COUPLED DOUBLE PHASE OBSTACLE SYSTEMS INVOLVING NONLOCAL FUNCTIONS AND MULTIVALUED CONVECTION TERMS

YONGJIAN LIU, VAN THIEN NGUYEN, PATRICK WINKERT, AND SHENGDA ZENG

Abstract. In this paper we study a new kind of coupled elliptic obstacle problems driven by double phase operators and with multivalued right-hand sides depending on the gradients of the solutions. Based on an abstract existence theorem for generalized mixed variational inequalities involving multivalued mappings due to Kenmochi [20], we prove the nonemptiness and compactness of the weak solution set of the coupled elliptic obstacle system.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$, we are concerned with the study of the following coupled double phase obstacle system

$$\begin{aligned} -\operatorname{div}\left(a_1(u_1)|\nabla u_1|^{p_1-2}\nabla u_1+\mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1\right) &\in f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) && \text{in } \Omega,\\ -\operatorname{div}\left(a_2(u_2)|\nabla u_2|^{p_2-2}\nabla u_2+\mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2\right) &\in f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) && \text{in } \Omega,\\ u_1(x)\leq \Phi_1(x) \text{ and } u_2(x) &\leq \Phi_2(x) && \text{on } \Omega,\\ u_1=u_2&=0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where, for $i=1,2$, $\Phi_i\colon \Omega\to\mathbb{R}$ are measurable obstacle functions, $f_i\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to 2^{\mathbb{R}}$ are multivalued convection functions, $a_i\colon L^{p_i^*}(\Omega)\to(0,+\infty)$ are nonlocal terms (see (H$_2$)-(H$_4$) for the precise assumptions) and the exponents p_i,q_i as well as the weight functions μ_i satisfy the following conditions:

(H$_1$): $1<p_i<N$, $p_i<q_i<p_i^*$ and $0\leq\mu_i(\cdot)\in L^\infty(\Omega)$ for $i=1,2$, where p_i^* is the critical exponent of p_i for $i=1,2$ given by

$$p_i^*:=\frac{Np_i}{N-p_i}. \tag{1.2}$$

The operators involved in problem (1.1) are called nonlocal double phase operators given by

$$\operatorname{div}\left(a_i(u_i)|\nabla u_i|^{p_i-2}\nabla u_i+\mu_i(x)|\nabla u_i|^{q_i-2}\nabla u_i\right),\quad u\in W_0^{1,\mathcal{H}_i}(\Omega), \tag{1.3}$$

with $W_0^{1,\mathcal{H}_i}(\Omega)$ being an appropriate Musielak-Orlicz Sobolev space for $i=1,2$. Note that if $a_i\equiv 1$ and $\mu_i\equiv 0$, the operators in (1.1) reduce to the p_i-Laplacians for $i=1,2$. If $a_i\equiv 1$, (1.3) become the usual double phase operators which are related to the energy functionals

$$\Psi_i(u)=\int_\Omega\left(|\nabla u|^{p_i}+\mu_i(x)|\nabla u|^{q_i}\right)\mathrm{d}x. \tag{1.4}$$

Functionals of the form (1.4) appeared for the first time as examples in models in order to describe strongly anisotropic materials in the context of homogenization and elasticity, see Zhikov [32], we refer also to applications in the study of duality theory and of the Lavrentiev gap phenomenon, see Zhikov [33, 34]. A first mathematical framework for such type of functionals in (1.4) has been done by Baroni-Colombo-Mingione [2], see also the related works by the same authors in [3, 4] and of De Filippis-Mingione [9] about nonautonomous integrals.

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In this paper, our main goal is to study the nonlocal obstacle system (1.1) involving multivalued convection in the right-hand sides concerning the nonemptiness and compactness of the weak solution set of (1.1). Note that if $\Phi_i(x) = +\infty$ for a. a. $x \in \Omega$, $a_i(u_i) = 1$ for all $u_i \in W_0^{1,\mathcal{H}_i}(\Omega)$, and f_i are single-valued operators for $i = 1, 2$, then problem (1.1) reduces to the one studied by Marino-Winkert [26]. However, the main method applied in the present paper is completely different from the one used in [26]. Indeed, we make use of an abstract existence theorem for generalized mixed variational inequalities involving multivalued mappings due to Kenmochi [20], but in [26], the authors applied the main surjectivity theorem for pseudomonotone operators to obtain the existence of a weak solution.

To the best of our knowledge, this is the first work dealing with nonlocal double phase systems with multivalued right-hand sides. However, even without nonlocal terms (that is, $a_i \equiv 1$ for $i = 1, 2$) and single-valued right-hand sides with convection, besides the work of Marino-Winkert [26] mentioned above, there exists only another paper recently published by Guarnotta-Livrea-Winkert [17] for nonlinear Neumann double phase systems with variable exponents by developing a sub-supersolution approach. In the case of a nonlocal problem with a single equations we refer to the current work of Liu-Zeng-Gasiński-Kim [25].

Finally, we mention some recent results for elliptic systems with convection term for p-Laplace or (p, q)-Laplace operator. We refer the works of Carl-Motreanu [5], Guarnotta-Marano [15], Guarnotta-Marano-Moussaoui [18], Guarnotta-Marano-Moussaoui [18] and Faria-Miyagaki-Pereira [11], see also Godoi-Miyagaki-Rodrigues [10] for Neumann systems without convection. For single equations involving the double phase operator with different type of right-hand sides we mention the following papers by Colasuonno-Squassina [7], Farkas-Winkert [12], Gasiński-Winkert [13, 14], Kim-Kim-Oh-Zeng [21], Liu-Dai [23], Liu-Migórski-Nguyen-Zeng [24], Perera-Squassina [28], Zeng-Bai-Gasiński-Winkert [29], Zeng-Rădulescu-Winkert [30, 31], Cen-Khan-Motreanu-Zeng [6] see also the references therein.

The paper is organized as follows. In Section 2 we recall some main properties of Musielak-Orlicz Sobolev spaces, the nonlocal double phase operator as well as the Dirichlet eigenvalue problem for the r-Laplacian ($1 < r < \infty$). In Section 3 we first state the hypotheses on the data of problem (1.1), formulate the definition of a weak solution and prove our main result about the nonemptiness and compactness of the weak solution set of system (1.1), see Theorem 3.4.

2. Preliminaries

In this section we recall some facts about Musielak-Orlicz Sobolev spaces and the properties of the double phase operator. To this end, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. We denote by $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_{r,\Omega}$ for any $1 \leq r \leq \infty$. Suppose that condition (H_1) holds and let $M(\Omega)$ be the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, then the Musielak-Orlicz space $L^{\mathcal{H}_i}(\Omega)$ is defined by

$$L^{\mathcal{H}_i}(\Omega) = \left\{ u \in M(\Omega) \,:\, \int_\Omega \left(|u|^{p_i} + \mu_i(x)|u|^{q_i} \right) \mathrm{d}x < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_i} = \inf \left\{ \tau > 0 \,:\, \int_\Omega \left(\left|\frac{u}{\tau}\right|^{p_i} + \mu_i(x) \left|\frac{u}{\tau}\right|^{q_i} \right) \mathrm{d}x \leq 1 \right\}$$

for $i = 1, 2$. The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_i}(\Omega)$ is defined by

$$W^{1,\mathcal{H}_i}(\Omega) = \left\{ u \in L^{\mathcal{H}_i}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}_i}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}_i} = \|\nabla u\|_{\mathcal{H}_i} + \|u\|_{\mathcal{H}_i},$$

where $\|\nabla u\|_{\mathcal{H}_i} = \| \, |\nabla u| \, \|_{\mathcal{H}_i}$ and $i = 1, 2$. Moreover, the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}_i}(\Omega)$ is denoted by $V_i := W_0^{1,\mathcal{H}_i}(\Omega)$ and from Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.12] we know that V_i are reflexive Banach spaces for $i = 1,2$. Due to Proposition 2.18 of Crespo-Blanco-Gasiński-Harjulehto-Winkert [8] we can equip V_i with the equivalent norm

$$\|u\|_{V_i} := \|\nabla u\|_{\mathcal{H}_i} \quad \text{for all } u \in V_i \text{ and } i = 1,2.$$

Furthermore, we define

$$L^{q_i}_{\mu_i}(\Omega) = \left\{ u \in M(\Omega) \,:\, \int_\Omega \mu_i(x)|u|^{q_i}\,\mathrm{d}x < +\infty \right\}$$

and endow it with the seminorm

$$\|u\|_{q_i,\mu_i,\Omega} = \left(\int_\Omega \mu_i(x)|u|^{q_i}\,\mathrm{d}x \right)^{\frac{1}{q_i}}$$

for $i = 1,2$.

The following proposition can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.13].

Proposition 2.1. *Let hypotheses* (H_1) *be satisfied and let*

$$\rho_{\mathcal{H}_i}(u) := \int_\Omega \mathcal{H}_i(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p_i} + \mu_i(x)|u|^{q_i}\right)\mathrm{d}x.$$

For $i = 1,2$ *we have the following assertions.*

(i) *If* $u \neq 0$, *then* $\|u\|_{\mathcal{H}_i} = \lambda$ *if and only if* $\rho_{\mathcal{H}_i}(\frac{u}{\lambda}) = 1$.
(ii) $\|u\|_{\mathcal{H}_i} < 1$ *(resp.* > 1, $= 1$*) if and only if* $\rho_{\mathcal{H}_i}(u) < 1$ *(resp.* > 1, $= 1$*).*
(iii) *If* $\|u\|_{\mathcal{H}_i} < 1$, *then* $\|u\|^{q_i}_{\mathcal{H}_i} \leqslant \rho_{\mathcal{H}_i}(u) \leqslant \|u\|^{p_i}_{\mathcal{H}_i}$.
(iv) *If* $\|u\|_{\mathcal{H}_i} > 1$, *then* $\|u\|^{p_i}_{\mathcal{H}_i} \leqslant \rho_{\mathcal{H}_i}(u) \leqslant \|u\|^{q_i}_{\mathcal{H}_i}$.
(v) $\|u\|_{\mathcal{H}_i} \to 0$ *if and only if* $\rho_{\mathcal{H}_i}(u) \to 0$.
(vi) $\|u\|_{\mathcal{H}_i} \to +\infty$ *if and only if* $\rho_{\mathcal{H}_i}(u) \to +\infty$.
(vii) $\|u\|_{\mathcal{H}_i} \to 1$ *if and only if* $\rho_{\mathcal{H}_i}(u) \to 1$.
(viii) *If* $u_n \to u$ *in* $L^{\mathcal{H}_i}(\Omega)$, *then* $\rho_{\mathcal{H}_i}(u_n) \to \rho_{\mathcal{H}_i}(u)$.

Moreover, from Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.16] we have the compact embedding

$$W^{1,\mathcal{H}_i}_0(\Omega) \hookrightarrow L^{r_i}(\Omega) \tag{2.1}$$

whenever $1 \leq r_1 < p_i^*$ with p_i^* being the critical Sobolev exponent given in (1.2) for $i = 1,2$.

For $i = 1,2$ let $\mathcal{E}_i \colon V_i \to V_i^*$ be defined by

$$\langle \mathcal{E}_i(u_i), v_i\rangle_{V_i} := \int_\Omega \left(|\nabla u_i|^{p_i-2}\nabla u_i + \mu_i(x)|\nabla u_i|^{q_i-2}\nabla u_i\right)\cdot \nabla v_i\,\mathrm{d}x \tag{2.2}$$

for all $u_i, v_i \in V_i$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{V_i}$ is the duality pairing between V_i and its dual space V_i^* for $i = 1,2$. The operators $\mathcal{E}_i \colon V_i \to V_i^*$ have the following properties, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 3.4] for $i = 1,2$.

Proposition 2.2. *Let hypotheses* (H_1) *be satisfied. Then, the operators defined in* (2.2) *are bounded, continuous, strictly monotone and of type* (S_+) *for* $i = 1,2$.

Now, let us consider the eigenvalue problem for the r-Laplacian with homogeneous Dirichlet boundary condition and $1 < r < \infty$ defined by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2}u \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

It is known that the first eigenvalue $\lambda_{1,r}$ of (2.3) is positive, simple, and isolated. Moreover, it can be variationally characterized through

$$\lambda_{1,r} = \inf_{u \in W^{1,r}(\Omega)} \left\{ \int_\Omega |\nabla u|^r\,dx : \int_\Omega |u|^r\,dx = 1 \right\}, \tag{2.4}$$

see Lê [22]. Hence, we get from (2.4) the inequality

$$\|u\|_{r,\Omega}^r \leq \left(\lambda_{1,r}^{-1}\right) \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W_0^{1,r}(\Omega). \tag{2.5}$$

3. Main results

In this section we state and prove our main result about the solvability of the system (1.1). First we are going to formulate our precise assumptions on the nonlocal terms, the obstacle functions and the right-hand sides of (1.1).

(H_2): $a_i\colon L^{p_i^*}(\Omega) \to (0,+\infty)$ are bounded and continuous such that $c_{a_i} := \inf_{u\in V_i} a_i(u) > 0$ for $i=1,2$ and $\Phi_i\colon \Omega \to \mathbb{R}$ are measurable functions for $i=1,2$.

(H_3): For $i=1,2$, the multivalued mappings $f_i\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to 2^{\mathbb{R}}$ are such that $0\notin f_i(x,0,0,0,0)$ for a. a. $x\in\Omega$, and fulfill the following conditions:

(i) for all $(s_1,s_2,\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ and for a. a. $x\in\Omega$, the sets $f_i(x,s_1,s_2,\eta_1,\eta_2)$ are nonempty, bounded, closed and convex in $\mathbb{R}$;

(ii) for all $(s_1,s_2,\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, the multivalued functions $x\mapsto f_i(x,s_1,s_2,\eta_1,\eta_2)$ are measurable in Ω, and $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \ni (s_1,s_2,\eta_1,\eta_2)\mapsto f(x,s_1,s_2,\eta_1,\eta_2)\subset\mathbb{R}$ are u.s.c. for a. a. $x\in\Omega$;

(iii) there exist constants

$$\alpha_{1,i},\alpha_{2,i},\alpha_{3,i},\alpha_{4,i},\alpha_{5,i},\alpha_{6,i},\beta_{1,i},\beta_{2,i},\beta_{3,i},\beta_{4,i},\beta_{5,i},\beta_{6,i},\beta_{7,i},\beta_{8,i}\geq 0$$

and functions $\delta_i\in L^{\frac{r_i}{r_i-1}}(\Omega)_+$ such that

$$\begin{aligned}|f_i(x,s_1,s_2,\eta_1,\eta_2)| &\leq \alpha_{1,i}|s_1|^{\beta_{1,i}}+\alpha_{2,i}|s_2|^{\beta_{2,i}}+\alpha_{3,i}|s_1|^{\beta_{3,i}}|s_2|^{\beta_{4,i}}+\alpha_{4,i}|\eta_1|^{\beta_{5,i}}\\ &\quad+\alpha_{5,i}|\eta_2|^{\beta_{6,i}}+\alpha_{6,i}|\eta_1|^{\beta_{7,i}}|\eta_2|^{\beta_{8,i}}+\delta_i(x)\end{aligned}$$

for a. a. $x\in\Omega$ and for all $(s_1,s_2,\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, where $1<r_i<p_i^*$ and the following compatibility conditions hold:

(I) $\beta_{1,1}\leq r_1-1$, (II) $\beta_{2,1}\leq \frac{r_2}{r_1'}$, (III) $\frac{\beta_{3,1}}{r_1}+\frac{\beta_{4,1}}{r_2}\leq\frac{1}{r_1'}$,

(IV) $\beta_{5,1}\leq\frac{p_1}{r_1'}$, (V) $\beta_{6,1}\leq\frac{p_2}{r_1'}$, (VI) $\frac{\beta_{7,1}}{p_1}+\frac{\beta_{8,1}}{p_2}\leq\frac{1}{r_1'}$,

(VII) $\beta_{1,2}\leq\frac{r_1}{r_2'}$, (VIII) $\beta_{2,2}\leq r_2-1$, (IX) $\frac{\beta_{3,2}}{r_1}+\frac{\beta_{4,2}}{r_2}\leq\frac{1}{r_2'}$,

(X) $\beta_{5,2}\leq\frac{p_1}{r_2'}$, (XI) $\beta_{6,2}\leq\frac{p_2}{r_2'}$, (XII) $\frac{\beta_{7,2}}{p_1}+\frac{\beta_{8,2}}{p_2}\leq\frac{1}{r_2'}$.

(H_4): There exist constants $\pi_i\geq 0$ and a function $0\leq\omega(\cdot)\in L^1(\Omega)$ satisfying the following inequality for a. a. $x\in\Omega$ and for all $(s_1,s_2,\eta_1,\eta_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$

$$\theta_1 s_1+\theta_2 s_2\leq\pi_1\left(|\eta_1|^{p_1}+|\eta_2|^{p_2}\right)+\pi_2\left(|s_1|^{p_1}+|s_2|^{p_2}\right)+\omega(x)$$

for all $\theta_i\in f_i(x,s_1,s_2,\eta_1,\eta_2)$ for $i=1,2$.

Next, we give the definition of a weak solution of the system (1.1).

Definition 3.1. *We say that a pair of functions $(u_1,u_2)\in K_1\times K_2$ is a weak solution of problem (1.1), if there exist functions $\xi_i\in L^{r_i'}(\Omega)$ with $\xi_i(x)\in f_i(x,u_1,u_2,,\nabla u_1,\nabla u_2)$ for a. a. $x\in\Omega$ such that the following inequalities hold*

$$\int_\Omega \left(a_i(u_i)|\nabla u_i|^{p_i-2}\nabla u_i+\mu_i(x)|\nabla u_i|^{q_i-2}\nabla u_i\right)\cdot\nabla(w_i-u_i)\,\mathrm{d}x\geq\int_\Omega\xi_i(x)(w_i-u_i)\,\mathrm{d}x$$

for all $w_i\in K_i$ where K_i are defined by

$$K_i:=\{u_i\in V_i\,:\,u_i(x)\leq\Phi_i(x)\text{ for a. a. }x\in\Omega\}$$

for $i=1,2$.

Remark 3.2. *From the choices of r_1, r_2 in* (H_2)(iii) *along with* (2.1) *we have the compact embedding*

$$(V_1 \times V_2, \|\cdot\|_{V_1} + \|\cdot\|_{V_2}) \hookrightarrow (L^{r_1}(\Omega) \times L^{r_2}(\Omega), \|\cdot\|_{r_1,\Omega} + \|\cdot\|_{r_2,\Omega}).$$

Remark 3.3. *The following functions satisfy hypothesis* (H_2)

$$a_1(u_1) := e^{\|u_1\|_{V_1}}, \quad a_1(u_1) := c_{a_1} + \|u_1\|_{V_1}, \quad a_1(u_1) := c_{a_1} + \ln(1 + \|u_1\|_{V_1})$$
$$a_2(u_2) := c_{a_2} + \frac{\|u_2\|_{V_2}^2}{1 + \|u_2\|_{p_2,\Omega}}, \quad a_2(u_2) := c_{a_2} + \|u_2\|_{p_2,\Omega}\|u_2\|_{V_2},$$
$$a_2(u_2) := e^{\|u_2\|_{V_2}} + \ln(1 + \|u_2\|_{\mu_2,q_2,\Omega})$$

for all $u_1 \in V_1$ and for all $u_2 \in V_2$ with $c_{a_1}, c_{a_2} > 0$.

The main result of this paper is stated by the next theorem.

Theorem 3.4. *Let hypotheses* (H_1)– (H_4) *be satisfied. Then the weak solution set of problem* (1.1) *is nonempty and compact in $V_1 \times V_2$ provided one of the following assertions is satisfied:*

(i) *a_1 and a_2 are coercive in V_1 and V_2, respectively;*
(ii) $\min\left\{c_{a_1} - \pi_1 - \pi_2\lambda_{1,p_1}^{-1}, c_{a_2} - \pi_1 - \pi_2\lambda_{1,p_2}^{-1}\right\} > 0$, *where λ_{1,p_i} is the first eigenvalue of the p_i-Laplace problem with homogeneous Dirichlet boundary condition for $i = 1, 2$.*

Proof. From hypotheses (H_3)(i), (ii) and the Yankov-von Neumann-Aumann selection theorem (see Papageorgiou-Winkert [27, Theorem 2.7.25]) it follows that for each $(u_1, u_2) \in V_1 \times V_2$ we find measurable selections $\xi_i \colon \Omega \to \mathbb{R}$ such that $\xi_i(x) \in f_i(x, u_1, u_2, \nabla u_1, \nabla u_2)$ for a. a. $x \in \Omega$. From hypotheses (H_3) along with Hölder's inequality we obtain

$$\begin{aligned}
\|\xi_1\|_{r_1',\Omega}^{r_1'} &= \int_\Omega |\xi_1(x)|^{r_1'} \,\mathrm{d}x \\
&\leq \int_\Omega \big(\alpha_{1,1}|u_1|^{\beta_{1,1}} + \alpha_{2,1}|u_2|^{\beta_{2,1}} + \alpha_{3,1}|u_1|^{\beta_{3,1}}|u_2|^{\beta_{4,1}} + \alpha_{4,1}|\nabla u_1|^{\beta_{5,1}} \\
&\qquad + \alpha_{5,1}|\nabla u_2|^{\beta_{6,1}} + \alpha_{6,1}|\nabla u_1|^{\beta_{7,1}}|\nabla u_2|^{\beta_{8,1}} + \delta_1(x)\big)^{r_1'} \,\mathrm{d}x \\
&\leq C_0\bigg(\|u_1\|_{\beta_{1,1}r_1',\Omega}^{\beta_{1,1}r_1'} + \|u_2\|_{\beta_{2,1}r_1',\Omega}^{\beta_{2,1}r_1'} + \|u_1\|_{r_1,\Omega}^{\beta_{3,1}r_1'}\|u_2\|_{\left(\frac{r_1}{\beta_{3,1}r_1'}\right)'\beta_{4,1}r_1',\Omega}^{\beta_{4,1}r_1'} + \|\nabla u_1\|_{\beta_{5,1}r_1',\Omega}^{\beta_{5,1}r_1'} \\
&\qquad + \|\nabla u_2\|_{\beta_{6,1}r_1',\Omega}^{\beta_{6,1}r_1'} + \|\delta_1\|_{r_1',\Omega}^{r_1'} + \|\nabla u_1\|_{p_1,\Omega}^{\beta_{7,1}r_1'}\|\nabla u_2\|_{\left(\frac{p_1}{\beta_{7,1}r_1'}\right)'\beta_{8,1}r_1',\Omega}^{\beta_{8,1}r_1'}\bigg) < \infty
\end{aligned} \tag{3.1}$$

for some $C_0 > 0$. Similarly, we can show that $\|\xi_2\|_{r_2',\Omega}^{r_2'} < \infty$ via using again Hölder's inequality and (H_3). Therefore, we can introduce the Nemytskii operators $\mathcal{F}_i \colon V_1 \times V_2 \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_i'}(\Omega)}$ of f_i defined by

$$\mathcal{F}_i(u, v) := \left\{\xi \in L^{r_i'}(\Omega) \,:\, \xi(x) \in f_i(x, u, v, \nabla u, \nabla v) \text{ for a. a. } x \in \Omega\right\},$$

which are well-defined and bounded for $i = 1, 2$.

We are going to show now that $\mathcal{F}_i \colon V_1 \times V_2 \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_i'}(\Omega)}$ are strongly-weakly u.s.c. for $i = 1, 2$. By symmetry, we only need to prove that $\mathcal{F}_1$ is strongly-weakly. Indeed, if we can prove that the set $\mathcal{F}_1^-(W)$ is closed for each weakly closed set $W \subset L^{r_1'}(\Omega)$ such that $\mathcal{F}_1^-(W) \neq \emptyset$, then we obtain the desired conclusion via employing Theorem 1.1.1 of Kamenskii-Obukhovskii-Zecca [19].

Assume that $W \subset L^{r_1'}(\Omega)$ is weakly closed such that $\mathcal{F}_1^-(W) \neq \emptyset$ and let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}_1^-(W)$ be a sequence such that $(u_n, v_n) \to (u, v)$ in $V_1 \times V_2$ with $(u, v) \in V_1 \times V_2$. Then, we are able to find a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset L^{r_1'}(\Omega)$ satisfying $\xi_n \in \mathcal{F}_1(u_n, v_n) \cap W$. From (3.1) it follows that $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $L^{r_1'}(\Omega)$. Without any loss of generality, we may assume that

$$\xi_n \xrightarrow{w} \xi \quad \text{in } L^{r_1'}(\Omega) \quad \text{for some } \xi \in L^{r_1'}(\Omega) \cap W$$

due to the weak closedness of W. Recall that $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \ni (u, \xi, v, \eta) \mapsto f_1(x, u, v, \xi, \eta) \subset \mathbb{R}$ is u.s.c. for a. a. $x \in \Omega$. Hence, we can apply the Aubin-Cellina convergence theorem (see Aubin-Cellina [1, Theorem 1, p.60]) to get that $\xi \in \mathcal{F}_1(u, v)$. Therefore, we have $(u, v) \in \mathcal{F}_1^-(W)$. Using Theorem 1.1.1 of Kamenskii-Obukhovskii-Zecca [19] proves that $\mathcal{F}_1$ is strongly-weakly closed.

Let I_{K_1} and I_{K_2} be the indicator functions of K_1 and K_2, respectively, and let $\iota_1 \colon V_1 \to L^{r_1}(\Omega)$ and $\iota_2 \colon V_2 \to L^{r_2}(\Omega)$ be the embedding operators of V_1 to $L^{r_1}(\Omega)$ and of V_2 to $L^{r_2}(\Omega)$, respectively. Invoking a standard procedure, it is not difficult to prove that $(u, v) \in K_1 \times K_2$ solves problem (1.1) if and only if it is a solution to the following mixed variational inequality: find $(u, v) \in V_1 \times V_2$ and

$$(u^*, v^*) \in \mathcal{U}(u, v) := (\mathcal{A}_1(u) - \iota_1^* \mathcal{F}_1(u, v), \mathcal{A}_2(v) - \iota_2^* \mathcal{F}_2(u, v))$$

such that

$$\langle (u^*, v^*), (w, z) - (u, v) \rangle + I_{K_1}(w) + I_{K_2}(z) - I_{K_1}(u) - I_{K_2}(v) \geq 0 \quad \text{for all } (w, z) \in K_1 \times K_2, \tag{3.2}$$

where $\langle (u^*, v^*), (w, z) \rangle := \langle u^*, w \rangle_{V_1} + \langle v^*, z \rangle_{V_2}$ stands for the duality paring between $V_1 \times V_1$ and $V_1^* \times V_2^*$ and $\mathcal{A}_1 \colon V_i \to V_i^*$ are defined by

$$\langle \mathcal{A}_i(u_i), v_i \rangle_{V_i} := \int_\Omega \left(a_i(u_i) |\nabla u_i|^{p_i - 2} \nabla u_i + \mu_i(x) |\nabla u_i|^{q_i - 2} \nabla u_i \right) \cdot \nabla v_i \, \mathrm{d}x$$

for $i = 1, 2$.

Next, we are going to apply Proposition 4.1 of Kenmochi [20] to prove the existence of a nontrivial weak solution of problem (3.2). From the closedness and convexity of f_i and the definition of the Nemytskii operators $\mathcal{F}_1$ for $i = 1, 2$, it is not hard to prove that for every $(u, v) \in V_1 \times V_2$, the set $\mathcal{U}(u, v)$ is nonempty, bounded, closed and convex in $V_1^* \times V_2^*$. Let $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset V_1 \times V_2$ and $\{(u_n^*, v_n^*)\}_{n \in \mathbb{N}} \subset V_1^* \times V_2^*$ be sequences such that

$$(u_n, v_n) \xrightarrow{w} (u, v) \quad \text{in } V_1 \times V_2 \quad \text{and} \quad \limsup_{n \to \infty} \langle (u_n^*, v_n^*), (u_n - u, v_n - v) \rangle \leq 0, \tag{3.3}$$

and $(u_n^*, v_n^*) \in \mathcal{U}(u_n, v_n)$. Then, we can find sequences $\{\xi_n\}_{n \in \mathbb{N}} \subset L^{r_1'}(\Omega)$ and $\{\eta_n\}_{n \in \mathbb{N}} \subset L^{r_2'}(\Omega)$ such that

$$u_n^* = \mathcal{A}_1(u_n) - \iota_1^* \xi_n \quad \text{and} \quad v_n^* = \mathcal{A}_2(v_n) - \iota_2^* \eta_n.$$

Recalling that $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded, we infer that $\{\xi_n\}_{n \in \mathbb{N}} \subset L^{r_1'}(\Omega)$ and $\{\eta_n\}_{n \in \mathbb{N}} \subset L^{r_2'}(\Omega)$ are bounded as well. So, we may suppose that

$$\xi_n \xrightarrow{w} \xi \quad \text{in } L^{r_1'}(\Omega) \quad \text{and} \quad \eta_n \xrightarrow{w} \eta \quad \text{in } L^{r_2'}(\Omega)$$

for some $(\xi, \eta) \in L^{r_1'}(\Omega) \times L^{r_2'}(\Omega)$ due to (2.1) and (H$_3$). The latter combined with the compactness of the embedding $V_1 \times V_2$ to $L^{r_1'}(\Omega) \times L^{r_2'}(\Omega)$ (see Remark 3.2) implies that

$$\langle (\iota_1^* \xi_n, \iota_2^* \eta_n), (u_n - u, v_n - v) \rangle \to 0 \quad \text{as } n \to \infty.$$

Hence, by (3.3), we have

$$\limsup_{n \to \infty} \langle (\mathcal{A}_1(u_n), \mathcal{A}_2(v_n)), (u_n - u, v_n - v) \rangle \leq 0.$$

However, from the boundedness of a_1 and a_2, we have

$$\limsup_{n \to \infty} \int_\Omega \left(a_1(u_n) |\nabla u_n|^{p_1 - 2} \nabla u_n + \mu_1(x) |\nabla u_n|^{q_1 - 2} \nabla u_n \right) \cdot \nabla (u_n - u) \, \mathrm{d}x \leq 0, \tag{3.4}$$

and

$$\limsup_{n \to \infty} \int_\Omega \left(a_2(v_n) |\nabla v_n|^{p_2 - 2} \nabla v_n + \mu_2(x) |\nabla v_n|^{q_2 - 2} \nabla v_n \right) \cdot \nabla (v_n - v) \, \mathrm{d}x \leq 0. \tag{3.5}$$

Then, from (3.4), we have

$$0 \geq \limsup_{n \to \infty} \int_\Omega \left(a_1(u_n) |\nabla u_n|^{p_1 - 2} \nabla u_n + \mu_1(x) |\nabla u_n|^{q_1 - 2} \nabla u_n \right) \cdot \nabla (u_n - u) \, \mathrm{d}x$$

$$
\begin{aligned}
&\geq \liminf_{n\to\infty} \int_\Omega \left(a_1(u_n) - \frac{c_{a_1}}{2}\right) |\nabla u|^{p_1-2}\nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x\\
&\quad + \limsup_{n\to\infty} \int_\Omega \left(\frac{c_{a_1}}{2}|\nabla u_n|^{p_1-2}\nabla u_n + \mu_1(x)|\nabla u_n|^{q_1-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&= \limsup_{n\to\infty} \int_\Omega \left(\frac{c_{a_1}}{2}|\nabla u_n|^{p_1-2}\nabla u_n + \mu_1(x)|\nabla u_n|^{q_1-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x.
\end{aligned}
$$

From the (S_+)-property of differential operator $\operatorname{div}\left(\frac{c_{a_1}}{2}|\nabla u_n|^{p_1-2}\nabla u_n + \mu_1(x)|\nabla u_n|^{q_1-2}\nabla u_n\right)$ (see Proposition 2.2) we conclude that $u_n \to u$ in V_1. Similarly, by using (3.5), we can show that $v_n \to v$ in V_2. Employing the strongly-weakly upper semicontinuity of $\mathcal{F}_1$ and $\mathcal{F}_2$ gives $\xi \in \mathcal{F}_1(u,v)$ and $\eta \in \mathcal{F}_2(u,v)$. Whereas, we use the continuity of a_1 and a_2 to find that

$$
\begin{aligned}
u_n^* = \mathcal{A}_1(u_n) - \iota_1^*\xi_n &\xrightarrow{\ w\ } u^* = \mathcal{A}_1(u) - \iota_1^*\xi \quad \text{in } V_1^*\\
v_n^* = \mathcal{A}_2(v_n) - \iota_2^*\eta_n &\xrightarrow{\ w\ } v^* = \mathcal{A}_2(v) - \iota_2^*\eta \quad \text{in } V_2^*.
\end{aligned}
$$

This means that the following equality holds

$$
\lim_{n\to\infty} \langle (u_n^*, v_n^*), (w,z) - (u_n, v_n)\rangle = \langle (u^*, v^*), (w,z) - (u,v)\rangle
$$

with $(u^*, v^*) \in \mathcal{U}(u,v)$ for all $(w,z) \in V_1 \times V_2$.

Now we are going to show that $\mathcal{U}$ is coercive. To this end, we distinguish between the cases (i) and (ii).

- Suppose first (i) is satisfied, that is, a_1 and a_2 are coercive. Then, for any $(u,v) \in V_1 \times V_2$ and $(\xi,\eta) \in (\mathcal{F}_1(u,v), \mathcal{F}_2(u,v))$ with $\|u\|_{V_1} > 1$, $\|v\|_{V_2} > 1$ and

$$
\min\left\{\left(a_1(v) - \pi_1 - \lambda_{1,p_1}^{-1}\pi_2\right), \left(a_2(v) - \pi_1 - \lambda_{1,p_2}^{-1}\pi_2\right)\right\} \geq 1
$$

we have by using (H_4), (2.5) for $r = p_1$ and $r = p_2$ as well as Proposition 2.1(iv)

$$
\begin{aligned}
&\langle (\mathcal{A}_1(u) - \iota_1^*\xi, \mathcal{A}_2(v) - \iota_2^*\eta), (u,v)\rangle\\
&\geq a_1(u)\|\nabla u\|_{p_1,\Omega}^{p_1} + \|\nabla u\|_{\mu_1,q_1,\Omega}^{q_1} + a_2(v)\|\nabla v\|_{p_2,\Omega}^{p_2} + \|\nabla v\|_{\mu_2,q_2,\Omega}^{q_2} - \int_\Omega \pi_1\left(|\nabla u|^{p_1} + |\nabla v|^{p_2}\right)\mathrm{d}x\\
&\quad - \int_\Omega \pi_2\left(|u|^{p_1} + |v|^{p_2}\right)\mathrm{d}x - \int_\Omega \omega(x)\,\mathrm{d}x\\
&\geq \left(a_1(u) - \pi_1 - \lambda_{1,p_1}^{-1}\pi_2\right)\|\nabla u\|_{p_1,\Omega}^{p_1} + \|\nabla u\|_{\mu_1,q_1,\Omega}^{q_1} + \left(a_2(v) - \pi_1 - \lambda_{1,p_2}^{-1}\pi_2\right)\|\nabla v\|_{p_2,\Omega}^{p_2}\\
&\quad + \|\nabla v\|_{\mu_2,q_2,\Omega}^{q_2} - \|\omega\|_{1,\Omega}\\
&\geq \left(a_1(u) - \pi_1 - \lambda_{1,p_1}^{-1}\pi_2\right)\|u\|_{V_1}^{p_1} + \left(a_2(v) - \pi_1 - \lambda_{1,p_2}^{-1}\pi_2\right)\|v\|_{V_2}^{p_2} - \|\omega\|_{1,\Omega}\\
&\geq \|u\|_{V_1}^{p_1} + \|v\|_{V_2}^{p_2} - \|\omega\|_{1,\Omega}.
\end{aligned}
$$

  This shows the coercivity in case (i).
- Let us now assume that the inequality

$$
\min\left\{c_{a_1} - \pi_1 - \pi_2\lambda_{1,p_1}^{-1}, c_{a_2} - \pi_1 - \pi_2\lambda_{1,p_2}^{-1}\right\} > 0
$$

  is satisfied. Then, for any $(u,v) \in V_1 \times V_2$ and $(\xi,\eta) \in (\mathcal{F}(u,v), \mathcal{G}(u,v))$ with $\|u\|_{V_1} > 1$ and $\|v\|_{V_2} > 1$ we have, similar to case (i), by applying (H_4), (H_2), (2.5) and Proposition 2.1(iv)

$$
\begin{aligned}
&\langle (\mathcal{A}_1(u) - \iota_1^*\xi, \mathcal{A}_2(v) - \iota_2^*\eta), (u,v)\rangle\\
&\geq \left(c_{a_1} - \pi_1 - \lambda_{1,p_1}^{-1}\pi_2\right)\|u\|_{V_1}^{p_1} + \left(c_{a_2} - \pi_1 - \lambda_{1,p_2}^{-1}\pi_2\right)\|v\|_{V_2}^{p_2} - \|\omega\|_{1,\Omega}\\
&\geq M_0\left(\|u\|_{V_1}^{p_1} + \|v\|_{V_2}^{p_2}\right) - \|\omega\|_{1,\Omega},
\end{aligned}
$$

  where $M_0 > 0$ is defined by

$$
M_0 := \min\left\{c_a - \pi_1 - \lambda_{1,p_1}^{-1}\pi_2, c_b - \pi_1 - \lambda_{1,p_2}^{-1}\pi_2, 1\right\}.
$$

So we have proved the coercivity also in this case, that is,

$$\frac{\langle(\mathcal{A}_1(u)-\iota_1^*\xi,\mathcal{A}_2(v)-\iota_2^*\eta),(u,v)\rangle}{\|u\|_{V_1}+\|v\|_{V_2}}\to+\infty \quad \text{as } \|u\|_{V_1}+\|v\|_{V_2}\to\infty.$$

Therefore, all conditions of Proposition 4.1 of Kenmochi [20] are fulfilled which implies that problem (1.1) has at least one nontrivial weak solution, because of $0\notin f_i(x,0,0,0,0)$ for a. a. $x\in\Omega$ and $i=1,2$.

Finally, we are going to prove that the solution set of problem (1.1) is compact. Assume that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is a sequence of solutions of problem (1.1). Hence, we can find $\xi_n\in\mathcal{F}_1(u_n,v_n)$ and $\eta_n\in\mathcal{F}_2(u_n,v_n)$ such that

$$\langle(\mathcal{A}_1(u_n)-\iota_1^*\xi_n,\mathcal{A}_2(v_n)-\iota_2^*\eta_n),(w-u_n,z-v_n)\rangle\geq 0 \tag{3.6}$$

for all $(w,z)\in K_1\times K_2$. By the coercivity of $\mathcal{U}$ and the boundedness of $\mathcal{A}_i$ for $i=1,2$, we easily obtain that $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset V_1\times V_2$ and $\{(\xi_n,\eta_n)\}_{n\in\mathbb{N}}\subset L^{r_1'}(\Omega)\times L^{r_2'}(\Omega)$ are bounded. So, there are functions $(u,v)\in K_1\times K_2$ and $(\xi,\eta)\in L^{r_1'}(\Omega)\times L^{r_2'}(\Omega)$ such that

$$(u_n,v_n)\overset{w}{\longrightarrow}(u,v)\quad\text{in } V_1\times V_2\quad\text{and}\quad(\xi_n,\eta_n)\overset{w}{\longrightarrow}(\xi,\eta)\quad\text{in } L^{r_1'}(\Omega)\times L^{r_2'}(\Omega).$$

Inserting $(u,v)\in K_1\times K_2$ into (3.6) and taking the lower upper limit as $n\to\infty$ for the resulting inequality yields

$$\limsup_{n\to\infty}\langle(\mathcal{A}_1(u_n)-\iota_1^*\xi_n,\mathcal{A}_2(v_n)-\iota_2^*\eta_n),(u_n-u,v_n-v)\rangle\leq 0.$$

Arguing as before, we can show that $u_n\to u$ in V_1 and $v_n\to v$ in V_2. However, the closedness of $\mathcal{F}_1$ and $\mathcal{F}_2$ reveals that $\xi\in\mathcal{F}_1(u,v)$ and $\eta\in\mathcal{F}_2(u,v)$. Passing to the limit as $n\to\infty$ in (3.6), we deduce that $(u,v)\in V_1\times V_2$ is a weak solution of problem (1.1). Consequently, the solution set of problem (1.1) is compact in $V_1\times V_2$. □

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(Y. Liu) Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, Guangxi, P.R. China
Email address: yongjianliuyln@163.com

(V.T. Nguyen) Department of Mathematics, FPT University, Education zone, Hoa Lac High Tech Park, Km29 Thang Long Highway, Thach That ward, Hanoi, Vietnam
Email address: Thiennv15@fe.edu.vn

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: winkert@math.tu-berlin.de

(S. Zeng) Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, P.R. China, & Jagiellonian University in Krakow, Faculty of Mathematics and Computer Science, ul. Lojasiewicza 6, 30-348 Krakow, Poland, & Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, Guangxi, P.R. China.
Email address: zengshengda@163.com