

# $W^{1,p}$ VERSUS $C^1$ : THE NONSMOOTH CASE INVOLVING CRITICAL GROWTH

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ABSTRACT. In this paper we study a class of generalized and not necessarily differentiable functionals of the form

$$J(u) = \int_{\Omega} G(x, \nabla u) dx - \int_{\Omega} j_1(x, u) dx - \int_{\partial\Omega} j_2(x, u) d\sigma$$

with functions  $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  that are only locally Lipschitz in the second argument and involving critical growth for the elements of their generalized gradients  $\partial j_k(x, \cdot)$ ,  $k = 1, 2$  even on the boundary  $\partial\Omega$ . We generalize the famous result of Brezis-Nirenberg [3] to a more general class of functionals and extend all the other generalizations of this result which has been published in the last decades.

## 1. INTRODUCTION

Consider the following functional  $\Phi: H_0^1(\Omega) \rightarrow \mathbb{R}$  defined by

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx$$

where  $F(x, s) = \int_0^s f(x, t) dt$  with a Carathéodory function  $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  that satisfies the growth condition

$$|f(x, u)| \leq C(1 + |u|^p) \quad \text{with } p \leq \frac{N+2}{N-2}.$$

It is well known that a local  $C_0^1(\overline{\Omega})$ -minimizer of  $\Phi$  is also a local  $H_0^1(\Omega)$ -minimizer of  $\Phi$ . Such a result is originally due to Brezis-Nirenberg [3] for functionals on  $H_0^1$  and the critical points of  $\Phi$  are weak solutions of the equation

$$\begin{aligned} -\Delta u &= f(x, u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\Delta$  denotes the well-known Laplace differential operator. An extension of the result of Brezis-Nirenberg to functionals related with the  $p$ -Laplace differential operator was done by García Azorero-Peral Alonso-Manfredi [6] who considered the functional  $J_p: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$J_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(x, u) dx$$

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where  $F(x, s) = \int_0^s f(x, t)dt$  and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following growth condition

$$|f(x, s)| \leq C(1 + |s|^{r-1}) \quad \text{with} \quad r < \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

A simpler proof than those in [6] but only in case  $p > 2$  was done by Guo-Zhang [11]. A nonsmooth version for functionals defined on  $W_0^{1,p}(\Omega)$  with  $p \geq 2$  has been studied by Motreanu-Papageorgiou [16].

The first paper concerning local minimizers of functional corresponding to nonlinear parametric Neumann problems was written by Motreanu-Motreanu-Papageorgiou [18]. Therein, the potential  $\Phi_0 : W_n^{1,p}(\Omega) \rightarrow \mathbb{R}$  is defined by

$$\Phi_0(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z F_0(z, x(z))dz, \quad 1 < p < \infty$$

with

$$W_n^{1,p}(\Omega) = \left\{ y \in W^{1,p}(\Omega) : \frac{\partial x}{\partial n} = 0 \right\},$$

where  $\frac{\partial x}{\partial n}$  is the outer normal derivative of  $u$  and  $F_0(z, x) = \int_0^x f_0(z, s)ds$ . The first result dealing with nonsmooth functionals defined on  $W_n^{1,p}(\Omega)$  for the case  $2 \leq p < \infty$  was proved by Barletta-Papageorgiou [2] while the general case  $1 < p < \infty$  has been treated by Iannizzotto-Papageorgiou [13]. The first result concerning functionals defined on  $W^{1,p}(\Omega)$  involving a boundary term was published by the third author in the smooth [21] and in the nonsmooth [22] case. Moreover, a singular functional  $I : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$I(u) = \frac{1}{p} \|u\|_{W_0^{1,p}(\Omega)}^p - \int_\Omega F(x, u^+) dx - \int_\Omega G(u^+) dx$$

with  $F(x, t) = \int_0^t f(x, s) ds$  and  $G(t) = \int_0^t g(s) ds$  with  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  being a singular term such that  $\lim_{t \rightarrow 0^+} g(t) = +\infty$  was studied by Giacomoni-Saoudi [10].

All the above mentioned works are related to the  $p$ -Laplace differential operator. A first result concerning local minimizers and nonhomogeneous operators was presented in the work of Motreanu-Papageorgiou [17] who studied functionals of the form

$$\varphi_0(u) = \int_\Omega G(x, \nabla u)dx - \int_\Omega F_0(x, u)dx, \quad u \in W_n^{1,p}(\Omega),$$

where  $G$  is the potential of a general nonhomogeneous operator. A prototype of such operator is the  $(p, q)$ -Laplace differential operator which is the sum of the  $p$ - and the  $q$ -Laplacian. A nonsmooth version of functionals related to nonhomogeneous operators defined on the space  $W^{1,p}(\Omega)$  has been studied by Gasiński-Papageorgiou [9].

Recently, Papageorgiou-Rădulescu [19] studied functionals that are not only related to nonhomogeneous operator but also have a boundary term and the potential term in the domain is related to a Carathéodory function that has critical growth. Namely, they considered the functional  $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi_0(u) = \int_\Omega G(Du)dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega F_0(z, u)dz,$$

where  $F_0(z, x) = \int_0^x f_0(z, s)ds$  and  $f_0(x, \cdot)$  has critical growth.

In this paper we are interested in a generalization of all the above mentioned results. The idea is to study functionals on  $W^{1,p}(\Omega)$  which are related to nonhomogeneous operators and involving boundary terms that allow critical growth also at the boundary.

To this end, let  $\Omega \subseteq \mathbb{R}^N$  with  $N > 1$  be a bounded domain with a  $C^{1,\alpha}$ -boundary  $\partial\Omega$  and consider the following functional  $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$J(u) = \int_{\Omega} G(x, \nabla u) dx - \int_{\Omega} j_1(x, u) dx - \int_{\partial\Omega} j_2(x, u) d\sigma, \quad (1.1)$$

where  $G(x, \cdot)$  is the primitive of a function  $a(x, \cdot)$  and the nonlinearities  $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are measurable in the first argument and locally Lipschitz in the second one, that is, for every  $s \in \mathbb{R}$  there exist a neighborhood  $U_{s,k}$  of  $s$  and a constant  $L_{s,k} \geq 0$  such that

$$|j_k(x, r) - j_k(x, t)| \leq L_{s,k} |r - t| \quad \text{for all } r, t \in U_{s,k}, \text{ for } k = 1, 2,$$

and for all  $x \in \Omega$  and for all  $x \in \partial\Omega$ , respectively. It is easy to see that  $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$  need not to be differentiable and clearly it corresponds to the following elliptic inclusion

$$\begin{aligned} -\operatorname{div} a(x, \nabla u) &\ni \partial j_1(x, u) && \text{in } \Omega, \\ a(x, \nabla u) \cdot \nu &\ni \partial j_2(x, \gamma u) && \text{on } \partial\Omega, \end{aligned}$$

where  $\nu(x)$  denotes the outer unit normal of  $\Omega$  at  $x \in \partial\Omega$  and  $\partial j_k(x, u)$ ,  $k = 1, 2$ , stands for Clarke's generalized gradient given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^\circ(x, s; r) \geq \xi r, \text{ for all } r \in \mathbb{R}\},$$

where the term  $j_k^\circ(x, s; r)$  denotes the generalized directional derivative of the locally Lipschitz function  $s \mapsto j_k(x, s)$  at  $s$  in the direction  $r$  defined by

$$j_k^\circ(x, s; r) = \limsup_{y \rightarrow s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t},$$

see Chapter 2 in Clarke [5]. Based on the Hahn-Banach theorem, the set  $\partial j_k(x, s)$  is nonempty. An element  $u \in \mathbb{R}$  is said to be a critical point of a locally Lipschitz function  $f: X \rightarrow \mathbb{R}$  if there holds

$$f^\circ(x; y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently,  $0 \in \partial f(x)$  (see Chang [4]).

## 2. PRELIMINARIES AND HYPOTHESES

For  $1 \leq p < \infty$ , we denote by  $L^p(\Omega)$  and  $L^p(\Omega, \mathbb{R}^N)$  the standard Lebesgue spaces equipped with the norm  $\|\cdot\|_p$  and, for  $1 < p < \infty$ ,  $W^{1,p}(\Omega)$  denotes the Sobolev spaces endowed with the norm  $\|\cdot\|_{1,p}$ . Duality pairing between  $W^{1,p}(\Omega)$  and  $W^{1,p}(\Omega)^*$  will be denoted by  $\langle \cdot, \cdot \rangle$ .

On the boundary  $\partial\Omega$  we consider the  $(N - 1)$ -dimensional Hausdorff (surface) measure  $\sigma$ . Having this measure, we can consider the boundary Lebesgue spaces  $L^q(\partial\Omega)$  for  $1 \leq q \leq \infty$  with norm  $\|\cdot\|_{q,\partial\Omega}$ . Furthermore, we know that there exists a unique linear, continuous map  $\gamma: W^{1,p}(\Omega) \rightarrow L^q(\partial\Omega)$  for  $1 \leq q \leq p_*$  called the trace map such that

$$\gamma(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}),$$

where  $p_*$  is the critical exponent on the boundary given by

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ \text{any } q \in (1, \infty) & \text{if } p \geq N. \end{cases} \quad (2.1)$$

Having the trace operator, we can talk about the boundary values for an arbitrary Sobolev function. Within the paper, we will omit the usage of the trace operator  $\gamma$ , for the sake of notational simplicity. Whenever considering the values of a Sobolev function on  $\partial\Omega$ , we understand that the trace operator is applied.

Furthermore, the Sobolev embedding theorem guarantees the existence of a linear, continuous map  $i: W^{1,p}(\Omega) \rightarrow L^{p^*}(\Omega)$  with the critical exponent in the domain given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \text{any } q \in (1, \infty) & \text{if } p \geq N. \end{cases} \quad (2.2)$$

For more information on the Sobolev embeddings we refer to Gasiński-Papageorgiou [7] or Adams [1].

For  $s \in \mathbb{R}$  we denote by  $s' = \frac{s}{s-1}$  its conjugate, the inner product in  $\mathbb{R}^N$  is denoted by  $\cdot$  and the norm of  $\mathbb{R}^N$  is given by  $|\cdot|$ . Moreover,  $\mathbb{R}_+ = [0, +\infty)$  and the Lebesgue measure is denoted by  $|\cdot|_N$ .

Next, let  $\vartheta \in C^1(0, \infty)$  be any function satisfying

$$0 < a_1 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq a_2 \quad \text{and} \quad a_3 t^{p-1} \leq \vartheta(t) \leq a_4 (t^{q-1} + t^{p-1}) \quad (2.3)$$

for all  $t > 0$ , with some constants  $a_i > 0$ ,  $i \in \{1, 2, 3, 4\}$  and for  $1 < q < p < \infty$ . The hypotheses on  $a: \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  are listed below.

H(a):  $a(x, \xi) = a_0(x, |\xi|)\xi$  with  $a_0 \in C(\bar{\Omega} \times \mathbb{R}_+)$  for all  $\xi \in \mathbb{R}^N$  and with  $a_0(x, t) > 0$  for all  $x \in \bar{\Omega}$ , for all  $t > 0$  and

(i)  $a_0 \in C^1(\bar{\Omega} \times (0, \infty))$ ,  $t \rightarrow ta_0(x, t)$  is strictly increasing in  $(0, \infty)$ ,  $\lim_{t \rightarrow 0^+} ta_0(x, t) = 0$  for all  $x \in \bar{\Omega}$  and

$$\lim_{t \rightarrow 0^+} \frac{ta_0'(x, t)}{a_0(x, t)} = c > -1 \quad \text{for all } x \in \bar{\Omega};$$

(ii)  $|\nabla_\xi a(x, \xi)| \leq a_5 \frac{\vartheta(|\xi|)}{|\xi|}$  for all  $x \in \bar{\Omega}$ , for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and for some  $a_5 > 0$ ;

(iii)  $\nabla_\xi a(x, \xi)y \cdot y \geq \frac{\vartheta(|\xi|)}{|\xi|}|y|^2$  for all  $x \in \bar{\Omega}$ , for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and for all  $y \in \mathbb{R}^N$ .

**Remark 2.1.** *The idea in the choice of the special structure in  $H(a)$  is the usage of the nonlinear regularity theory due to Lieberman [14] coupled with the nonlinear maximum principle of Pucci-Serrin [20] as well as Zhang [23] when considering certain differential equations. If we set*

$$G_0(x, t) = \int_0^t a_0(x, s) s ds,$$

then  $G_0 \in C^1(\bar{\Omega} \times \mathbb{R}_+)$  and the function  $G_0(x, \cdot)$  is increasing and strictly convex for all  $x \in \bar{\Omega}$ . We set  $G(x, \xi) = G_0(x, |\xi|)$  for all  $(x, \xi) \in \bar{\Omega} \times \mathbb{R}^N$  and obtain that

$G \in C^1(\bar{\Omega} \times \mathbb{R}^N)$  and that the function  $\xi \rightarrow G(x, \xi)$  is convex. Moreover, we easily derive that

$$\nabla_{\xi} G(x, \xi) = (G_0)'_t(x, |\xi|) \frac{\xi}{|\xi|} = a_0(x, |\xi|)\xi = a(x, \xi)$$

for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and  $\nabla_{\xi} G(x, 0) = 0$ . In other words,  $G(x, \cdot)$  occurs to be the primitive of  $a(x, \cdot)$ . Combining this with convexity of  $G(x, \cdot)$  and the fact that  $G(x, 0) = 0$  for all  $x \in \bar{\Omega}$  we get

$$G(x, \xi) \leq a(x, \xi) \cdot \xi \quad \text{for all } (x, \xi) \in \bar{\Omega} \times \mathbb{R}^N. \quad (2.4)$$

The following lemma summarizes some properties of the function  $a: \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ .

**Lemma 2.2.** *If hypotheses  $H(a)$  hold, then:*

- (i)  $a \in C(\bar{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\bar{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$  and for all  $x \in \bar{\Omega}$  the map  $\xi \rightarrow a(x, \xi)$  is continuous, strictly monotone and so maximal monotone as well;
- (ii) there exists  $a_6 > 0$ , such that  $|a(x, \xi)| \leq a_6(1 + |\xi|^{p-1})$  for all  $x \in \bar{\Omega}$  and  $\xi \in \mathbb{R}^N$ ;
- (iii)  $a(x, \xi) \cdot \xi \geq \frac{a_3}{p-1}|\xi|^p$  for all  $x \in \bar{\Omega}$  and for all  $\xi \in \mathbb{R}^N$ .

Lemma 2.2 together with (2.4) allow to obtain the following growth estimates on  $G(x, \cdot)$ .

**Corollary 2.3.** *If hypotheses  $H(a)$  hold, then there exists  $a_7 > 0$  such that*

$$\frac{a_3}{p(p-1)}|\xi|^p \leq G(x, \xi) \leq a_7(1 + |\xi|^p)$$

for all  $x \in \bar{\Omega}$  and  $\xi \in \mathbb{R}^N$ .

The nonlinear operator  $A: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  defined by

$$\langle A(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx \quad \text{for all } u, \varphi \in W^{1,p}(\Omega), \quad (2.5)$$

possesses the following useful properties (see Gasiński-Papageorgiou [7]).

**Proposition 2.4.** *If hypotheses  $H(a)$  hold and the operator  $A: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  is defined by (2.5), then  $A$  is bounded, monotone, continuous, hence maximal monotone and of type  $(S_+)$ .*

The following examples expose some operators fitting in our setting.

**Example 2.5.** *In the definitions of the operators  $a$ , we drop the dependence on  $x$  just for simplicity. All the following maps satisfy hypotheses  $H(a)$ :*

- (i) *If  $a(\xi) = |\xi|^{p-2}\xi$  with  $1 < p < \infty$ , then the corresponding operator is the classical  $p$ -Laplacian*

$$\Delta_p u = \operatorname{div} (|\nabla u|^{p-2} \nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

*In this case  $G(\xi) = \frac{1}{p}|\xi|^p$  for all  $\xi \in \mathbb{R}^N$ .*

- (ii) *If  $a(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$  with  $1 < q < p < \infty$  and  $\mu > 0$  then the corresponding operator is the so called weighted  $(p, q)$ -Laplacian defined by  $\Delta_p u + \mu\Delta_q u$  for all  $u \in W^{1,p}(\Omega)$ . In this case  $G(\xi) = \frac{1}{p}|\xi|^p + \frac{\mu}{q}|\xi|^q$  for all  $\xi \in \mathbb{R}^N$ .*

(iii) If  $a(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}} \xi$  with  $1 < p < \infty$ , then this map represents the generalized  $p$ -mean curvature differential operator defined by

$$\operatorname{div} \left[ (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] \quad \text{for all } u \in W^{1,p}(\Omega).$$

In this case  $G(\xi) = \frac{1}{p}(1 + |\xi|^2)^{\frac{p}{2}}$  for all  $\xi \in \mathbb{R}^N$ .

Next, let us give the hypotheses on the nonsmooth potentials  $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ .

- H( $j_1$ ) (i)  $x \mapsto j_1(x, s)$  is measurable in  $\Omega$  for all  $s \in \mathbb{R}$ ;  
(ii)  $s \mapsto j_1(x, s)$  is locally Lipschitz for almost all  $x \in \Omega$ ;  
(iii) for some constant  $c_1 > 0$  and  $1 < q_1 \leq p^*$  (where  $p^*$  is the given in (2.2)), we have

$$|\xi_1| \leq c_1(1 + |s|^{q_1-1})$$

for almost all  $x \in \Omega$  and for all  $\xi_1 \in \partial j_1(x, s)$ .

- H( $j_2$ ) (i)  $x \mapsto j_2(x, s)$  is measurable in  $\partial\Omega$  for all  $s \in \mathbb{R}$ ;  
(ii)  $s \mapsto j_2(x, s)$  is locally Lipschitz for almost all  $x \in \partial\Omega$ ;  
(iii) for some constant  $c_2 > 0$  and  $1 < q_2 \leq p_*$  (where  $p_*$  is given in (2.1)), we have

$$|\xi_2| \leq c_2(1 + |s|^{q_2-1})$$

for almost all  $x \in \partial\Omega$  and all  $\xi_2 \in \partial j_2(x, s)$ ;

- (iv) for any  $u \in W^{1,p}(\Omega)$  and  $\xi_3 \in \partial j_2(x, u)$  we have

$$|\xi_3(x_1) - \xi_3(x_2)| \leq L|x_1 - x_2|^\alpha,$$

for all  $x_1, x_2$  in  $\partial\Omega$  with  $\alpha \in (0, 1]$ .

### 3. MAIN RESULT

The following main result of this paper gives an answer about the relation between local Sobolev and Hölder minimizers of functionals of type  $J$  given in (1.1). We point out again that our functional is more general than the functionals of all the other cited papers above because we have a general, nonhomogeneous operator and we allow critical growth even on the boundary.

**Theorem 3.1.** *Let  $\Omega \subseteq \mathbb{R}^N$  with  $N > 1$  be a bounded domain with a  $C^{1,\alpha}$ -boundary  $\partial\Omega$  and let the assumptions  $H(a)$ ,  $H(j_1)$ , and  $H(j_2)$  be satisfied. If  $u_0 \in W^{1,p}(\Omega)$  is a local  $C^1(\overline{\Omega})$ -minimizer of  $J$ , that is, there exists  $\rho_0 > 0$  such that*

$$J(u_0) \leq J(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

*then  $u_0 \in C^{1,\eta}(\overline{\Omega})$  for some  $\eta \in (0, 1)$  and  $u_0$  is a local  $W^{1,p}(\Omega)$ -minimizer of  $J$ , that is, there exists  $\rho_1 > 0$  such that*

$$J(u_0) \leq J(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

*Proof.* First, from hypotheses  $H(a)$ ,  $H(j_1)$ ,  $H(j_2)$  and Hu-Papageorgiou [12, p. 313] we know that the functional  $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$  is locally Lipschitz continuous. Let  $h \in C^1(\overline{\Omega})$  and let  $t > 0$  be small. Since  $u_0$  is a local  $C^1(\overline{\Omega})$ -minimizer of  $J$ , we have

$$0 \leq \frac{J(u_0 + th) - J(u_0)}{t}.$$

This implies

$$0 \leq J^\circ(u_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$

Notice that the function  $h \mapsto J^\circ(u_0; h)$  is upper semicontinuous and  $C^1(\overline{\Omega})$  is dense in  $W^{1,p}(\Omega)$ , hence

$$0 \leq J^\circ(u_0; h) \quad \text{for all } h \in W^{1,p}(\Omega).$$

Obviously, we have

$$0 \in \partial J(u_0).$$

This means that there exist functions  $g_1 \in L^{q'_1}(\Omega)$  with  $g_1(x) \in \partial j_1(x, u_0(x))$  for almost all  $x \in \Omega$  and  $g_2 \in L^{q'_2}(\partial\Omega)$  with  $g_2(x) \in \partial j_2(x, u_0(x))$  for almost all  $x \in \partial\Omega$  such that

$$\int_{\Omega} a(x, \nabla u_0) \cdot \nabla v dx = \int_{\Omega} g_1 v dx + \int_{\partial\Omega} g_2 v d\sigma \quad \text{for all } v \in W^{1,p}(\Omega). \quad (3.1)$$

Equation (3.1) stands for the weak formulation of the following nonhomogeneous Neumann boundary value problem

$$-\operatorname{div} a(x, \nabla u_0) = g_1 \quad \text{in } \Omega, \quad a(x, \nabla u_0) \cdot \nu = g_2 \quad \text{on } \partial\Omega.$$

It follows from Marino-Winkert [15, Theorem 3.1] that  $u_0 \in L^\infty(\overline{\Omega})$ . This combined with the regularity results due to Lieberman [14] implies the existence of  $\eta \in (0, 1)$  and  $M > 0$  such that

$$u_0 \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_0\|_{C^{1,\eta}(\overline{\Omega})} \leq M. \quad (3.2)$$

To obtain our thesis, we need to show that  $u_0$  is also a local minimizer of  $J$  in the  $W^{1,p}(\Omega)$ -norm. For this purpose, consider the minimizing problem

$$m_0^\varepsilon = \inf_{h \in \overline{B}_\varepsilon} J(u_0 + h), \quad (3.3)$$

where

$$\overline{B}_\varepsilon = \{h \in W^{1,p}(\Omega) \mid \|h\|_{1,p} \leq \varepsilon\}.$$

Arguing by contradiction, assume that  $u_0$  is not a local minimizer of the functional  $J$  in the  $W^{1,p}(\Omega)$ -topology. Then we find  $\varepsilon_0 \in (0, 1]$  such that

$$m_0^\varepsilon < J(u_0) \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \quad (3.4)$$

Fix  $\varepsilon \in (0, \varepsilon_0)$  and let  $\{h_n\}_{n \geq 1} \subset \overline{B}_\varepsilon$  be a minimizing sequence for (3.3), that is,

$$\lim_{n \rightarrow \infty} J(u_0 + h_n) = m_0^\varepsilon. \quad (3.5)$$

From (3.4) we see that  $\|\nabla h_n\|_p$  is bounded and since  $u \rightarrow \|\nabla u\|_p + \|u\|_{p^*}$  is an equivalent norm on  $W^{1,p}(\Omega)$  (we can also use the norm  $u \rightarrow \|\nabla u\|_p + \|u\|_{p^*, \partial\Omega}$ ), it is clear that the sequence  $\{h_n\}_{n \geq 1} \subseteq \overline{B}_\varepsilon$  is bounded in  $W^{1,p}(\Omega)$  and so we can assume that

$$\begin{aligned} h_n &\rightharpoonup h_\varepsilon \quad \text{in } W^{1,p}(\Omega), \text{ in } L^{p^*}(\Omega) \text{ and in } L^{p^*}(\partial\Omega), \\ h_n(x) &\rightarrow h_\varepsilon(x) \quad \text{for a.a. } x \in \Omega \text{ and for a.a. } x \in \partial\Omega, \end{aligned} \quad (3.6)$$

by the Sobolev and the trace embedding theorem, respectively.

Applying the Extended Fatou Lemma (see, Gasiński-Papageorgiou [8, Theorem A.2.8]), we can obtain that  $\varphi$  is sequentially weakly semicontinuous. From (3.5) and (3.6) it follows that

$$m_0^\varepsilon = \inf_{h \in \overline{B}_\varepsilon} J(u_0 + h) \leq J(u_0 + h_\varepsilon) \leq \liminf_{n \rightarrow \infty} J(u_0 + h_n) \leq \lim_{n \rightarrow \infty} J(u_0 + h_n) = m_0^\varepsilon,$$

and hence, due to (3.4),  $h_\varepsilon \neq 0$ .

We are now in the position to apply the nonsmooth Lagrange multiplier rule, see Clarke [5, Theorem 1 and Proposition 13], which guarantees the existence of a multiplier  $\lambda_\varepsilon \geq 0$  such that

$$0 \in \partial J(u_0 + h_\varepsilon) + \lambda_\varepsilon K(h_\varepsilon),$$

where the function  $K: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  is defined by

$$\langle K(h_\varepsilon), v \rangle = \int_{\Omega} |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v dx + \int_{\Omega} |h_\varepsilon|^{p-2} h_\varepsilon v dx \quad \text{for all } v \in W^{1,p}(\Omega).$$

Therefore, there exist  $\hat{g}_1 \in L^{q_1}(\Omega)$  and  $\hat{g}_2 \in L^{q_2}(\partial\Omega)$  with  $\hat{g}_1(x) \in \partial j_1(x, (u_0 + h_\varepsilon)(x))$  for a.a.  $x \in \Omega$  and  $\hat{g}_2(x) \in \partial j_2(x, (u_0 + h_\varepsilon)(x))$  for a.a.  $x \in \partial\Omega$  such that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla(u_0 + h_\varepsilon)) \cdot \nabla v dx - \int_{\Omega} \hat{g}_1 v dx - \int_{\partial\Omega} \hat{g}_2 v d\sigma \\ & + \lambda_\varepsilon \int_{\Omega} |h_\varepsilon|^{p-2} h_\varepsilon v dx + \lambda_\varepsilon \int_{\Omega} |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v dx = 0 \end{aligned} \quad (3.7)$$

for all  $v \in W^{1,p}(\Omega)$ . We need to prove that  $h_\varepsilon$  belongs to  $L^\infty(\Omega)$  and hence to  $C^{1,\eta}(\overline{\Omega})$  for some  $\eta \in (0, 1)$  due to the regularity results due to Lieberman [14]. To end this, let us consider three cases for the multiplier  $\lambda_\varepsilon$ .

**Case 1.**  $\lambda_\varepsilon = 0$  with  $\varepsilon \in (0, 1]$

In this case equation (3.7) becomes

$$\int_{\Omega} a(x, \nabla(u_0 + h_\varepsilon)) \cdot \nabla v dx = \int_{\Omega} \hat{g}_1 v dx + \int_{\partial\Omega} \hat{g}_2 v d\sigma \quad \text{for all } v \in W^{1,p}(\Omega).$$

As before, by applying the a priori results of Marino-Winkert [15, Theorem 3.1], the regularity results due to Lieberman [14, Theorem 2] and the fact that  $u_0 \in C^{1,\eta}$  for some  $\eta \in (0, 1)$  gives

$$h_\varepsilon \in C^{1,\hat{\eta}}(\overline{\Omega}) \quad \text{and} \quad \|h_\varepsilon\|_{C^{1,\hat{\eta}}(\overline{\Omega})} \leq M \quad (3.8)$$

for some  $\hat{\eta} \in (0, 1)$  and  $M > 0$ .

**Case 2.**  $0 < \lambda_\varepsilon \leq 1$  with  $\varepsilon \in (0, 1]$

Multiplying (3.1) by  $\lambda_\varepsilon > 0$  and adding this to (3.7) results in

$$\begin{aligned} & \int_{\Omega} a(x, \nabla(u_0 + h_\varepsilon)) \cdot \nabla v dx + \lambda_\varepsilon \int_{\Omega} a(x, \nabla u_0) \cdot \nabla v dx \\ & + \lambda_\varepsilon \int_{\Omega} |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v dx \\ & = \int_{\Omega} (-\lambda_\varepsilon |h_\varepsilon|^{p-2} h_\varepsilon + \hat{g}_1 + \lambda_\varepsilon g_1) v dx + \int_{\partial\Omega} (\hat{g}_2 + \lambda_\varepsilon g_2) v d\sigma. \end{aligned} \quad (3.9)$$

Now we introduce the map  $T_\varepsilon: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  defined by

$$T_\varepsilon(x, \xi) = a(x, \xi) + \lambda_\varepsilon a(x, H(x)) + \lambda_\varepsilon |\xi - H(x)|^{p-2} (\xi - H(x))$$



for all  $\xi \in \mathbb{R}^N$  and for almost all  $x \in \Omega$ , where  $H(x) = \nabla u_0(x)$  and  $H \in C^\eta(\overline{\Omega}; \mathbb{R}^N)$  for some  $\eta \in (0, 1)$ , thanks to (3.2). Since  $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is continuous (see Lemma 2.2(i)), let  $m_H = \max_{x \in \overline{\Omega}} |a(x, H(x))| = \max_{x \in \overline{\Omega}} |a(x, \nabla u_0(x))|$ . It is easy to see that  $T_\varepsilon \in C(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^N)$ . On the other side, we can apply Lemma 2.2(iii) and Young's inequality to obtain

$$\begin{aligned}
 & T_\varepsilon(x, \xi) \cdot \xi \\
 &= a(x, \xi) \cdot \xi + \lambda_\varepsilon a(x, H(x)) \cdot \xi + \lambda_\varepsilon |\xi - H(x)|^{p-2} (\xi - H(x)) \cdot \xi \\
 &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_\varepsilon |a(x, H(x))| \cdot |\xi| + \lambda_\varepsilon |\xi - H(x)|^p \\
 &\quad - \lambda_\varepsilon |\xi - H(x)|^{p-2} (\xi - H(x)) \cdot H(x) \\
 &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_\varepsilon m_H |\xi| - \lambda_\varepsilon |\xi - H(x)|^{p-1} |H(x)| \\
 &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_\varepsilon m_H |\xi| - \lambda_\varepsilon m_H |\xi - H(x)|^{p-1} \\
 &\geq \frac{a_3}{p-1} |\xi|^p - \delta |\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta),
 \end{aligned}$$

where  $\delta = \frac{a_3}{2(p-1)}$  and  $d_1(\lambda_\varepsilon, m_H, \delta) > 0$  is a constant, which is independent of  $\xi$ . Hence, we have

$$T_\varepsilon(x, \xi) \cdot \xi \geq \frac{a_3}{2(p-1)} |\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta)$$

for all  $\xi \in \mathbb{R}^N$  and for almost all  $x \in \Omega$ . This means that  $T_\varepsilon$  satisfies a strong ellipticity condition. Notice that equation (3.9) can be written in the form

$$\begin{aligned}
 -\operatorname{div}(T_\varepsilon(x, \nabla(u_0 + h_\varepsilon))) &= -\lambda_\varepsilon |h_\varepsilon|^{p-2} h_\varepsilon + \hat{g}_1 + \lambda_\varepsilon g_1 && \text{in } \Omega, \\
 T_\varepsilon(x, \nabla(u_0 + h_\varepsilon)) \cdot \nu &= \hat{g}_2 + \lambda_\varepsilon g_2 && \text{on } \partial\Omega,
 \end{aligned} \tag{3.10}$$

Now are able to apply the again the results of Marino-Winkert [15, Theorem 3.1] which gives  $u_0 + h_\varepsilon \in L^\infty(\overline{\Omega})$ . However,  $u_0 \in C^{1,\eta}(\overline{\Omega})$  leads to  $h_\varepsilon \in L^\infty(\overline{\Omega})$ . Moreover, by using (2.3) and hypothesis H(a)(ii), we obtain

$$\begin{aligned}
 & |\nabla_\xi T_\varepsilon(x, \xi)| \\
 &\leq |\nabla_\xi a(x, \xi)| + \lambda_\varepsilon \left| \nabla_\xi [|\xi - H(x)|^{p-2} (\xi - H(x))] \right| \\
 &\leq a_5 \frac{\vartheta(|\xi|)}{|\xi|} + b_1 + b_2 |\xi|^{p-2} \\
 &\leq a_5 a_4 (1 + 2|\xi|^{p-2}) + b_1 + b_2 |\xi|^{p-2} \\
 &= (2a_4 a_5 + b_2) |\xi|^{p-2} + b_1 + a_4 a_5
 \end{aligned} \tag{3.11}$$

for all  $\xi \in \mathbb{R}^N \setminus \{0\}$ , for a.a.  $x \in \Omega$  and for some  $b_1, b_2 > 0$  which are independent of  $\xi$ . In the same way, applying (2.3) and hypothesis H(a)(iii) leads to

$$\begin{aligned}
& \nabla_\xi T_\varepsilon(x, \xi)y \cdot y \\
&= \nabla_\xi a(x, \xi)y \cdot y + \lambda_\varepsilon \nabla_\xi [|\xi - H(x)|^{p-2}(\xi - H(x))]y \cdot y \\
&\geq \frac{\vartheta(|\xi|)}{|\xi|} |y|^2 + \lambda_\varepsilon |\xi - H(x)|^{p-2} |y|^2 \\
&\quad + \lambda_\varepsilon (p-2) |\xi - H(x)|^{p-4} (\xi - H(x)) \cdot y \\
&\geq c_1 |\xi|^{p-2} |y|^2 + \lambda_\varepsilon \min\{1, p-1\} |\xi - H(x)|^{p-2} |y|^2 \\
&\geq c_1 |\xi|^{p-2} |y|^2.
\end{aligned} \tag{3.12}$$

Finally, since  $h_\varepsilon \in L^\infty(\Omega)$  satisfies (3.10) and because of H(a), (3.11), (3.12) along with hypotheses H(j<sub>1</sub>) and H(j<sub>2</sub>) we are able to apply the regularity results due to Lieberman [14] which gives (3.8) in Case 2 as well.

**Case 3.**  $\lambda_\varepsilon > 1$  with  $\varepsilon \in (0, 1]$

Multiplying (3.1) by  $-1$  and adding this to (3.7) results in

$$\begin{aligned}
& \int_\Omega a(x, \nabla(u_0 + h_\varepsilon)) \cdot \nabla v dx - \int_\Omega a(x, \nabla u_0) \cdot \nabla v dx \\
&\quad + \lambda_\varepsilon \int_\Omega |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v dx \\
&= \int_\Omega (\hat{g}_1 - g_1 - \lambda_\varepsilon |h_\varepsilon|^{p-2} h_\varepsilon) v(x) dx + \int_{\partial\Omega} (\hat{g}_2 - g_2) d\sigma.
\end{aligned} \tag{3.13}$$

As before, we define a map  $T_\varepsilon: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  by

$$T_\varepsilon(x, \xi) = \frac{1}{\lambda_\varepsilon} (a(x, H(x) + \xi) - a(x, H(x))) + |\xi|^{p-2} \xi$$

for all  $\xi \in \mathbb{R}^N$  and for almost all  $x \in \Omega$  where  $H(x) = \nabla u_0(x)$  with  $H \in C^\eta(\bar{\Omega}; \mathbb{R}^N)$  for some  $\eta \in (0, 1)$  because of (3.2). Applying the notation for  $T_\varepsilon$  we can rewrite (3.13) in the following sense

$$\begin{aligned}
-\operatorname{div}(T_\varepsilon(x, \nabla h_\varepsilon)) &= \frac{1}{\lambda_\varepsilon} (\hat{g}_1 - g_1) - |h_\varepsilon|^{p-2} h_\varepsilon && \text{in } \Omega, \\
T_\varepsilon(x, \nabla h_\varepsilon) \cdot \nu &= \frac{1}{\lambda_\varepsilon} (\hat{g}_2 - g_2) && \text{on } \partial\Omega.
\end{aligned}$$

As before we can easily show that

$$\begin{aligned}
\nabla_\xi T_\varepsilon(x, \xi)y \cdot y &\geq b_3 |\xi|^{p-2} |y|^2, \\
T_\varepsilon(x, \xi) \cdot \xi &\geq b_4 |\xi|^p + b_5, \\
|\nabla_\xi T_\varepsilon(x, \xi)| &\leq b_6 |\xi|^{p-2} + b_7,
\end{aligned}$$

for some positive constants  $b_3, b_4, b_5, b_6, b_7$ . Finally applying Marino-Winkert [15, Theorem 3.1] and Lieberman [14, Theorem 2] we reach again (3.8) in Case 3.

Let  $\varepsilon \downarrow 0$ . By the compactness of the embedding  $C^{1, \hat{\eta}}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$  (see Adams [1, p.11]), there exists a subsequence  $\{h_{\varepsilon_n}\}_{n \geq 1}$  of  $\{h_\varepsilon\}$  and a function  $h^* \in C^1(\bar{\Omega})$  such that

$$h_{\varepsilon_n} \rightarrow h^* \quad \text{in } C^1(\bar{\Omega}).$$

Note that  $h_{\varepsilon_n} \in \overline{B}_{\varepsilon_n}$  which gives  $h^* = 0$ . Therefore, we are able to find  $N_0 \in \mathbb{N}$  large enough such that

$$\|h_{\varepsilon_n}\|_{C^1(\overline{\Omega})} \leq r_1 \quad \text{for all } n \geq N_0.$$

Because  $u_0$  is a minimizer of  $J$  in the  $C^1(\overline{\Omega})$ -topology, we have

$$J(u_0) \leq J(u_0 + h_{\varepsilon_n}).$$

However, by the choice of  $\{h_{\varepsilon_n}\}_{n \geq 1}$ , it holds

$$J(u_0 + h_{\varepsilon_n}) = m_{\varepsilon_n}^0 < J(u_0).$$

which is a contradiction. Therefore we conclude that  $u_0$  is a local minimizer of  $J$  in the  $W^{1,p}(\Omega)$ -topology.  $\square$

Let us comment on the case where the functional is smooth. Let  $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be Carathéodory functions, that means, we assume measurability in the first argument and continuity in the second one. We define  $F(x, s) = \int_0^s f(x, t)dt$ ,  $H(x, s) = \int_t^s h(x, t)dt$  and consider the functional  $I: W^{1,p}(\Omega) \rightarrow \mathbb{R}$  given by

$$I(u) = \int_{\Omega} G(x, \nabla u)dx - \int_{\Omega} F(x, u)dx - \int_{\partial\Omega} H(x, u)d\sigma. \quad (3.14)$$

Of course,  $I \in C^1(W^{1,p}(\Omega))$ . For the functions  $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  we suppose the existence of constants  $c_1, c_2 > 0$  such that

$$\begin{aligned} |f(x, s)| &\leq c_1 (1 + |s|^{q_1-1}) \quad \text{for a.a. } x \in \Omega, \\ |h(x, s)| &\leq c_2 (1 + |s|^{q_2-1}) \quad \text{for a.a. } x \in \partial\Omega, \end{aligned} \quad (3.15)$$

for all  $s \in \mathbb{R}$  and for  $1 < q_1 \leq p^*$  as well as  $1 < q_2 \leq p_*$ . Moreover,  $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the condition

$$|h(x, s) - h(y, t)| \leq L [|x - y|^\alpha + |s - t|^\alpha], \quad |g(x, s)| \leq L \quad (3.16)$$

for all  $(x, s), (y, t) \in \partial\Omega \times [-M_0, M_0]$  with  $\alpha \in (0, 1]$  and constants  $M_0 > 0$  and  $L \geq 0$ .

Then, Theorem 3.1 states the following for the functional  $I: W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined in (3.14).

**Theorem 3.2.** *Let  $\Omega \subseteq \mathbb{R}^N$  with  $N > 1$  be a bounded domain with a  $C^{1,\alpha}$ -boundary  $\partial\Omega$  and let the assumptions  $H(a)$ , (3.15) and (3.16) be satisfied. If  $u_0 \in W^{1,p}(\Omega)$  is a local  $C^1(\overline{\Omega})$ -minimizer of  $I$ , that is, there exists  $\rho_0 > 0$  such that*

$$I(u_0) \leq I(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

*then  $u_0 \in C^{1,\eta}(\overline{\Omega})$  for some  $\eta \in (0, 1)$  and  $u_0$  is a local  $W^{1,p}(\Omega)$ -minimizer of  $I$ , that is, there exists  $\rho_1 > 0$  such that*

$$I(u_0) \leq I(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

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