GENERAL COMPARISON PRINCIPLE FOR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. We study quasilinear elliptic variational-hemivariational inequalities involving general Leray-Lions operators. The novelty of this paper is to provide existence and comparison results whereby only a local growth condition on Clarke's generalized gradient is required. Based on these results, in the second part the theory is extended to discontinuous variational-hemivariational inequalities.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. By $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega), 1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $(W^{1,p}(\Omega))^*$ and $W^{-1,q}(\Omega)$, respectively, where q is the Hölder conjugate satisfying $1/p + 1/q = 1$. We consider the following elliptic variational-hemivariational inequality. Find $u \in K$ such that

$$\begin{aligned} &\langle Au + F(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v - u)dx \\ &\quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K, \end{aligned} \tag{1.1}$$

where $j_k^o(x, s; r), k = 1, 2$ denotes the generalized directional derivative of the locally Lipschitz functions $s \mapsto j_k(x, s)$ at s in the direction r given by

$$j_k^o(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t}, k = 1, 2$$

(cf. [16, Chapter 2]). We denote by K a closed convex subset of $W^{1,p}(\Omega)$, and A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)).$$

The operator F stands for the Nemytskij operator associated with some Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x), \nabla u(x)).$$

Furthermore, we denote the trace operator by $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is known to be linear, bounded, and even compact.

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The aim of this paper is to establish the method of sub- and supersolutions for problem (1.1). We prove the existence of solutions between a given pair of sub-supersolution assuming only a local growth condition of Clarke's generalized gradient, which extends results recently obtained by Carl in [6]. To complete our findings, we also give the proof for the existence of extremal solutions of problem (1.1) for a fixed ordered pair of sub- and supersolutions in case A has the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)).$$

In the second part we consider (1.1) with a discontinuous Nemytskij operator F involved, which extends results in [31] and partly of [8]. Let us consider next some special cases of problem (1.1), where we suppose $A = -\Delta_p$.

(1) If $K = W^{1,p}(\Omega)$ and j_k are smooth, problem (1.1) reduces to

$$\begin{aligned} &\langle -\Delta_p u + F(u), v\rangle + \int_\Omega j_1'(\cdot, u) v dx \\ &\quad + \int_{\partial\Omega} j_2'(\cdot, \gamma u)\gamma v d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega), \end{aligned}$$

which is equivalent to the weak formulation of the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + F(u) + j_1'(u) &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + j_2'(\gamma u) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\partial u/\partial \nu$ denotes the conormal derivative of u. The method of sub- and supersolution for this kind of problems is a special case of [5].

(2) For $f \in V_0^*$, $K \subset W_0^{1,p}(\Omega)$ and $j_2 = 0$, (1.1) corresponds to the variational-hemivariational inequality given by

$$\langle -\Delta_p u + f, v - u\rangle + \int_\Omega j_1^o(\cdot, u; v - u) dx \geq 0, \quad \forall v \in K,$$

which has been discussed in detail in [4].

(3) If $K \subset W_0^{1,p}(\Omega)$ and $j_k = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K: \quad \langle -\Delta_p u + F(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

whose method of sub- and supersolution has been developed in [9, Chapter 5].

(4) Let $K = W_0^{1,p}(\Omega)$ or $K = W^{1,p}(\Omega)$ and j_k not necessarily smooth. Then problem (1.1) is a hemivariational inequality, which contains for $K = W_0^{1,p}(\Omega)$ as a special case the following Dirichlet problem for the elliptic inclusion:

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

and for $K = W^{1,p}(\Omega)$ the elliptic inclusion

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \partial j_2(\cdot, u) &\ni 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where the multivalued functions $s \mapsto \partial j_k(x, s), k = 1, 2$ stand for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j_k(x, s), k = 1, 2$ given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}. \tag{1.4}$$

Problems of the form (1.2) and (1.3) have been studied in [12] and [5], respectively.

Existence results for variational-hemivariational inequalities with or without the method of sub- and supersolutions have been obtained under different structure and regularity conditions on the nonlinear functions by various authors. For example, we refer to [3, 10, 11, 19, 21, 24, 26, 28]. In case that K is the whole space $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, respectively, problem (1.1) reduces to a hemivariational inequality which has been treated in [2, 14, 17, 18, 20, 23, 25, 27, 29].
Comparison principles for general elliptic operators A, including the negative p-Laplacian $-\Delta_p$, Clarke's generalized gradient $s \mapsto \partial j(x, s)$, satisfying a one-sided growth condition in the form

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1} \tag{1.5}$$

for all $\xi_i \in \partial j(x, s_i), i = 1, 2$, for a.a. $x \in \Omega$, and for all s_1, s_2 with $s_1 < s_2$, can be found in [9]. Inspired by results recently obtained in [12] and [13], we prove the existence of (extremal) solutions for the variational-hemivariational inequality (1.1) within a sector of an ordered pair of sub- and supersolutions $\underline{u}, \overline{u}$ without assuming a one-sided growth condition on Clarke's gradient of the form (1.5).

2. Notation of sub- and supersolution

For functions $u, v : \Omega \to \mathbb{R}$ we use the notation $u \wedge v = \min(u, v), u \vee v = \max(u, v), K \wedge K = \{u \wedge v : u, v \in K\}, K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K, u \vee K = \{u\} \vee K$ and introduce the following definitions:

Definition 2.1. *A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (1.1) if the following holds:*

(1) $F(\underline{u}) \in L^q(\Omega)$;
(2) $\langle A\underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j_1^o(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0, \quad \forall w \in \underline{u} \wedge K.$

Definition 2.2. *A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (1.1) if the following holds:*

(1) $F(\overline{u}) \in L^q(\Omega)$;
(2) $\langle A\overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j_1^o(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \geq 0, \quad \forall w \in \overline{u} \vee K.$

In order to prove our main results, we additionally suppose the following assumptions:

$$\underline{u} \vee K \subset K, \qquad \overline{u} \wedge K \subset K. \tag{2.1}$$

3. Preliminaries and hypotheses

Let $1 < p < \infty, 1/p + 1/q = 1$, and assume for the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, i = 1, \ldots, N$ the following conditions.

(A1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, that is, is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s, ξ) for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1})$$

for a.a. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^N a_i(x, s, \xi)\xi_i \geq c_1|\xi|^p - k_1(x)$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

Condition (A1) implies that $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au, \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$, and assumption (A3) is a coercivity type condition.

Let $[\underline{u}, \overline{u}]$ be an ordered pair of sub- and supersolutions of problem (1.1). We impose the following hypotheses on j_k and the nonlinearity f in problem (1.1).

(j1) $x \mapsto j_1(x, s)$ and $x \mapsto j_2(x, s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$.

(j2) $s \mapsto j_1(x, s)$ and $s \mapsto j_2(x, s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

(j3) There are functions $L_1 \in L^q_+(\Omega)$ and $L_2 \in L^q_+(\partial\Omega)$ such that for all $s \in [\underline{u}(x), \overline{u}(x)]$ the following local growth conditions hold:

$$\eta \in \partial j_1(x, s) : |\eta| \leq L_1(x), \quad \text{for a.a. } x \in \Omega,$$
$$\xi \in \partial j_2(x, s) : |\xi| \leq L_2(x), \quad \text{for a.a. } x \in \partial\Omega.$$

(F1) (i) $x \mapsto f(x, s, \xi)$ is measurable in Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.
(ii) $(s, \xi) \mapsto f(x, s, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for a.a. $x \in \Omega$.

(iii) There exist a constant $c_2 > 0$ and a function $k_3 \in L^q_+(\Omega)$ such that

$$|f(x,s,\xi)| \le k_3(x) + c_2|\xi|^{p-1}$$

for a.a. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $s \in [\underline{u}(x), \overline{u}(x)]$.

Note that the associated Nemytskij operator F defined by $F(u)(x) = f(x, u(x), \nabla u(x))$ is continuous and bounded from $[\underline{u}, \overline{u}] \subset W^{1,p}(\Omega)$ to $L^q(\Omega)$ (cf. [32]). We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \le v$ if and only if $v - u \in L^p_+(\Omega)$, where $L^p_+(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.
Based on an approach in [12], the main idea in our considerations is to modify the functions j_k. First we set for $k = 1, 2$

$$\alpha_k(x) := \min\{\xi : \xi \in \partial j_k(x, \underline{u}(x))\}, \quad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \overline{u}(x))\}. \tag{3.1}$$

By means of (3.1) we introduce the mappings $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{j}_k(x,s) = \begin{cases} j_k(x, \underline{u}(x)) + \alpha_k(x)(s - \underline{u}(x)), & \text{if } s < \underline{u}(x), \\ j_k(x,s), & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ j_k(x, \overline{u}(x)) + \beta_k(x)(s - \overline{u}(x)), & \text{if } s > \overline{u}(x). \end{cases} \tag{3.2}$$

The following lemma provides some properties of the functions $\widetilde{j}_1$ and $\widetilde{j}_2$.

Lemma 3.1. *Let the assumptions in (j1)–(j3) be satisfied. Then the modified functions $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ have the following qualities.*

($\widetilde{j}$1) *$x \mapsto \widetilde{j}_1(x,s)$ and $x \mapsto \widetilde{j}_2(x,s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$ and $s \mapsto \widetilde{j}_1(x,s)$ and $s \mapsto \widetilde{j}_2(x,s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.*

($\widetilde{j}$2) *Let $\partial\widetilde{j}_k(x,s)$ be Clarke's generalized gradient of $s \mapsto \widetilde{j}_k(x,s)$. Then for all $s \in \mathbb{R}$ the following estimates hold true:*

$$\eta \in \partial\widetilde{j}_1(x,s) : |\eta| \le L_1(x), \quad \text{for a.a. } x \in \Omega,$$
$$\xi \in \partial\widetilde{j}_2(x,s) : |\xi| \le L_2(x), \quad \text{for a.a. } x \in \partial\Omega.$$

($\widetilde{j}$3) *Clarke's generalized gradient of $s \mapsto \widetilde{j}_1(x,s)$ and $s \mapsto \widetilde{j}_2(x,s)$ are given by*

$$\partial\widetilde{j}_k(x,s) = \begin{cases} \alpha_k(x) & \text{if } s < \underline{u}(x), \\ \partial\widetilde{j}_k(x, \underline{u}(x)) & \text{if } s = \underline{u}(x), \\ \partial j_k(x,s) & \text{if } \underline{u}(x) < s < \overline{u}(x), \\ \partial\widetilde{j}_k(x, \overline{u}(x)) & \text{if } s = \overline{u}(x), \\ \beta_k(x) & \text{if } s > \overline{u}(x), \end{cases}$$

and the inclusions $\partial\widetilde{j}_k(x, \underline{u}(x)) \subset \partial j_k(x, \underline{u}(x))$ and $\partial\widetilde{j}_k(x, \overline{u}(x)) \subset \partial j_k(x, \overline{u}(x))$ are valid for $k = 1, 2$.

Proof. With a view to the assumptions (j1)–(j3) and the definition of $\widetilde{j}_k$ in (3.2), one verifies the lemma in few steps. □

With the aid of Lemma 3.1, we introduce the integral functionals J_1 and J_2 defined on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively, given by

$$J_1(u) = \int_\Omega \widetilde{j}_1(x, u(x))dx, \quad u \in L^p(\Omega)$$
$$J_2(v) = \int_{\partial\Omega} \widetilde{j}_2(x, v(x))d\sigma, \quad v \in L^p(\partial\Omega).$$

Due to the conditions ($\widetilde{j}$1)–($\widetilde{j}$2) and Lebourg's mean value theorem (see [16, Chapter 2]), the functionals $J_1 : L^p(\Omega) \to \mathbb{R}$ and $J_2 : L^p(\partial\Omega) \to \mathbb{R}$ are well-defined and Lipschitz continuous on bounded sets of $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. This implies among others that Clarke's generalized gradients $\partial J_1 : L^p(\Omega) \to 2^{L^q(\Omega)}$ and $\partial J_2 : L^p(\partial\Omega) \to 2^{L^q(\partial\Omega)}$ are well-defined, too. Furthermore, by means of Aubin-Clarke's theorem (see [16]), for $u \in L^p(\Omega)$ and $v \in L^p(\partial\Omega)$ we get

$$\begin{aligned} &\eta \in \partial J_1(u) \Longrightarrow \eta \in L^q(\Omega) \text{ with } \eta(x) \in \partial\widetilde{j}_1(x, u(x)) \text{ for a.a. } x \in \Omega, \\ &\xi \in \partial J_2(v) \Longrightarrow \xi \in L^q(\partial\Omega) \text{ with } \xi(x) \in \partial\widetilde{j}_2(x, v(x)) \text{ for a.a. } x \in \partial\Omega. \end{aligned} \tag{3.3}$$

An important tool in our considerations is the following surjectivity result for multivalued pseudomonotone mappings perturbed by maximal monotone operators in reflexive Banach spaces.

Theorem 3.2. *Let X be a real reflexive Banach space with the dual space X^*, $\Phi : X \to 2^{X^*}$ a maximal monotone operator, and $u_0 \in \operatorname{dom}(\Phi)$. Let $A : X \to 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasibounded or Φ_{u_0} is strongly quasibounded. Assume further that $A : X \to 2^{X^*}$ is u_0-coercive, that is, there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in \operatorname{graph}(A)$ one has $\langle u^*, u - u_0\rangle \geq c(\|u\|_X)\|u\|_X$. Then $A + \Phi$ is surjective, that is,* $\operatorname{range}(A + \Phi) = X^*$.

The proof of the theorem can be found for example in [30, Theorem 2.12]. The notation A_{u_0} and Φ_{u_0} stands for $A_{u_0}(u) := A(u_0 + u)$ and $\Phi_{u_0}(u) := \Phi(u_0 + u)$, respectively. Note that any bounded operator is, in particular, also quasibounded and strongly quasibounded. For more details we refer to [30]. The next proposition provides a sufficient condition to prove the pseudomonotonicity of multivalued operators and plays an important part in our argumentations. The proof is presented, for example, in [30, Chapter 2].

Proposition 3.3. *Let X be a reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u \in X$ one has that $A(u)$ is a nonempty, closed and convex subset of X^*;*
(ii) *$A : X \to 2^{X^*}$ is bounded;*
(iii) *if $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if $\limsup\langle u_n^*, u_n - u\rangle \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.

We denote by $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\gamma^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ the adjoint operators of the imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the trace operator

$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, respectively, given by

$$\langle i^*\eta, \varphi\rangle = \int_\Omega \eta\varphi dx, \quad \forall \varphi \in W^{1,p}(\Omega),$$
$$\langle \gamma^*\xi, \varphi\rangle = \int_{\partial\Omega} \xi\gamma\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Next, we introduce the following multivalued operators:

$$\Phi_1(u) := (i^* \circ \partial J_1 \circ i)(u), \qquad \Phi_2(u) := (\gamma^* \circ \partial J_2 \circ \gamma)(u), \tag{3.4}$$

where i, i^*, γ, γ^* are defined as mentioned above. The operators $\Phi_k, k = 1, 2$, have the following properties (see e.g. [5, Lemma 3.1 and Lemma 3.2]).

Lemma 3.4. *The multivalued operators $\Phi_1 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ and $\Phi_2 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ are bounded and pseudomonotone.*

Let $b : \Omega \times \mathbb{R} \to \mathbb{R}$ be the cutoff function related to the given ordered pair $\underline{u}, \overline{u}$ of sub- and supersolutions defined by

$$b(x, s) = \begin{cases} (s - \overline{u}(x))^{p-1}, & \text{if } s > \overline{u}(x), \\ 0, & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ -(\underline{u}(x) - s)^{p-1}, & \text{if } s < \underline{u}(x). \end{cases} \tag{3.5}$$

Clearly, the mapping b is a Carathéodory function satisfying the growth condition

$$|b(x, s)| \le k_4(x) + c_3|s|^{p-1} \tag{3.6}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, where $k_4 \in L^q_+(\Omega)$ and $c_3 > 0$. Furthermore, elementary calculations show the following estimate

$$\int_\Omega b(x, u(x))u(x)dx \ge c_4\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega), \tag{3.7}$$

where c_4 and c_5 are some positive constants. Due to (3.6) the associated Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b(x, u(x)) \tag{3.8}$$

is bounded and continuous. Since the embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, the composed operator $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is completely continuous.

For $u \in W^{1,p}(\Omega)$, we define the truncation operator T with respect to the functions $\underline{u}$ and $\overline{u}$ given by

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x). \end{cases}$$

The mapping T is continuous and bounded from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ which follows from the fact that the functions $\min(\cdot, \cdot)$ and $\max(\cdot, \cdot)$ are continuous from $W^{1,p}(\Omega)$ to itself and that T can be represented as $Tu = \max(u, \underline{u}) + \min(u, \overline{u}) - u$ (cf. [22]). Let $F \circ T$ be the composition of the Nemytskij operator F and T given by

$$(F \circ T)(u)(x) = f(x, Tu(x), \nabla Tu(x)).$$

Due to hypothesis (F1)(iii), the mapping $F \circ T : W^{1,p}(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{F} : i^* \circ (F \circ T) : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, and consider the multivalued operator

$$\widetilde{A} = A_T u + \widehat{F} + \lambda \widehat{B} + \Phi_1 + \Phi_2 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}, \tag{3.9}$$

where λ is a constant specified later, and the operator A_T is given by

$$\langle A_T u, \varphi \rangle = -\sum_{i=1}^{N} \int_\Omega a_i(x, Tu, \nabla u) \frac{\partial \varphi}{\partial x_i} dx.$$

We are going to prove the following properties for the operator $\widetilde{A}$.

Lemma 3.5. *The operator $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded, pseudomonotone, and coercive for λ sufficiently large.*

Proof. The boundedness of $\widetilde{A}$ follows directly from the boundedness of the specific operators A_T, $\widehat{F}$,$\widehat{B}$,Φ_1 and Φ_2. As seen above, the operator $\widehat{B}$ is completely continuous and thus pseudomonotone. The elliptic operator $A_T + \widehat{F}$ is pseudomonotone because of hypotheses (A1), (A2), and (F1), and in view of Lemma 3.4 the operators Φ_1 and Φ_2 are bounded and pseudomonotone as well. Since pseudomonotonicity is invariant under addition, we conclude that $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded and pseudomonotone. To prove the coercivity of $\widetilde{A}$, we have to find the existence of a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\lim_{s \to +\infty} c(s) = +\infty, \tag{3.10}$$

such that for all $u \in W^{1,p}(\Omega)$ and $u^* \in \widetilde{A}(u)$ the following holds

$$\langle u^*, u - u_0 \rangle \geq c(\|u\|_{W^{1,p}(\Omega)}) \|u\|_{W^{1,p}(\Omega)}, \tag{3.11}$$

for some $u_0 \in K$. Let $u^* \in \widetilde{A}(u)$, that is, u^* is of the form

$$u^* = (A_T + \widehat{F} + \lambda \widehat{B})(u) + i^* \eta + \gamma^* \xi,$$

where $\eta \in L^q(\Omega)$ with $\eta(x) \in \partial \widetilde{j}_1(x, u(x))$ for a.a. $x \in \Omega$ and $\xi \in L^q(\partial\Omega)$ with $\xi(x) \in \partial \widetilde{j}_2(x, u(x))$ for a.a. $x \in \partial\Omega$. Applying (A1), (A3), (F1)(iii), (3.7), and ($\widetilde{j}$2), the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and Young's inequality yield

$$
\begin{aligned}
&\langle u^*, u - u_0\rangle \\
&= \langle (A_T + \widehat{F} + \lambda \widehat{B})(u) + i^*\eta + \gamma^*\xi, u - u_0\rangle \\
&= \int_\Omega \sum_{i=1}^N a_i(x, Tu, \nabla u) \frac{\partial u - \partial u_0}{\partial x_i} dx \\
&\qquad + \int_\Omega (f(\cdot, Tu, \nabla Tu)(u - u_0) + \lambda b(x, u)(u - u_0))dx \\
&\qquad + \int_\Omega \eta(u - u_0))dx + \int_{\partial\Omega} \xi\gamma(u - u_0)d\sigma \\
&\geq c_1\|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - d_1\|u\|^{p-1}_{L^p(\Omega)} - d_2\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_3 \\
&\qquad - \varepsilon\|\nabla u\|^p_{L^p(\Omega)} - c(\varepsilon)\|u\|^p_{L^p(\Omega)} - d_5\|u\|_{L^p(\Omega)} - d_6\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_7 \\
&\qquad + \lambda c_4\|u\|^p_{L^p(\Omega)} - \lambda c_5 - d_8 - d_9\|u\|^{p-1}_{L^p(\Omega)} - d_{10}\|u\|_{L^p(\Omega)} - d_{11} \\
&\qquad - d_{12}\|u\|_{L^p(\partial\Omega)} - d_{13} \\
&= (c_1 - \varepsilon)\|\nabla u\|^p_{L^p(\Omega)} + (\lambda c_4 - c(\varepsilon))\|u\|^p_{L^p(\Omega)} - d_{14}\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_{15}\|u\|^{p-1}_{L^p(\Omega)} \\
&\qquad - d_{16}\|u\|_{L^p(\Omega)} - d_{17},
\end{aligned}
$$

where d_j are some positive constants. Choosing $\varepsilon < c_1$ and λ such that $\lambda > c(\varepsilon)/c_4$ yields the estimate

$$\langle u^*, u - u_0\rangle \geq d_{18}\|u\|^p_{W^{1,p}(\Omega)} - d_{19}\|u\|^{p-1}_{W^{1,p}(\Omega)} - d_{20}\|u\|_{W^{1,p}(\Omega)} - d_{21}.$$

Setting $c(s) = d_{18}s^{p-1} - d_{19}s^{p-2} - d_{20} - d_{21}/s$ for $s > 0$ and $c(0) = 0$ provides the estimate in (3.11) satisfying (3.10). This proves the coercivity of A and completes the proof of the lemma. □

4. Main results

Theorem 4.1. *Let hypotheses (A1)–(A3),(j1)–(j3), and (F1) be satisfied, and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$, respectively, satisfying $\underline{u} \leq \overline{u}$ and (2.1). Then, there exists a solution of (1.1) in the order interval $[\underline{u}, \overline{u}]$.*

Proof. Let $I_K : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the indicator function corresponding to the closed convex set $K \neq \emptyset$ given by

$$I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K, \end{cases}$$

which is known to be proper, convex, and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows. Find $u \in K$ such that

$$
\begin{aligned}
&\langle Au + F(u), v - u\rangle + I_K(v) - I_K(u) + \int_\Omega j_1^\circ(\cdot, u; v - u)dx \\
&\qquad + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in W^{1,p}(\Omega)
\end{aligned}
\tag{4.1}
$$

By using the operators $A_T, \widehat{F}, \widehat{B}$ and the functions $\widetilde{j}_1, \widetilde{j}_2$ introduced in Section 3, we consider the following auxiliary problem. Find $u \in K$ such that

$$\begin{aligned} &\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v-u\rangle + I_K(v) - I_K(u) + \int_\Omega \widetilde{j}_1^o(\cdot, u; v-u)dx \\ &\quad + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \end{aligned} \tag{4.2}$$

for all $v \in W^{1,p}(\Omega)$. Consider now the multivalued operator

$$\widetilde{A} + \partial I_K : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*},$$

where $\widetilde{A}$ is as in (3.9), and $\partial I_K : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is the subdifferential of the indicator function I_K which is known to be a maximal monotone operator (cf. [30, Page 20]). Lemma 3.5 provides that $\widetilde{A}$ is bounded, pseudomonotone, and coercive. Applying Theorem 3.2 proves the surjectivity of $\widetilde{A} + \partial I_K$ meaning that range$(\widetilde{A} + \partial I_K) = (W^{1,p}(\Omega))^*$. Since $0 \in (W^{1,p}(\Omega))^*$, there exist a solution $u \in K$ of the inclusion

$$\widetilde{A}(u) + \partial I_K(u) \ni 0. \tag{4.3}$$

This implies the existence of $\eta^* \in \Phi_1(u), \xi^* \in \Phi_2(u)$, and $\theta^* \in \partial I_K(u)$ such that

$$A_T u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^* = 0, \quad \text{ in } (W^{1,p}(\Omega))^*, \tag{4.4}$$

where it holds in view of (3.3) and (3.4) that

$$\eta^* = i^*\eta \quad \text{and} \quad \xi^* = \gamma^*\xi$$

with

$$\eta \in L^q(\Omega),\ \eta(x) \in \partial\widetilde{j}_1(x, u(x)) \quad \text{ as well as } \quad \xi \in L^q(\partial\Omega),\ \xi(x) \in \partial\widetilde{j}_2(x, \gamma u(x)).$$

Due to the Definition of Clarke's generalized gradient $\partial\widetilde{j}_k(\cdot, u), k = 1, 2$, one gets

$$\begin{aligned} \langle \eta^*, \varphi\rangle &= \int_\Omega \eta(x)\varphi(x)dx \leq \int_\Omega \widetilde{j}_1^o(x, u(x); \varphi(x))dx, \quad \forall \varphi \in W^{1,p}(\Omega), \\ \langle \xi^*, \varphi\rangle &= \int_{\partial\Omega} \xi(x)\gamma\varphi(x)d\sigma \leq \int_{\partial\Omega} \widetilde{j}_2^o(x, \gamma u(x); \gamma\varphi(x))d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{4.5}$$

Moreover, we have the following estimate:

$$\langle \theta^*, v-u\rangle \leq I_K(v) - I_K(u), \quad \forall v \in W^{1,p}(\Omega). \tag{4.6}$$

From (4.4) we conclude

$$\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^*, \varphi\rangle = 0, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Using the estimates in (4.5) and (4.6) to the equation above where φ is replaced by $v - u$, yields for all $v \in W^{1,p}(\Omega)$

$$\begin{aligned} 0 &= \langle A_T - \Delta_p u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^*, v-u\rangle \\ &\leq \langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v-u\rangle + I_K(v) - I_K(u) \\ &\quad + \int_\Omega \widetilde{j}_1^o(\cdot, u; v-u)dx + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma. \end{aligned}$$

Hence, we obtain a solution u of the auxiliary problem (4.2) which is equivalent to the problem. Find $u \in K$ such that

$$\begin{aligned} &\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v - u\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; v - u)dx \\ &\quad + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K. \end{aligned} \tag{4.7}$$

In the next step we have to show that any solution u of (4.7) belongs to $[\underline{u}, \overline{u}]$. By Definition 2.2 and by choosing $w = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+ \in \overline{u} \vee K$, we obtain

$$\langle A\overline{u} + F(\overline{u}), (u - \overline{u})^+\rangle + \int_\Omega j_1^o(\cdot, \overline{u}; (u - \overline{u})^+)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\overline{u}; \gamma(u - \overline{u})^+)d\sigma \geq 0,$$

and selecting $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ in (4.7) provides

$$\begin{aligned} &\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), -(u - \overline{u})^+\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; -(u - \overline{u})^+)dx \\ &+ \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; -\gamma(u - \overline{u})^+)d\sigma \geq 0. \end{aligned}$$

Adding these inequalities yields

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega (a_i(x, \overline{u}, \nabla\overline{u}) - a_i(x, Tu, \nabla u))\frac{\partial(u - \overline{u})^+}{\partial x_i}dx \\ &\quad + \int_\Omega (F(\overline{u}) - (F \circ T)(u))(u - \overline{u})^+ dx \\ &\quad + \int_\Omega (j_1^o(\cdot, \overline{u}; 1) + \widetilde{j}_1^o(\cdot, u; -1))(u - \overline{u})^+ dx \\ &\quad + \int_{\partial\Omega} (j_2^o(\cdot, \gamma\overline{u}; 1) + \widetilde{j}_2^o(\cdot, \gamma u; -1))\gamma(u - \overline{u})^+ d\sigma \\ &\geq \lambda \int_\Omega B(u)(u - \overline{u})^+ dx. \end{aligned} \tag{4.8}$$

Let us analyze the specific integrals in (4.8). By using (A2) and the definition of the truncation operator, we obtain

$$\begin{aligned} &\int_\Omega (a_i(x, \overline{u}, \nabla\overline{u}) - a_i(x, Tu, \nabla u))\frac{\partial(u - \overline{u})^+}{\partial x_i}dx \leq 0, \\ &\int_\Omega (F(\overline{u}) - (F \circ T)(u))(u - \overline{u})^+ dx = 0. \end{aligned} \tag{4.9}$$

Furthermore, we consider the third integral of (4.8) in case $u > \overline{u}$, otherwise it would be zero. Applying (1.4) and (3.2) proves

$$\begin{aligned}
&\widetilde{j}_1^o(x, u(x); -1)\\
&= \limsup_{s \to u(x), t \downarrow 0} \frac{\widetilde{j}_1(x, s-t) - \widetilde{j}_1(x, s)}{t}\\
&= \limsup_{s \to u(x), t \downarrow 0} \frac{j_1(x, \overline{u}(x)) + \beta_1(x)(s - t - \overline{u}(x)) - j_1(x, \overline{u}(x)) - \beta_1(x)(s - \overline{u}(x))}{t}\\
&= \limsup_{s \to u(x), t \downarrow 0} \frac{-\beta_1(x) t}{t}\\
&= -\beta_1(x).
\end{aligned} \tag{4.10}$$

Proposition 2.1.2 in [16] along with (3.1) shows

$$j_1^o(x, \overline{u}(x); 1) = \max\{\xi : \xi \in \partial j_1(x, \overline{u}(x))\} = \beta_1(x). \tag{4.11}$$

In view of (4.10) and (4.11) we obtain

$$\int_\Omega (j_1^o(\cdot, \overline{u}; 1) + \widetilde{j}_1^o(\cdot, u; -1))(u - \overline{u})^+ dx = \int_\Omega (\beta_1(x) - \beta_1(x))(u - \overline{u})^+ dx = 0, \tag{4.12}$$

and analog to this calculation

$$\int_{\partial\Omega} (j_2^o(\cdot, \gamma\overline{u}; 1) + \widetilde{j}_2^o(\cdot, \gamma u; -1))\gamma(u - \overline{u})^+ d\sigma = 0. \tag{4.13}$$

Due to (4.9), (4.12) and (4.13), we immediately realize that the left-hand side in (4.8) is nonpositive. Thus, we have

$$\begin{aligned}
0 &\geq \lambda \int_\Omega B(u)(u - \overline{u})^+ dx\\
&= \lambda \int_\Omega b(\cdot, u)(u - \overline{u})^+ dx\\
&= \lambda \int_{\{x : u(x) > \overline{u}(x)\}} (u - \overline{u})^p dx\\
&= \lambda \int_\Omega ((u - \overline{u})^+)^p dx\\
&\geq 0,
\end{aligned}$$

which implies $(u - \overline{u})^+ = 0$ and hence, $u \leq \overline{u}$. The proof for $\underline{u} \leq u$ is done in a similar way. So far we have shown that any solution of the inclusion (4.3) (which is a solution of (4.2) as well) belongs to the interval $[\underline{u}, \overline{u}]$. The latter implies $A_T u = Au$, $B(u) = 0$ and $(F \circ T)(u) = F(u)$, and thus from (4.3) it follows

$$\langle Au + F(u) + i^*\eta + \gamma^*\xi, v - u \rangle \geq 0, \quad \forall v \in K,$$

where $\eta(x) \in \partial\widetilde{j}_1(x, u(x)) \subset \partial j_1(x, u(x))$ and $\xi(x) \in \partial\widetilde{j}_2(x, \gamma u(x)) \subset \partial j_2(x, \gamma u(x))$, which proves that $u \in [\underline{u}, \overline{u}]$ is also a solution of our original problem (1.1). This completes the proof of the theorem. □

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the order interval $[\underline{u}, \overline{u}]$. In addition, we will assume that K has lattice structure, that is, K fulfills

$$K \vee K \subset K, \qquad K \wedge K \subset K. \tag{4.14}$$

We are going to show that $\mathcal{S}$ possesses the smallest and greatest element with respect to the given partial ordering.

Theorem 4.2. *Let the hypothesis of Theorem 4.1 be satisfied. Then the solution set $\mathcal{S}$ is compact.*

Proof. First, we are going to show that $\mathcal{S}$ is bounded in $W^{1,p}(\Omega)$. Let $u \in \mathcal{S}$ be a solution of (4.1), and notice that $\mathcal{S}$ is $L^p(\Omega)$-bounded because of $\underline{u} \le u \le \overline{u}$. This implies $\gamma\underline{u} \le \gamma u \le \gamma\overline{u}$, and thus, u is also bounded in $L^p(\partial\Omega)$. Choosing a fixed $v = u_0 \in K$ in (4.1) delivers

$$\langle Au + F(u), u_0 - u\rangle + \int_\Omega j_1^o(\cdot, u; u_0 - u)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma u_0 - \gamma u)d\sigma \ge 0.$$

Using (A1), (j3), (F1)(iii), Proposition 2.1.2 in [16], and Young's inequality yields

$$\begin{aligned}
\langle Au, u\rangle \le& \int_\Omega \sum_{i=1}^N |a_i(x, u, \nabla u)| \left|\frac{\partial u_0}{\partial x_i}\right| dx \\
&+ \int_\Omega |f(x, u, \nabla u)||u_0 - u|dx \\
&+ \int_\Omega \max\{\eta(u_0 - u) : \eta \in \partial j_1(x, u)\}dx \\
&+ \int_{\partial\Omega} \max\{\xi(u_0 - u) : \xi \in \partial j_2(x, u)\}d\sigma \\
\le& \int_\Omega \sum_{i=1}^N (k_0 + c_0|u|^{p-1} + c_0|\nabla u|^{p-1})|\nabla u_0|dx \\
&+ \int_\Omega (k_3 + c_2|\nabla u|^{p-1})|u_0 - u|dx \\
&+ \int_\Omega L_1|u_0 - u|dx + \int_{\partial\Omega} L_2|\gamma u_0 - \gamma u|d\sigma \\
\le& e_1 + e_2\|u\|^{p-1}_{L^p(\Omega)} + e_3\|\nabla u\|^{p-1}_{L^p(\Omega)} + e_4 + e_5\|u\|_{L^p(\Omega)} + e_6\|\nabla u\|^{p-1}_{L^p(\Omega)} \\
&+ \varepsilon\|\nabla u\|^p_{L^p(\Omega)} + c(\varepsilon)\|u\|^p_{L^p(\Omega)} + e_7 + e_8\|u\|_{L^p(\Omega)} + e_9 + e_{10}\|u\|_{L^p(\partial\Omega)} \\
\le& \varepsilon\|\nabla u\|^p_{L^p(\Omega)} + e_{11}\|\nabla u\|^{p-1}_{L^p(\Omega)} + e_{12}\|\nabla u\|_{L^p(\Omega)} + e_{13},
\end{aligned}$$

where the left-hand side fulfills the estimate

$$\langle Au, u\rangle \ge c_1\|\nabla u\|^p_{L^p(\Omega)} - k_1.$$

Thus, one has

$$(c_1 - \varepsilon)\|\nabla u\|^p_{L^p(\Omega)} \le e_{11}\|\nabla u\|^{p-1}_{L^p(\Omega)} + e_{13},$$

where the choice $\varepsilon < c_1$ proves that $\|\nabla u\|_{L^p(\Omega)}$ is bounded. Hence, we obtain the boundedness of u in $W^{1,p}(\Omega)$. Let $(u_n) \subset \mathcal{S}$. Since $W^{1,p}(\Omega), 1 < p < \infty$, is reflexive, there exists a weakly convergent subsequence, not relabelled, which yields along with the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega),\\ &u_n \to u \quad \text{in } L^p(\Omega) \quad \text{and a.e. pointwise in } \Omega,\\ &\gamma u_n \to \gamma u \quad \text{in } L^p(\partial\Omega) \quad \text{and a.e. pointwise in } \partial\Omega. \end{aligned} \tag{4.15}$$

As u_n solves (4.1), in particular, for $v = u \in K$, we obtain

$$\begin{aligned} &\langle Au_n, u_n - u\rangle \\ &\le \langle F(u_n), u - u_n\rangle + \int_\Omega j_1^o(\cdot, u_n; u - u_n)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma. \end{aligned} \tag{4.16}$$

Since $(s, r) \mapsto j_k^o(x, s; r), k = 1, 2$, is upper semicontinuous and due to Fatou's Lemma, we get from (4.16)

$$\begin{aligned} \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le \underbrace{\limsup_{n\to\infty}\langle F(u_n), u - u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^o(\cdot, u_n; u - u_n)}_{\le j_1^o(\cdot,u,0)=0} dx \\ + \int_{\partial\Omega} \underbrace{\limsup_{n\to\infty} j_2^o(\cdot, \gamma u_n; \gamma u - \gamma u_n)}_{\le j_2^o(\cdot,\gamma u,\gamma 0)=0} d\sigma \le 0. \end{aligned} \tag{4.17}$$

The elliptic operator A satisfies the (S_+)-property, which due to (4.17) and (4.15) implies

$$u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

Replacing u by u_n in (1.1) yields the following inequality:

$$\begin{aligned} \langle Au_n + F(u_n), v - u_n\rangle + \int_\Omega j_1^o(\cdot, u_n; v - u_n)dx \\ + \int_{\partial\Omega} j_2^o(\cdot, \gamma u_n; \gamma v - \gamma u_n)d\sigma \ge 0, \quad \forall v \in K. \end{aligned} \tag{4.18}$$

Passing to the limes superior in (4.18) and using Fatou's Lemma, the strong convergence of (u_n) in $W^{1,p}(\Omega)$, and the upper semicontinuity of $(s, r) \to j_k^o(x, s; r), k = 1, 2$, we have

$$\langle Au + F(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K.$$

Hence, $u \in \mathcal{S}$. This shows the compactness of the solution set $\mathcal{S}$. □

In order to prove the existence of extremal elements of the solution set $\mathcal{S}$, we drop the u-dependence of the operator A. Then, our assumptions read as follows.

(A1') Each $a_i(x, \xi)$ satisfies Carathéodory conditions, that is, is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for a.a. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x, \xi)| \le k_0(x) + |\xi|^{p-1}$$

for a.a. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2') The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N}(a_i(x,\xi)-a_i(x,\xi'))(\xi_i-\xi_i')>0$$

for a.a. $x\in\Omega$, and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$.

(A3') A constant $c_1>0$ and a function $k_1\in L^1(\Omega)$ exist such that

$$\sum_{i=1}^{N}a_i(x,\xi)\xi_i\geq c_1|\xi|^p-k_1(x)$$

for a.a. $x\in\Omega$, and for all $\xi\in\mathbb{R}^N$.

Then the operator $A:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ acts in the following way:

$$\langle Au,\varphi\rangle=\int_\Omega\sum_{i=1}^{N}a_i(x,\nabla u)\frac{\partial\varphi}{\partial x_i}dx.$$

Let us recall the definition of a directed set.

Definition 4.3. *Let $(\mathcal{P},\leq)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be upward directed if for each pair $x,y\in\mathcal{C}$ there is a $z\in\mathcal{C}$ such that $x\leq z$ and $y\leq z$. Similarly, $\mathcal{C}$ is downward directed if for each pair $x,y\in\mathcal{C}$ there is a $w\in\mathcal{C}$ such that $w\leq x$ and $w\leq y$. If $\mathcal{C}$ is both upward and downward directed, it is called directed.*

Theorem 4.4. *Let hypotheses (A1')–(A3') and (j1)–(j3) be fulfilled, and assume that (F1) and (4.14) are valid. Then the solution set $\mathcal{S}$ of problem (1.1) is a directed set.*

Proof. By Theorem 4.1, we have $\mathcal{S}\neq\emptyset$. Let $u_1,u_2\in\mathcal{S}$ be given solutions of (1.1), and let $u_0=\max\{u_1,u_2\}$. We have to show that there is a $u\in\mathcal{S}$ such that $u_0\leq u$. Our proof is mainly based on an approach developed recently in [13] which relies on a properly constructed auxiliary problem. Let the operator $\widehat{B}$ be given basically as in (3.5)-(3.8) with the following slight change:

$$b(x,s)=\begin{cases}(s-\overline{u}(x))^{p-1}, & \text{if } s>\overline{u}(x),\\ 0, & \text{if } \underline{u}(x)\leq s\leq\overline{u}(x),\\ -(u_0(x)-s)^{p-1}, & \text{if } s<u_0(x).\end{cases}$$

We introduce truncation operators T_j related to u_j and modify the truncation operator T as follows. For $j=1,2$, we define

$$T_ju(x)=\begin{cases}\overline{u}(x), & \text{if } u(x)>\overline{u}(x),\\ u(x), & \text{if } u_j(x)\leq u(x)\leq\overline{u}(x),\\ u_j(x), & \text{if } u(x)<u_j(x).\end{cases}$$

$$Tu(x)=\begin{cases}\overline{u}(x), & \text{if } u(x)>\overline{u}(x),\\ u(x), & \text{if } u_0(x)\leq u(x)\leq\overline{u}(x),\\ u_0(x), & \text{if } u(x)<u_0(x),\end{cases}$$

and we set

$$Gu(x) = f(x, Tu(x), \nabla Tu(x)) - \sum_{j=1}^{2} |f(x, Tu(x), \nabla Tu(x)) - f(x, T_j u(x), \nabla T_j u(x))|$$

as well as

$$\widehat{F} : i^* \circ G : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*.$$

Moreover, we define

$$\alpha_{k,j}(x) := \min\{\xi : \xi \in \partial j_k(x, u_j(x))\}, \qquad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \overline{u}(x))\},$$

$$\alpha_{k,0}(x) := \begin{cases} \alpha_{k,1}(x), & \text{if } x \in \{u_1 \geq u_2\}, \\ \alpha_{k,2}(x), & \text{if } x \in \{u_2 > u_1\} \end{cases}$$

for $k, j = 1, 2$, and introduce the functions $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{j}_k(x, s) = \begin{cases} j_k(x, u_0(x)) + \alpha_{k,0}(x)(s - u_0(x)), & \text{if } s < u_0(x), \\ j_k(x, s), & \text{if } u_0(x) \leq s \leq \overline{u}(x), \\ j_k(x, \overline{u}(x)) + \beta_k(x)(s - \overline{u}(x)), & \text{if } s > \overline{u}(x). \end{cases} \tag{4.19}$$

Furthermore, we define the functions $h_{1,j} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h_{2,j} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ for $j = 0, 1, 2$ as follows:

$$h_{k,0}(x, s) = \begin{cases} \alpha_{k,0}(x), & \text{if } s \leq u_0(x), \\ \alpha_{k,0}(x) + \dfrac{\beta_k(x) - \alpha_{k,0}(x)}{\overline{u}(x) - u_0(x)}(s - u_0(x)), & \text{if } u_0(x) < s < \overline{u}(x), \\ \beta_k(x), & \text{if } s \geq \overline{u}(x), \end{cases}$$

and for $j = 1, 2$

$$h_{k,j}(x, s) = \begin{cases} \alpha_{k,j}(x), & \text{if } s \leq u_j(x), \\ \alpha_{k,j}(x) + \dfrac{\alpha_{k,0}(x) - \alpha_{k,j}(x)}{u_0(x) - u_k(x)}(s - u_j(x)), & \text{if } u_j(x) < s < u_0(x), \\ h_{k,0}(x, s), & \text{if } s \geq u_0(x), \end{cases}$$

where $k = 1, 2$. (Note that for $k = 2$ we understand the functions above being defined on $\partial\Omega$.) Apparently, the mappings $(x, s) \mapsto h_{k,j}(x, s)$ are Carathéodory functions which are piecewise linear with respect to s. Let us introduce the Nemytskij operators $H_1 : L^p(\Omega) \to L^q(\Omega)$ and $H_2 : L^p(\partial\Omega) \to L^q(\partial\Omega)$ defined by

$$H_1 u(x) = \sum_{j=1}^{2} |h_{1,j}(x, u(x)) - h_{1,0}(x, u(x))|,$$

$$H_2 u(x) = \sum_{j=1}^{2} |h_{2,j}(x, \gamma(u(x))) - h_{2,0}(x, \gamma(u(x)))|.$$

Due to the compact imbedding $W^{1,p}(\Omega) \to L^p(\Omega)$, and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, the operators $\widetilde{H}_1 = i^* \circ H_1 \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ and $\widetilde{H}_2 = \gamma^* \circ H_2 \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ are bounded and completely continuous and thus pseudomonotone. Now, we consider the following auxiliary variational-hemivariational inequality. Find $u \in K$ such that

$$\begin{aligned} &\langle Au + \widehat{F}(u) + \lambda \widehat{B}(u), v-u\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; v-u)dx - \langle \widetilde{H}_1 u, v-u\rangle \\ &+ \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma - \langle \widetilde{H}_2 \gamma u, \gamma v - \gamma u\rangle \geq 0, \end{aligned} \tag{4.20}$$

for all $v \in K$. The construction of the auxiliary problem (4.20) including the functions H_k and G is inspired by a very recent approach introduced by Carl and Motreanu in [13]. The first part of the proof of Theorem 4.1 delivers the existence of a solution u of (4.20), since all calculations in Section 3 are still valid. In order to show that the solution set $\mathcal{S}$ of (1.1) is upward directed, we have to verify that a solution u of (4.20) satisfies $u_l \leq u \leq \overline{u}, l = 1, 2$. By assumption $u_l \in \mathcal{S}$, that is, u_l solves

$$\begin{aligned} u_l \in K: \quad &\langle Au_l + F(u_l), v - u_l\rangle + \int_\Omega j_1^o(\cdot, u_l; v - u_l)dx \\ &+ \int_{\partial\Omega} j_2^o(\cdot, \gamma u_l; \gamma v - \gamma u_l)d\sigma \geq 0, \end{aligned}$$

for all $v \in K$. Selecting $v = u \wedge u_l = u_l - (u_l - u)^+ \in K$ in the inequality above yields

$$\begin{aligned} &\langle Au_l + F(u_l), -(u_l - u)^+\rangle + \int_\Omega j_1^o(\cdot, u_l; -(u_l-u)^+)dx \\ &+ \int_{\partial\Omega} j_2^o(\cdot, \gamma u_l; -\gamma(u_l - u)^+)d\sigma \geq 0. \end{aligned} \tag{4.21}$$

Taking the special test function $v = u \vee u_l = u + (u_l - u)^+ \in K$ in (4.20), we get

$$\begin{aligned} &\langle Au + \widehat{F}(u) + \lambda \widehat{B}(u), (u_l - u)^+\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; (u_l - u)^+)dx \\ &- \langle \widetilde{H}_1, (u_l - u)^+\rangle + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma(u_l - u)^+)d\sigma - \langle \widetilde{H}_2 \gamma u, \gamma(u_l - u)^+\rangle \geq 0. \end{aligned} \tag{4.22}$$

Adding (4.21) and (4.22) yields

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N (a_i(x,\nabla u) - a_i(x,\nabla u_l)) \frac{\partial (u_l-u)^+}{\partial x_i} dx \\
&+ \int_\Omega \Bigg(f(x,Tu),\nabla Tu) - f(x,u_l,\nabla u_l) \\
&\qquad\qquad - \sum_{j=1}^2 |f(x,Tu,\nabla Tu) - f(x,T_ju,\nabla T_ju)| \Bigg) (u_l-u)^+ dx \\
&+ \int_\Omega \Big(\widetilde{j}_1^o(\cdot,u;1) + j_1^o(\cdot,u_l;-1) - \sum_{j=1}^2 |h_{1,j}(x,u) - h_{1,0}(x,u)| \Big) (u_l-u)^+ dx \\
&+ \int_{\partial\Omega} \Bigg(\widetilde{j}_2^o(\cdot,\gamma u;1) + j_2^o(\cdot,\gamma u_l;-1) \\
&\qquad\qquad - \sum_{j=1}^2 |h_{2,j}(x,\gamma u) - h_{2,0}(x,\gamma u)| \Bigg) \gamma(u_l-u)^+ d\sigma \\
&\geq -\lambda \int_\Omega B(u)(u_l-u)^+ dx.
\end{aligned} \tag{4.23}$$

The condition (A2') implies directly

$$\int_\Omega \sum_{i=1}^N (a_i(x,\nabla u) - a_i(x,\nabla u_l)) \frac{\partial (u_l-u)^+}{\partial x_i} dx \leq 0, \tag{4.24}$$

and the second integral can be estimated to obtain

$$\begin{aligned}
&\int_\Omega \Bigg(f(x,Tu,\nabla Tu) - f(x,u_l,\nabla u_l) \\
&\qquad - \sum_{j=1}^2 |f(x,Tu,\nabla Tu) - f(x,T_ju,\nabla T_ju)| \Bigg) (u_l-u)^+ dx \\
&\leq \int_\Omega \Bigg(f(x,Tu,\nabla Tu) - f(x,u_l,\nabla u_l) \\
&\qquad\qquad - |f(x,Tu,\nabla Tu) - f(x,T_lu,\nabla T_lu)| \Bigg) (u_l-u)^+ dx \\
&= \int_{\{x\in\Omega : u_l(x) > u(x)\}} \Bigg(f(x,Tu,\nabla Tu) - f(x,u_l,\nabla u_l) \\
&\qquad\qquad - |f(x,Tu,\nabla Tu) - f(x,u_l,\nabla u_l)| \Bigg) (u_l-u) dx \\
&\leq 0.
\end{aligned} \tag{4.25}$$

In order to investigate the third integral, we make use of some auxiliary calculation. In view of (4.19) we have for $u_l(x) > u(x)$

$$\begin{aligned}
&\widetilde{j}_1^{o}(x,u(x);1)\\
&= \limsup_{s\to u(x),t\downarrow 0} \frac{\widetilde{j}_1(x,s+t)-\widetilde{j}_1(x,s)}{t}\\
&= \limsup_{s\to u(x),t\downarrow 0} \bigg[\frac{j_1(x,u_0(x))+\alpha_{1,0}(x)(s+t-u_0(x))}{t}\\
&\qquad\qquad + \frac{-j_1(x,u_0(x))-\alpha_{1,0}(x)(s-u_0(x))}{t}\bigg]\\
&= \limsup_{s\to u(x),t\downarrow 0} \frac{\alpha_{1,0}(x)t}{t}\\
&= \alpha_{1,0}(x).
\end{aligned} \tag{4.26}$$

Applying Proposition 2.1.2 in [16] and (3.1) results in

$$\begin{aligned}
j_1^{o}(x,u_l(x);-1) &= \max\{-\xi : \xi \in \partial j_1(x,u_l(x))\}\\
&= -\min\{\xi : \xi \in \partial j_1(x,u_l(x))\}\\
&= -\alpha_{1,l}(x).
\end{aligned} \tag{4.27}$$

Furthermore, we have in case $u_l(x) > u(x)$

$$\begin{aligned}
h_{1,l}(x,u(x)) &= \alpha_{1,l}(x),\\
h_{1,0}(x,u(x)) &= \alpha_{1,0}(x).
\end{aligned} \tag{4.28}$$

Thus, we get

$$\begin{aligned}
&\int_\Omega \Big(\widetilde{j}_1^{o}(\cdot,u;1)+j_1^{o}(\cdot,u_l;-1)-\sum_{j=1}^{2}|h_{1,j}(x,u)-h_{1,0}(x,u)|\Big)(u_l-u)^+dx\\
&\le \int_\Omega \Big(\widetilde{j}_1^{o}(\cdot,u;1)+j_1^{o}(\cdot,u_l;-1)-|h_{1,l}(x,u)-h_{1,0}(x,u)|\Big)(u_l-u)^+dx\\
&= \int_{\{x\in\Omega: u_l(x)>u(x)\}} (\alpha_{1,0}(x)-\alpha_{1,l}(x)-|\alpha_{1,l}(x)-\alpha_{1,0}(x)|)(u_l-u)^+dx\\
&\le 0.
\end{aligned} \tag{4.29}$$

The same result can be proven for the boundary integral meaning

$$\begin{aligned}
\int_{\partial\Omega} \Bigg(&\widetilde{j}_2^{o}(\cdot,\gamma u;1)+j_2^{o}(\cdot,\gamma u_l;-1)\\
&-\sum_{j=1}^{2}|h_{2,j}(x,\gamma u)-h_{2,0}(x,\gamma u)|\Bigg)\gamma(u_l-u)^+d\sigma \le 0.
\end{aligned} \tag{4.30}$$

Applying (4.24)–(4.30) to (4.23) yields

$$
\begin{aligned}
0 &\geq -\lambda \int_\Omega B(u)(u_l - u)^+ dx \\
&= -\lambda \int_{\{x \in \Omega : u_l(x) > u(x)\}} -(u_0 - u)^{p-1}(u_l - u) dx \\
&\geq \lambda \int_\Omega ((u_l - u)^+)^p dx \\
&\geq 0,
\end{aligned}
$$

and hence, $(u_l - u)^+ = 0$ meaning that $u_l \leq u$ for $l = 1, 2$. This proves $u_0 = \max\{u_1, u_2\} \leq u$. The proof for $u \leq \overline{u}$ can be shown in a similar way. More precisely, we obtain a solution $u \in K$ of (4.20) satisfying $\underline{u} \leq u_0 \leq u \leq \overline{u}$ which implies $\widehat{F}(u) = f(\cdot, u, \nabla u)$, $\widehat{B}(u) = 0$ and $H_1(u) = H_2(\gamma u) = 0$. The same arguments as at the end of the proof of Theorem 4.1 apply, which shows that u is in fact a solution of problem (1.1) belonging to the interval $[u_0, \overline{u}]$. Thus, the solution set $\mathcal{S}$ is upward directed. Analogously, one proves that $\mathcal{S}$ is downward directed. □

Theorem 4.2 and Theorem 4.4 allow us to formulate the next theorem about the existence of extremal solutions.

Theorem 4.5. *Let the hypotheses of Theorem 4.4 be satisfied. Then the solution set $\mathcal{S}$ possesses extremal elements.*

Proof. Since $\mathcal{S} \subset W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ is separable, $\mathcal{S}$ is also separable, that is, there exists a countable, dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. We construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$ and select $u_{n+1} \in \mathcal{S}$ such that

$$\max(z_n, u_n) \leq u_{n+1} \leq \overline{u}.$$

By Theorem 4.4, the element u_{n+1} exists because $\mathcal{S}$ is upward directed. Moreover, we can choose by Theorem 4.2 a convergent subsequence (denoted again by u_n) with $u_n \to u$ in $W^{1,p}(\Omega)$ and $u_n(x) \to u(x)$ a. e. in Ω. Since (u_n) is increasing, the entire sequence converges in $W^{1,p}(\Omega)$ and further, $u = \sup u_n$. One sees at once that $Z \subset [\underline{u}, u]$ which follows from

$$\max(z_1, \ldots, z_n) \leq u_{n+1} \leq u, \ \forall n,$$

and the fact that $[\underline{u}, u]$ is closed in $W^{1,p}(\Omega)$ implies

$$\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u].$$

Therefore, as $u \in \mathcal{S}$, we conclude that u is the greatest element in $\mathcal{S}$. The existence of the smallest solution of (1.1) in $[\underline{u}, \overline{u}]$ can be proven in a similar way. □

Remark 4.6. *If A depends on s, we have to require additional assumptions. For example, if A satisfies in s a monotonicity condition, the existence of extremal solutions can be shown, too. In case $K = W^{1,p}(\Omega)$, a Lipschitz condition with respect to s is sufficient for proving extremal solutions. For more details we refer to* [9].

5. Generalization to discontinuous Nemytskij operators

In this section, we will extend our problem in (1.1) to include discontinuous nonlinearities f of the form $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. We consider again the elliptic variational-hemivariational inequality

$$\begin{aligned} &\langle Au + F(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v-u)dx \\ &\quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K, \end{aligned} \tag{5.1}$$

where all denotations of Section 1 are valid. Here, F denotes the Nemytskij operator given by

$$F(u)(x) = f(x, u(x), u(x), \nabla u(x)),$$

where we will allow f to depend discontinuously on its third argument. The aim of this section is to deal with discontinuous Nemytskij operators $F : [\underline{u}, \overline{u}] \subset W^{1,p}(\Omega) \to L^q(\Omega)$ by combining the results of Section 4 with an abstract fixed point result for not necessarily continuous operators, cf. [7, Theorem 1.1.1]. This will extend recent results obtained in [31]. Let us recall the definitions of sub- and supersolutions.

Definition 5.1. *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (5.1) if the following holds:*

(1) $F(\underline{u}) \in L^q(\Omega)$;

(2) $\langle A\underline{u}+F(\underline{u}), w-\underline{u}\rangle + \int_\Omega j_1^o(\cdot, \underline{u}; w-\underline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0 \quad \forall w \in \underline{u} \wedge K$.

Definition 5.2. *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (5.1) if the following holds:*

(1) $F(\overline{u}) \in L^q(\Omega)$;

(2) $\langle A\overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j_1^o(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \geq 0, \quad \forall v \in \overline{u} \vee K$.

The conditions for Clarke's generalized gradient $s \mapsto \partial j_k(x, s)$ and the functions $j_k : \Omega \times \mathbb{R} \to \mathbb{R},\ k = 1, 2$, are the same as in (j1)–(j3). We only change the property (F1) to the following.

(F2) (i) $x \mapsto f(x, r, u(x), \xi)$ is measurable for all $r \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, and for all measurable functions $u : \Omega \to \mathbb{R}$.

(ii) $(r, \xi) \mapsto f(x, r, s, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$.

(iii) $s \mapsto f(x, r, s, \xi)$ is decreasing for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and for a.a. $x \in \Omega$.

(iv) There exist a constant $c_2 > 0$ and a function $k_2 \in L^q_+(\Omega)$ such that

$$|f(x, r, s, \xi)| \leq k_2(x) + c_0|\xi|^{p-1},$$

for a.a. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $r, s \in [\underline{u}(x), \overline{u}(x)]$.

By [1] the mapping $x \mapsto f(x, u(x), u(x), \nabla u(x))$ is measurable for $x \mapsto u(x)$ measurable, however, the associated Nemytskij operator $F : W^{1,p}(\Omega) \subset L^p(\Omega) \to L^q(\Omega)$ is not necessarily continuous. An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this lemma can be found in [7, Theorem 1.1.1].

Lemma 5.3. *Let P be a subset of an ordered normed space, $G : P \to P$ an increasing mapping and $G[P] = \{Gx \mid x \in P\}$.*

(1) *If $G[P]$ has a lower bound in P and the increasing sequences of $G[P]$ converge weakly in P, then G has the least fixed point x_*, and $x_* = \min\{x \mid Gx \le x\}$.*
(2) *If $G[P]$ has an upper bound in P and the decreasing sequences of $G[P]$ converge weakly in P, then G has the greatest fixed point x^*, and $x^* = \max\{x \mid x \le Gx\}$.*

Our main result of this section is the following theorem.

Theorem 5.4. *Assume that hypotheses (A1')–(A3'),(j1)–(j3),(F2), and (4.14) are valid, and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of (5.1) satisfying $\underline{u} \le \overline{u}$ and (2.1). Then there exist extremal solutions u^* and u_* of (5.1) with $\underline{u} \le u_* \le u^* \le \overline{u}$.*

Proof. We consider the following auxiliary problem:

$$\begin{aligned} u \in K: \quad & \langle Au + F_z(u), v-u\rangle + \int_\Omega j_1^o(\cdot, u; v-u)dx \\ & \qquad + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K, \end{aligned} \tag{5.2}$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$, and we define the set $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \overline{u}]$, and z is a supersolution of (5.1) satisfying $z \wedge K \subset K\}$. On H we introduce the fixed point operator $L : H \to K$ by $z \mapsto u^* =: Lz$, that is, for a given supersolution $z \in H$, the element Lz is the greatest solution of (5.2) in $[\underline{u}, z]$ and thus, it holds $\underline{u} \le Lz \le z$ for all $z \in H$. This implies $L : H \to [\underline{u}, \overline{u}] \cap K$. Because of (4.14), Lz is also a supersolution of (5.2) satisfying

$$\langle ALz + F_z(Lz), w - Lz\rangle + \int_\Omega j_1^o(\cdot, Lz; w - Lz)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma Lz; \gamma w - \gamma Lz)d\sigma \ge 0,$$

for all $w \in Lz \vee K$. By the monotonicity of f with respect to its third argument, $Lz \le z$, and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$ we obtain

$$\begin{aligned} 0 &\le \langle ALz + F_z(Lz), (v-Lz)^+\rangle + \int_\Omega j_1^o(\cdot, Lz; (v-Lz)^+)dx \\ &\quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma Lz; \gamma(v-Lz)^+)d\sigma \\ &\le \langle ALz + F_{Lz}(Lz), (v-Lz)^+\rangle + \int_\Omega j_1^o(\cdot, Lz; (v-Lz)^+)dx \\ &\quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma Lz; \gamma(v-Lz)^+)d\sigma, \end{aligned}$$

for all $v \in K$. Consequently, Lz is a supersolution of (5.1). This shows $L : H \to H$. Let $v_1, v_2 \in H$ and assume that $v_1 \leq v_2$. Then we have the following.

$$\begin{aligned} Lv_1 \in & [\underline{u}, v_1] \text{ is the greatest solution of} \\ & \langle Au + F_{v_1}(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v-u)dx \\ & \quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K \end{aligned} \tag{5.3}$$

$$\begin{aligned} Lv_2 \in & [\underline{u}, v_2] \text{ is the greatest solution of} \\ & \langle Au + F_{v_2}(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v-u)dx \\ & \quad + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K. \end{aligned} \tag{5.4}$$

Since $v_1 \leq v_2$, it follows that $Lv_1 \leq v_2$, and due to (4.14), Lv_1 is also a subsolution of (5.3), that is, (5.3) holds, in particular, for $v \in Lv_1 \wedge K$, that is,

$$\begin{aligned} 0 \geq & \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ & - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma, \end{aligned}$$

for all $v \in K$. Using the monotonicity of f with respect to its third argument s yields

$$\begin{aligned} 0 \geq & \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ & - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma \\ \geq & \langle ALv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ & - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma \end{aligned}$$

for all $v \in K$. Hence, Lv_1 is a subsolution of (5.4). By Theorem 4.5, we know there exists a greatest solution of (5.4) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (5.4) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \leq Lv_2$. This shows that L is increasing. In the last step we have to prove that any decreasing sequence of $L(H)$ converges weakly in H. Let $(u_n) = (Lz_n) \subset L(H) \subset H$ be a decreasing sequence. Then $u_n(x) \searrow u(x)$ for a.a. $x \in \Omega$ for some $u \in [\underline{u}, \overline{u}]$. The boundedness of u_n in $W^{1,p}(\Omega)$ can be shown similarly as in Section 4. Thus the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ along with the monotony of u_n as well as the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ implies

$$\begin{aligned} & u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ & u_n \to u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ & \gamma u_n \to \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned}$$

Since $u_n \in K$, it follows $u \in K$. From (5.2) with u replaced by u_n and v by u, and using the fact that $(s, r) \mapsto j_k^o(x, s; r)$, $k = 1, 2$, is upper semicontinuous, we obtain by applying Fatou's Lemma

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle Au_n, u_n-u\rangle\\
&\le \limsup_{n\to\infty}\langle F_{z_n}(u_n), u-u_n\rangle + \limsup_{n\to\infty}\int_\Omega j_1^o(\cdot,u_n;u-u_n)dx\\
&\quad + \limsup_{n\to\infty}\int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma u-\gamma u_n)d\sigma\\
&\le \underbrace{\limsup_{n\to\infty}\langle F_{z_n}(u_n), u-u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^o(\cdot,u_n;u-u_n)}_{\le j_1^o(\cdot,u;0)=0}dx\\
&\quad + \int_{\partial\Omega}\underbrace{\limsup_{n\to\infty} j_2^o(\cdot,\gamma u_n;\gamma u-\gamma u_n)}_{\le j_2^o(\cdot,\gamma u;\gamma 0)=0}d\sigma\\
&\le 0.
\end{aligned}
$$

The S_+-property of A provides the strong convergence of (u_n) in $W^{1,p}(\Omega)$. As $Lz_n = u_n$ is also a supersolution of (5.2) Definition 5.2 yields

$$
\begin{aligned}
&\langle Au_n + F_{z_n}(u_n), (v-u_n)^+\rangle + \int_\Omega j_1^o(\cdot,u_n;(v-u_n)^+)dx\\
&\quad + \int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma \ge 0
\end{aligned}
$$

for all $v \in K$. Due to $z_n \ge u_n \ge u$ and the monotonicity of f we get

$$
\begin{aligned}
0 &\le \langle Au_n + F_{z_n}(u_n), (v-u_n)^+\rangle + \int_\Omega j_1^o(\cdot,u_n;(v-u_n)^+)dx\\
&\quad + \int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma\\
&\le \langle Au_n + F_{u}(u_n), (v-u_n)^+\rangle + \int_\Omega j_1^o(\cdot,u_n;(v-u_n)^+)dx\\
&\quad + \int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma
\end{aligned}
$$

for all $v \in K$ and since the mapping $u \mapsto u^+ = \max(u,0)$ is continuous from $W^{1,p}(\Omega)$ to itself (cf. [22]), we can pass to the upper limit on the right hand side for $n \to \infty$. This yields

$$
\begin{aligned}
&\langle Au + F_u(u), (v-u)^+\rangle + \int_\Omega j_1^o(\cdot,u;(v-u)^+)dx\\
&\quad + \int_{\partial\Omega} j_2^o(\cdot,\gamma u;\gamma(v-u)^+)dx \ge 0, \quad \forall v\in K,
\end{aligned}
$$

which shows that u is a supersolution of (5.1), that is, $u \in H$. As $\overline{u}$ is an upper bound of $L(H)$, we can apply Lemma 5.3, which yields the existence of a greatest fixed point u^* of L in H. This implies that u^* must be the greatest solution of (5.1) in $[\underline{u}, \overline{u}]$. By analogous reasoning, one shows the existence of a smallest solution u_* of (5.1). This completes the proof of the theorem. □

Remark. Sub- and supersolutions of problem (5.1) have been constructed in [15] under the conditions (A1')–(A3'), (j1)–(j2) and (F2)(i)–(F2)(iii), where the gradient dependence of f has been dropped, meaning $f(x, r, s) := f(x, r, s, \xi)$. Further, it is assumed that $A = -\Delta_p$ which is the negative $p-$Laplacian defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{where} \quad \nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N).$$

The coefficients $a_i, i = 1, \ldots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2} \xi_i.$$

Thus, hypothesis (A1') is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2') is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see [9, Page 37]) and (A3') is satisfied with $c_1 = 1$ and $k_1 = 0$. The construction is done by using solutions of simple auxiliary elliptic boundary value problems and the eigenfunction of the $p-$Laplacian which belongs to its first eigenvalue.

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