{d}x+\int_\Gamma |\tilde{u}_1|^p\,\mathrm{d}S},$$

see (2.5). Thus we obtain

$$\int_\Omega |\tilde{u}_1|^{p-1}w\,\mathrm{d}x+\int_\Gamma |\tilde{u}_1|^{p-1}w\,\mathrm{d}S=0.$$

Hence, $w\in W$ and our claim is true.

We may assume that K_J is finite, otherwise we would have found infinite number of critical points of J which are solutions of problem (1.1). Now, let

$$B_\rho=\{u\in X\ :\ \|u\|\leq\rho\}$$

and choose $\rho\in(0,1)$ small enough such that $K_J\cap\overline{B}_\rho=\{0\}$. Furthermore, let $\varepsilon>0$ small enough such that the hypothesis (H)(iv) holds, that is,

$$\tilde{\lambda}_1(p)\,|t|^p\leq\theta\,|t|^p\leq pF(x,t)\leq\tilde{\lambda}\,|t|^p\leq\tilde{\lambda}_2(p)\,|t|^p\,, \tag{3.19}$$

$$\tilde{\lambda}_1(p)\,|t|^p\leq\theta\,|t|^p\leq pG(x,t)\leq\tilde{\lambda}\,|t|^p\leq\tilde{\lambda}_2(p)\,|t|^p\,, \tag{3.20}$$

for all $|t| \leq \varepsilon$. Recall that all norms are equivalent on a finite-dimensional normed space, see, for example, Papageorgiou-Winkert [34, Proposition 3.1.17, p.183]. Thus making $\rho \in (0,1)$ smaller if necessary, we can obtain that $\|u\| \leq \rho$ implies

$$|u| \leq \varepsilon \quad \text{for all } u \in V. \tag{3.21}$$

Then for $t\tilde{u}_1 = u \in V \cap \overline{B}_\rho$ with $t \in (0,1)$, by (3.19), (3.20) and (3.21) we have

$$\begin{aligned}
&J(u)\\
&\leq \frac{\tilde{\lambda}_1(p)t^p}{p}\left(\int_\Omega |\tilde{u}_1|^p \,\mathrm{d}x + \int_\Gamma |\tilde{u}_1|^p \,\mathrm{d}S\right) + \frac{t^q}{q}\int_\Omega a(x)(|\nabla \tilde{u}_1|^q + |\tilde{u}_1|^q)\,\mathrm{d}x\\
&\quad - \frac{\theta t^p}{p}\left(\int_\Omega |\tilde{u}_1|^p \,\mathrm{d}x + \int_\Gamma |\tilde{u}_1|^p \,\mathrm{d}S\right)\\
&= \frac{t^q}{q}\int_\Omega a(x)(|\nabla \tilde{u}_1|^q + |\tilde{u}_1|^q)\,\mathrm{d}x - \frac{t^p}{p}\left(\theta - \tilde{\lambda}_1(p)\right)\left(\int_\Omega |\tilde{u}_1|^p \,\mathrm{d}x + \int_\Gamma |\tilde{u}_1|^p \,\mathrm{d}S\right)\\
&= c_1 t^q - c_2 t^p \quad \text{for some } c_1, c_2 > 0.
\end{aligned}$$

Taking $\rho \in (0,1)$ small enough yields

$$J|_{V\cap \overline{B}_\rho} < 0 \tag{3.22}$$

since $1 < p < q$. Recall that $X = V \oplus W$. It is clear that

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geq \tilde{\lambda}_2(p)\left(\int_\Omega |u|^p \,\mathrm{d}x + \int_\Gamma |u|^p \,\mathrm{d}S\right) \quad \text{for all } u \in W. \tag{3.23}$$

Then for all $u \in W \cap \overline{B}_\rho \setminus \{0\}$, by hypothesis (H)(iv) and (3.23), we have

$$\begin{aligned}
&J(u)\\
&= \int_\Omega \left(\frac{1}{p}(|\nabla u|^p + |u|^p) + \frac{a(x)}{q}(|\nabla u|^q + |u|^q)\right)\mathrm{d}x - \int_{\Omega\cap\{|u|<\varepsilon\}} F(x,u)\,\mathrm{d}x\\
&\quad - \int_{\Omega\cap\{|u|\geqslant\varepsilon\}} F(x,u)\,\mathrm{d}x - \int_{\Gamma\cap\{|u|<\varepsilon\}} G(x,u)\,\mathrm{d}S - \int_{\Gamma\cap\{|u|\geqslant\varepsilon\}} G(x,u)\,\mathrm{d}S\\
&\geq \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\tilde{\lambda}}{p}\left(\int_\Omega |u|^p \,\mathrm{d}x + \int_\Gamma |u|^p \,\mathrm{d}S\right) + \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x\\
&\quad + \frac{1}{q}\int_\Omega a(x)(|\nabla u|^q + |u|^q)\,\mathrm{d}x - \int_{\Omega\cap\{|u|\geqslant\varepsilon\}} F(x,u)\,\mathrm{d}x - \int_{\Gamma\cap\{|u|\geqslant\varepsilon\}} G(x,u)\,\mathrm{d}S\\
&\geq \frac{1}{p}\left(1 - \frac{\tilde{\lambda}}{\tilde{\lambda}_2(p)}\right)\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega a(x)(|\nabla u|^q + |u|^q)\,\mathrm{d}x\\
&\quad + \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - c_1\int_\Omega |u|^{r_1}\,\mathrm{d}x - c_2\int_\Omega |u|^{r_2}\,\mathrm{d}x\\
&\geq \frac{1}{q}\left(1 - \frac{\tilde{\lambda}}{\tilde{\lambda}_2(p)}\right)\varrho(u) - c_3\|u\|^{r_1} - c_4\|u\|^{r_2}\\
&\geq \frac{1}{q}\left(1 - \frac{\tilde{\lambda}}{\tilde{\lambda}_2(p)}\right)\|u\|^q - c_3\|u\|^{r_1} - c_4\|u\|^{r_2}
\end{aligned}$$

Since $r_1, r_2 > q$ and $\rho \in (0,1)$ is sufficiently small, we have

$$J|_{W\cap\overline{B}_\rho\setminus\{0\}} > 0. \tag{3.24}$$

Now let

$$U=\bar{B}_\rho,\quad E_0=V\cap\partial B_\rho\quad\text{and}\quad E=V\cap\overline{B}_\rho.$$

Then we have $0\notin E_0\subseteq E\subseteq U$ and from (3.22) as well as (3.24) we obtain $E_0\cap W=\emptyset$.

From (3.18), for every $u\in X$, we can write it in the form

$$u=v+w\quad\text{with } v\in V \text{ and } w\in W.$$

Let $h\colon[0,1]\times(X\setminus W)\to X\setminus W$ defined by

$$h(t,u)=(1-t)u+t\rho\frac{v}{\|v\|}\quad\text{for all } [0,1] \text{ and for all } u\in X\setminus W.$$

This implies

$$h(0,u)=u\quad\text{and}\quad h(1,u)=\rho\frac{v}{\|v\|}\in E_0.$$

By Papageorgiou-Rădulescu [29, Definition 5.3.10], we know that E_0 is a deformation retract of $X\setminus W$. So we have that

$$i_*\colon H_0(E_0)\to H_0(X\setminus\{0\})$$

is an isomorphism, see Eilenberg-Steenrod [13, Theorem 11.5] and Papageorgiou-Rădulescu-Repovš [32, Remark 6.1.6]. Moreover, $E=V\cap\overline{B}_\rho$ is contractible. Hence $H_0(E,E_0)=0$ due to Eilenberg-Steenrod [13, Theorem 11.5]. Let $j_*\colon H_0(E_0)\to H_0(E)$, then we have $\dim\operatorname{im} j_*=1$, see Eilenberg-Steenrod [13, Remark 6.1.26]. Therefore, we have

$$\dim\operatorname{im} i_*-\dim\operatorname{im} j_*=2-1=1.$$

Then we obtain that $J\colon X\to\mathbb{R}$ has a local $(1,1)$-linking at 0, see Definition 2.6. $\square$

By Proposition 3.3 and Theorem 6.6.17 of Papageorgiou-Rădulescu-Repovš [32], we know that

$$\dim C_1(J,0)\geq 1.$$

Based on the results above, we are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First, since $W^{1,\mathcal{H}}(\Omega)\hookrightarrow L^{r_1}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)\hookrightarrow L^{r_2}(\partial\Omega)$ are compact due to Proposition 2.1, we know that $J\colon X\to\mathbb{R}$ is sequentially weakly lower semicontinuous. From Proposition 3.2 we conclude that $J\colon X\to\mathbb{R}$ is coercive as well. Therefore, by Proposition 2.5, we deduce that there exists $u_1\in X$ such that

$$J(u_1)=\min\{J(u)\,:\,u\in X\}.$$

By the proof of Proposition 3.3, we see that

$$J(u_1)<0=J(0),$$

which implies that $u_1\neq 0$ and $u_1\in K_J$, that is, $u_1\in K_J$ is a nontrivial solution of problem (1.1). Moreover, it follows from Propositions 2.7, 3.1 and 3.3 that there exists $u_2\in K_J$ such that $u_2\notin\{0,u_1\}$, which implies that u_2 is the second nontrivial solution of problem (1.1). From Theorem 3.1 of Gasiński-Winkert [15] we conclude that u_1 and u_2 are bounded. $\square$

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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