STRONGLY SINGULAR PROBLEMS WITH UNBALANCED GROWTH

MARCOS T.O. PIMENTA AND PATRICK WINKERT

Abstract. In this paper we study strongly singular problems with Dirichlet boundary condition on bounded domains given by

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)=\frac{h(x)}{(u^+)^r}\quad \text{in } \Omega,$$

where $1<p<N$, $p<q<p^*=\frac{Np}{N-p}$, $0\leq \mu(\cdot)\in L^\infty(\Omega)$, $1<r$ and $h\in L^1(\Omega)$ with $h(x)>0$ for a.a. $x\in\Omega$. Since the exponent r is larger than one, the corresponding energy functional is not continuous anymore and so the related Nehari manifold

$$\mathcal{N}=\left\{u\in W_0^{1,\mathcal{H}}(\Omega)\colon \|\nabla u\|_p^p+\|\nabla u\|_{q,\mu}^q-\int_\Omega h(x)\left(u^+\right)^{1-r}\,\mathrm{d}x=0\right\}$$

is not closed in the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$. Instead we are minimizing the energy functional over the constraint set

$$\mathcal{M}=\left\{u\in W_0^{1,\mathcal{H}}(\Omega)\colon \|\nabla u\|_p^p+\|\nabla u\|_{q,\mu}^q-\int_\Omega h(x)\left(u^+\right)^{1-r}\,\mathrm{d}x\geq 0\right\},$$

which turns out to be closed in $W_0^{1,\mathcal{H}}(\Omega)$ and prove the existence of at least one weak solution. Our result is even new in the case when the weight function μ is away from zero.

1. Introduction

In 1991, Lazer-McKenna [13] studied the strongly singular problem

$$\Delta u(x)+p(x)u(x)^{-\gamma}=0\quad \text{in } \Omega,\qquad u\big|_{\partial\Omega}=0, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 1$, with a $C^{2+\alpha}$-boundary for $\alpha\in(0,1)$, the function p is of class $C^\alpha(\overline{\Omega})$ and strictly positive in $\overline{\Omega}$ and $\gamma>0$. The authors proved that there exists a H_0^1-solution of (1.1) if and only if $\gamma<3$. In particular, if $\gamma>1$, then there is no solution belonging to $C^1(\overline{\Omega})$. Note that in case $N=1$, (1.1) occurs in different kind of problems in fluid mechanics and pseudoplastic flow, see the works of Nachman-Callegari [17] and Stuart [25]. Crandall-Rabinowitz-Tartar [5] proved that a solution of (1.1) exists if the domain Ω is of class C^3 and there is no solution in $C^1(\overline{\Omega})$ if $\gamma>1$, see also the work of Gatica-Oliker-Waltman [9] in case Ω is the open unit ball in $\mathbb{R}^N$.

In 2010 Boccardo-Orsina [1] proved existence, regularity and nonexistence results for problems of type

$$\Delta u=\frac{f(x)}{u^\gamma}\quad \text{in } \Omega,\qquad u\big|_{\partial\Omega}=0, \tag{1.2}$$

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where $\gamma > 0$ and f is a nonnegative function belonging to certain Lebesgue spaces. The authors showed several existence results for problem (1.2) even in the cases $\gamma = 1$ and $\gamma > 1$. We also refer to the results of Sun-Wu-Long [23] and Yang [27] in the semilinear case but for $\gamma < 1$.

An extension of the classical existence result of Lazer-McKenna [13] has been done by Sun-Zhang [24] (see also Sun [22]) who considered the problem

$$-\Delta u = \frac{h(x)}{u^p} \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0, \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded regular domain with $N \geq 3$, $p > 1$ and h is a positive L^1-function. It is shown that problem (1.3) admits a unique H_0^1-solution if and only if there exists $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega h(x)|u_0|^{1-p}\,\mathrm{d}x < +\infty.$$

A similar treatment for Kirchhoff problems has been used by Li [14].

Motivated by the works in [14] and [24], in this paper we are going to study the following strongly singular problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) = \frac{h(x)}{(u^+)^r} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $u^+ = \max(u, 0)$ and the operator involved is the so-called double phase operator. In addition, we suppose the following assumptions on the data:

(H$_1$) $1 < p < N$, $p < q < p^* = \frac{Np}{N-p}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;
(H$_2$) $1 < r$ and $h \in L^1(\Omega)$ with $h(x) > 0$ for a.a. $x \in \Omega$.

We call a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ a weak solution of problem (1.4) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega h(x)u^{-r}\varphi\,\mathrm{d}x$$

is satisfied for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ provided all the integrals converge.

Our main result is the following one.

Theorem 1.1. *Let hypotheses* (H$_1$)–(H$_2$) *be satisfied and assume that there exists* $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$ *such that*

$$\int_\Omega h(x)\left(u_0^+\right)^{r-1}\mathrm{d}x < +\infty.$$

Then, problem (1.4) *admits at least one positive solution.*

The difficulties in the proof of Theorem 1.1 lie in the fact that the exponent r is larger than one, so we have a strongly singular problem. To be more precise, due to this fact, the related energy functional is not continuous anymore since the function

$$u \mapsto \int_\Omega h(x)\left(u^+\right)^{1-r}\mathrm{d}x$$

is not continuous on $W_0^{1,\mathcal{H}}(\Omega)$. This implies, in particular, that the related Nehari manifold to problem (1.4) given by

$$\mathcal{N} = \left\{u \in W_0^{1,\mathcal{H}}(\Omega)\colon \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega h(x)\left(u^+\right)^{1-r}\mathrm{d}x = 0\right\}$$

is not a closed set in the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ as it is the case when $r \in (0,1)$. Indeed, if $r \in (0,1)$, a standard proceeding is the splitting of the Nehari manifold into three disjoint parts and minimizing the energy functional over two of them to get two positive solutions with different energy sign, see, for example, the papers of Chen-Kuo-Wu [3] for degenerate Kirchhoff Laplacian problems with sign-changing weight, Fiscella-Mishra [7] for fractional Kirchhoff problems, Kumar-Rădulescu-Sreenadh [12] for critical (p,q)-equations, Papageorgiou-Repovš-Vetro [19] for weighted (p,q)-Laplacian, Liu-Dai-Papageorgiou-Winkert [16] for double phase problems and Tang-Chen [26] for ground state solutions of Schrödinger type, see also the references therein.

As already mentioned, this treatment is not possible in our case due to the appearance of the strongly singular term. Instead of this, we consider the constraint set

$$\mathcal{M} = \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \colon \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega h(x)\left(u^+\right)^{1-r} \mathrm{d}x \geq 0 \right\},$$

which is a closed set in $W_0^{1,\mathcal{H}}(\Omega)$, see Lemma 3.2. The idea is to minimize the corresponding energy functional of problem (1.4) over the constraint set $\mathcal{M}$ to get a positive solution. This is done by using Ekeland's Variational Principle to get a sequence with limit $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$ and this limit turns out to be an element of the Nehari manifold $\mathcal{N}$ (see Proposition 3.4) which contains all nontrivial weak solutions of problem (1.4).

The main arguments we use here are strongly influenced by Sun-Zhang [24], however, some technical difficulties arrise. For example, since we are dealing with a non-homogeneous operator, for each $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that $\int_\Omega h(x)(u^+)^{1-r}\,\mathrm{d}x < +\infty$, we cannot explicitly express $t_u > 0$ in terms of u, see Lemma 3.1. This complicates the proofs of some estimates involving this number, especially some crucial estimates in Proposition 3.4.

As far as we know our treatment has not been used before for double phase problems and also not for (p,q)-problems. In general, there are only few papers for strongly singular problems in the context of nonlinear operators. In order to mention some of them, we refer to the works of Carvalho-Goncalves-Silva-Santos [2] for a type of Brézis-Oswald problem to Φ-Laplacian operators with strongly singular terms, Gambera-Guarnotta [8] for strongly singular convective elliptic equations in $\mathbb{R}^N$ driven by a nonhomogeneous operator, Giachetti-Oliva-Petitta [10] for 1-Laplacian problems with strongly singular right-hand sides and Papageorgiou-Rădulescu-Zhang [18] for strongly singular (p,q)-problems. The methods used in these papers are different from our treatment.

This paper is organized as follows. In Section 2 we recall the main properties of Musielak-Orlicz Sobolev spaces and of the double phase operator. The proof of Theorem 1.1 is then given in Section 3 by studying the constraint set $\mathcal{M}$ and minimizing the energy functional of (1.4) over $\mathcal{M}$.

2. Preliminaries

In this section we recall the definition of the Musielak-Orlicz Sobolev spaces and the double phase operator including their properties. For more information on these spaces and the operators we refer to Colasuonno-Squassina

[4], Crespo-Blanco-Gasiński-Harjulehto-Winkert [6], Harjulehto-Hästö [11], Liu-Dai [15], Papageorgiou-Winkert [20] and Perera-Squassina [21].

To this end, we denote by Ω a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and with a Lipschitz boundary $\partial\Omega$. For $1 \leq r \leq \infty$, $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ represent the usual Lebesgue spaces equipped with the norm $\|\cdot\|_r$. Moreover, we denote by $W_0^{1,r}(\Omega)$ the corresponding Sobolev space endowed with the equivalent norm $\|\nabla\cdot\|_r$ for $r \in (1,\infty)$. Next, we introduce the nonlinear function $\mathcal{H}\colon \Omega \times [0,\infty) \to [0,\infty)$ given by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q,$$

whereby we assume hypothesis (H_1). Denoting by $M(\Omega)$ the set of all measurable functions $u\colon \Omega \to \mathbb{R}$, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega)\colon \rho(u) < +\infty\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\lambda > 0\colon \rho\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

whereas ρ stands for the modular function given as

$$\rho(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x.$$

Furthermore, $L^q_\mu(\Omega)$ stands for the seminormed space

$$L^q_\mu(\Omega) = \left\{u \in M(\Omega)\colon \int_\Omega \mu(x)|u|^q\,\mathrm{d}x < +\infty\right\},$$

equipped with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Similarly, we can define $L^q_\mu(\Omega;\mathbb{R}^N)$. The related Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

is equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}.$$

Finally, the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}}(\Omega)$. We know that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces and we can equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\| = \|\nabla u\|_{\mathcal{H}},$$

see the paper of Crespo-Blanco-Gasiński-Harjulehto-Winkert [6].

The relation between the modular ρ and the norm $\|\cdot\|_{\mathcal{H}}$ has been proven in the paper of Liu-Dai [15].

Proposition 2.1. *Let* (H_1) *be satisfied,* $\lambda > 0$ *and* $y \in L^{\mathcal{H}}(\Omega)$*. Then the following hold:*

(i) *If* $y \neq 0$*, then* $\|y\|_{\mathcal{H}} = \lambda$ *if and only if* $\rho(\frac{y}{\lambda}) = 1$*;*
(ii) $\|y\|_{\mathcal{H}} < 1$ *(resp.* > 1, $= 1$*) if and only if* $\rho(y) < 1$ *(resp.* > 1, $= 1$*);*
(iii) *If* $\|y\|_{\mathcal{H}} < 1$*, then* $\|y\|^q_{\mathcal{H}} \leq \rho(y) \leq \|y\|^p_{\mathcal{H}}$*;*

(iv) *If* $\|y\|_{\mathcal{H}} > 1$, *then* $\|y\|_{\mathcal{H}}^p \leq \rho(y) \leq \|y\|_{\mathcal{H}}^q$;
(v) $\|y\|_{\mathcal{H}} \to 0$ *if and only if* $\rho(y) \to 0$;
(vi) $\|y\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho(y) \to +\infty$.

Again, from Crespo-Blanco-Gasiński-Harjulehto-Winkert [6, Proposition 2.16] we know that

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$$

is continuous for all $r \in [1, p^*]$ and compact for all $r \in [1, p^*)$.

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. Obviously, $|u| = u^+ + u^-$ and $u = u^+ - u^-$. Further, it is known that $u^\pm \in W_0^{1,\mathcal{H}}(\Omega)$ whenever $u \in W_0^{1,\mathcal{H}}(\Omega)$. In the following, the Lebesgue measure of a set $K \subseteq \mathbb{R}^N$ will be denoted by $|K|_N$.

Let us now summarize the properties of the our operator. To this end, let $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$ with the duality pairing $\langle \cdot\,, \cdot\rangle$ of $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The following proposition can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [6, Theorem 3.3].

Proposition 2.2. *Under hypotheses*(H_1), *the operator* A *is bounded, continuous, strictly monotone and satisfies the* (S_+)*-property, i.e., if*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$$

hold, then we have $u_n \to u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$.

3. Proof of Theorem 1.1

First, we introduce the energy functional $I\colon W_0^{1,\mathcal{H}}(\Omega) \to (-\infty, +\infty]$ associated to problem (1.4) given by

$$I(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_{q,\mu}^q - \frac{1}{1-r}\int_\Omega h(x)\left(u^+\right)^{1-r} \mathrm{d}x.$$

Next, let us consider the Nehari manifold related to problem (1.4) which is defined by

$$\begin{aligned}\mathcal{N} &= \left\{u \in W_0^{1,\mathcal{H}}(\Omega)\colon \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega h(x)\left(u^+\right)^{1-r} \mathrm{d}x = 0\right\}\\ &= \left\{u \in W_0^{1,\mathcal{H}}(\Omega)\colon \varphi_u'(1) = 0\right\},\end{aligned}$$

where $\varphi_u\colon [0,\infty) \to \mathbb{R}$ defined by

$$\varphi_u(t) = I(tu) = \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^q}{q}\|\nabla u\|_{q,\mu}^q - \frac{t^{1-r}}{1-r}\int_\Omega h(x)\left(u^+\right)^{1-r} \mathrm{d}x$$

for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that $\int_\Omega h(x)(u^+)^{1-r}\,\mathrm{d}x < +\infty$, is the fibering map.

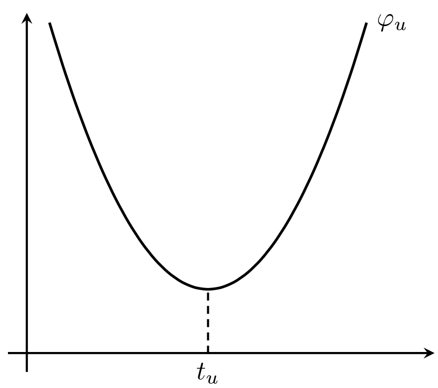
Since $1 < r$, the function $u \mapsto \int_\Omega h(x)(u^+)^{1-r}\,\mathrm{d}x$ is not continuous on $W_0^{1,\mathcal{H}}(\Omega)$ and so is I. Hence, we cannot prove that $\mathcal{N}$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$. This is the reason why we deal with another constraint to the problem. More specifically, let us define

$$\begin{aligned}\mathcal{M} &= \left\{u \in W_0^{1,\mathcal{H}}(\Omega) \colon \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega h(x)\left(u^+\right)^{1-r} \,\mathrm{d}x \geq 0\right\}\\ &= \left\{u \in W_0^{1,\mathcal{H}}(\Omega) \colon \varphi_u'(1) \geq 0\right\}.\end{aligned}$$

In the next results, we study some general aspects of $\mathcal{N}$ and $\mathcal{M}$.

Lemma 3.1. *For each $u \in W_0^{1,\mathcal{H}}(\Omega)$, such that $\int_\Omega h(x)(u^+)^{1-r}\,\mathrm{d}x < +\infty$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover,*

$$I(t_u u) = \min_{t>0} I(tu).$$



Proof. Note that φ_u is $C^1((0,+\infty))$ and, for $t > 0$,

$$\varphi_u'(t) = t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,\mu}^q - t^{-r}\int_\Omega h(x)\left(u^+\right)^{1-r}\,\mathrm{d}x.$$

Obviously, $tu \in \mathcal{N}$ if and only if $\varphi_u'(t) = 0$ which is equivalent to

$$\int_\Omega h(x)\left(u^+\right)^{1-r}\,\mathrm{d}x = t^{p+r-1}\|\nabla u\|_p^p + t^{q+r-1}\|\nabla u\|_{q,\mu}^q. \tag{3.1}$$

If we define

$$g(t) := t^{p+r-1}\|\nabla u\|_p^p + t^{q+r-1}\|\nabla u\|_{q,\mu}^q,$$

it is easily seen that $g(0) = 0$ and $\lim_{t\to+\infty} g(t) = +\infty$. Then, there exists $t_u > 0$ such that (3.1) holds true, which is equivalent to $t_u u \in \mathcal{N}$. On the other hand, we have

$$g'(t) = (p+r-1)t^{p+r-2}\|\nabla u\|_p^p + (q+r-1)t^{q+r-2}\|\nabla u\|_{q,\mu}^q > 0$$

for all $t > 0$. Hence, g is strictly increasing and so $t_u > 0$ is unique. □

Next, we show that $\mathcal{M}$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$.

Lemma 3.2. *$\mathcal{M}$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be such that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Then, Fatou's Lemma implies that, up to a subsequence,

$$\|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega h(x)(u^+)^{1-r}\,\mathrm{d}x$$

$$\geq \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \int_\Omega h(x)\big(u_n^+\big)^{1-r}\,\mathrm{d}x + o_n(1)$$
$$\geq 0 + o_n(1).$$

Passing to the limit in both sides, we have that $u \in \mathcal{M}$. □

Now we prove that the set $\mathcal{M}$ is bounded away from the origin.

Lemma 3.3. *There exists $\delta > 0$ such that $\|u\| \geq \delta$ for all $u \in \mathcal{M}$.*

Proof. We assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that

$$u_n \to 0 \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \text{ as } n \to +\infty.$$

Then, by the reverse Hölder inequality, it follows

$$\begin{aligned}\left(\int_\Omega h(x)^{\frac{1}{r}}\,\mathrm{d}x\right)^r \cdot \left(\int_\Omega u_n^+\,\mathrm{d}x\right)^{1-r} &\leq \int_\Omega h(x)\big(u_n^+\big)^{1-r}\,\mathrm{d}x\\ &\leq \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q\\ &= o_n(1).\end{aligned}$$

This, in turn, implies that

$$\frac{1}{\|u_n^+\|_1^{r-1}} = o_n(1).$$

Therefore, $\|u_n^+\|_1 \to +\infty$ as $n \to \infty$ and so we get a contradiction. □

Note that, if $u \in \mathcal{M}$, then

$$\begin{aligned}I(u) &\geq \frac{1}{q}\left(\|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q\right) - \frac{1}{1-r}\int_\Omega h(x){u^+}^{1-r}\,\mathrm{d}x\\ &\geq \left(\frac{1}{q} - \frac{1}{1-r}\right)\int_\Omega h(x)\big(u^+\big)^{1-r}\,\mathrm{d}x.\end{aligned}$$

Hence, I is bounded from below on $\mathcal{M}$. Moreover, from Lemma 3.2 we know that $\mathcal{M}$ is closed and so it is a complete metric space. On the other hand, I is a proper (since by assumption, $\int_\Omega h(x)(u_0^+)^{q-1}\,\mathrm{d}x < +\infty$) and a lower semicontinuous functional over $\mathcal{M}$. Thereby, Ekeland's Variational Principle implies that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that

$$\lim_{n\to+\infty} I(u_n) = \inf_{\mathcal{M}} I$$

and

$$I(u_n) - I(v) \leq \frac{1}{n}\|u_n - v\| \quad \text{for all } v \in \mathcal{M}. \tag{3.2}$$

Since $I(u) = I(|u|)$, we can assume, without any loss of generality, that $u_n \geq 0$ a.e. in Ω and for all $n \in \mathbb{N}$. Moreover, since $r > 1$, we have

$$\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x \leq \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q < +\infty,$$

which implies that $u_n > 0$ a.e. in Ω.

Furthermore, by Proposition 2.1 (iii) and (iv), it holds

$$\begin{aligned}\frac{1}{q}\min\{\|\nabla u_n\|_{\mathcal{H}}^p, \|\nabla u_n\|_{\mathcal{H}}^q\} \leq \frac{1}{q}\rho(\nabla u) &= \frac{1}{q}\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q\right)\\ &\leq I(u_n) = \inf_{\mathcal{M}} I + o_n(1).\end{aligned}$$

Hence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. So, we can assume, up to a subsequence if necessary, that there exists $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } W_0^{1,\mathcal{H}}(\Omega),\\ u_n &\to u_0 \quad \text{in } L^s(\Omega) \text{ for } 1 \le s < p^*\\ u_n &\to u_0 \quad \text{a.e. in } \Omega. \end{aligned}$$

Note that, even though we are minimizing the functional I on $\mathcal{M}$, we want to show that the minimizer actually belongs to $\mathcal{N}$, since all the nontrivial solutions of (1.4) belong to $\mathcal{N}$. In the next result, we prove this crucial fact.

Proposition 3.4. *It holds that $u_0 \in \mathcal{N}$. Moreover, we have*

$$\int_\Omega \left(|\nabla u_0|^{p-2}\nabla u_0 + \mu(x)|\nabla u_0|^{q-2}\nabla u_0\right)\cdot\nabla\varphi\,\mathrm{d}x \ge \int_\Omega h(x)u_0^{-r}\varphi\,\mathrm{d}x$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\varphi \ge 0$.

Proof. We will split the proof into two cases. In the first one we assume that $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}\setminus\mathcal{N}$. Let us fix $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\varphi \ge 0$ and $n \in \mathbb{N}$.

Since $u_n \notin \mathcal{N}$, we have for $t > 0$ that

$$\int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x \le \int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x < \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q.$$

Choosing $t > 0$ sufficiently small, it follows that

$$\int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x < \|\nabla(u_n + t\varphi)\|_p^p + \|\nabla(u_n + t\varphi)\|_{q,\mu}^q,$$

which implies that $u_n + t\varphi \in \mathcal{M}$. Hence, from (3.2), we obtain

$$\begin{aligned} \frac{t}{n}\|\varphi\| &\ge I(u_n) - I(u_n + t\varphi)\\ &= \frac{1}{p}\left(\|\nabla u_n\|_p^p - \|\nabla(u_n + t\varphi)\|_p^p\right) + \frac{1}{q}\left(\|\nabla u_n\|_{q,\mu}^q - \|\nabla(u_n + t\varphi)\|_{q,\mu}^q\right)\\ &\quad - \frac{1}{1-r}\int_\Omega \left(h(x)u_n^{1-r} - h(x)(u_n + t\varphi)^{1-r}\right)\,\mathrm{d}x. \end{aligned}$$

Dividing the last inequality by $t > 0$ and calculating the lim inf as $t \to 0^+$, we have

$$\begin{aligned} &\frac{\|\varphi\|}{n} + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\\ &\ge \liminf_{t\to 0^+}\frac{1}{r-1}\int_\Omega \frac{\left(h(x)u_n^{1-r} - h(x)(u_n + t\varphi)^{1-r}\right)}{t}\,\mathrm{d}x\\ &\ge \int_\Omega h(x)u_n^{-r}\varphi\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

Recall that $u_n \rightharpoonup u_0$ in $W_0^{1,\mathcal{H}}(\Omega)$. Then the weak lower semicontinuity of the norm $\|\nabla\cdot\|_p^p$ and the seminorm $\|\nabla\cdot\|_{q,\mu}^q$ along with Fatou's Lemma imply that

$$\begin{aligned} \inf_{\mathcal{M}} I &= \lim_{n\to+\infty} I(u_n)\\ &\ge \liminf_{n\to+\infty}\left(\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{q}\|\nabla u_n\|_{q,\mu}^q\right) - \frac{1}{1-r}\liminf_{n\to+\infty}\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x\\ &\ge \frac{1}{p}\|\nabla u_0\|_p^p + \frac{1}{q}\|\nabla u_0\|_{q,\mu}^q - \frac{1}{1-r}\int_\Omega h(x)u_0^{1-r}\,\mathrm{d}x. \end{aligned}$$

This, in turn, implies that $\int_\Omega h(x)u_0^{1-r}\,\mathrm{d}x < +\infty$. Hence, by Lemma 3.1, there exists $t_{u_0} > 0$, such that $t_{u_0}u_0 \in \mathcal{N}$. Hence,

$$\inf_{\mathcal{M}} I \geq I(u_0) \geq I(t_{u_0}u_0) \geq \inf_{\mathcal{N}} I \geq \inf_{\mathcal{M}} I. \tag{3.4}$$

Hence, all the inequalities in (3.4) turn into equalities. This implies that $\rho(u_n) \to \rho(u)$, for a subsequence if necessary, and so, since the modular function ρ is uniformly convex, we get from Proposition 2.1 (v) that $u_n \to u_0$ in $W_0^{1,\mathcal{H}}(\Omega)$. Moreover, $t_{u_0} = 1$ and $u_0 \in \mathcal{N}$.

Now, letting $n \to +\infty$ in (3.3) and using again Fatou's Lemma, we obtain

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\varphi\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla\varphi\,\mathrm{d}x \geq \int_\Omega h(x)u_0^{-r}\varphi\,\mathrm{d}x \tag{3.5}$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$. In particular, (3.5) implies that $u_0 \in \mathcal{M}$.

Now we consider the case in which, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$. Again, let us fix $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$ and $n \in \mathbb{N}$. For all $t \geq 0$, we have

$$\int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x \leq \int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x < +\infty. \tag{3.6}$$

Then, by Lemma 3.1, there exists a unique $f_{n,\varphi}(t) > 0$ such that $f_{n,\varphi}(t)(u_n + t\varphi) \in \mathcal{N}$, that is,

$$\begin{aligned} &f_{n,\varphi}(t)^p\|\nabla(u_n + t\varphi)\|_p^p + f_{n,\varphi}(t)^q\|\nabla(u_n + t\varphi)\|_{q,\mu}^q \\ &= f_{n,\varphi}(t)^{1-r}\int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x. \end{aligned}$$

Hence,

$$\begin{aligned} &f_{n,\varphi}(t)^{p-1+r}\|\nabla(u_n + t\varphi)\|_p^p + f_{n,\varphi}(t)^{q-1+r}\|\nabla(u_n + t\varphi)\|_{q,\mu}^q \\ &= \int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x. \end{aligned} \tag{3.7}$$

Now, let us consider a sequence $t_k \to 0^+$ such that there exists the limit

$$\lim_{k\to+\infty} \frac{f_{n,\varphi}(t_k) - 1}{t_k} =: f'_{n,\varphi}(0) \in [-\infty, +\infty]. \tag{3.8}$$

Note that, up to a subsequence, the sequence $\{f_{n,\varphi}(t_k)\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Indeed, otherwise, $f_{n,\varphi}(t_k) \to +\infty$ as $k \to +\infty$. Then, by (3.7) and Lebesgue's Convergence Theorem, it would follow that

$$\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x = +\infty,$$

which is a contradiction to (3.6). Hence there exists a nonnegative $L \in \mathbb{R}$ such that, up to a subsequence,

$$f_{n,\varphi}(t_k) \to L \quad \text{as } k \to +\infty.$$

Then, passing to the limit in (3.7) as $t_k \to 0^+$ and using Lebesgue's Convergence Theorem, we have that

$$\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x = L^{p-1+r}\|\nabla u_n\|_p^p + L^{q-1+r}\|\nabla u_n\|_{q,\mu}^q. \tag{3.9}$$

But since $u_n \in \mathcal{N}$,

$$\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x = \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q. \tag{3.10}$$

Thus, from (3.9) and (3.10), it follows that $L = 1$. Hence, from now on, the sequence $\{t_k\}_{k\in\mathbb{N}}$ is going to be considered such that (3.8) and also

$$\lim_{k\to+\infty} f_{n,\varphi}(t_k) = 1.$$

hold. Moreover, we assume, without any loss of generality, that either $f_{n,\varphi}(t_k) > 1$ for all $k \in \mathbb{N}$ or $f_{n,\varphi}(t_k) < 1$ for all $k \in \mathbb{N}$. This implies

$$\lim_{k\to+\infty} \operatorname{sgn}(f_{n,\varphi}(t_k) - 1) = \pm 1. \tag{3.11}$$

Moreover, due to $u_n \in \mathcal{N}$, it follows that $f_{n,\varphi}(0) = 1$.

Let us now assume that there exists $C > 0$ such that

$$-C \le f'_{n,\varphi}(0) \le C \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

From now on, for the sake of simplicity, we substitute t_k for t and by $t \to 0^+$, we mean $t_k \to 0^+$. Then, from (3.2) and the fact that $f_{n,\varphi}(t)(u_n + t\varphi) \in \mathcal{M}$, we have

$$\begin{aligned} I(u_n) - I(f_{n,\varphi}(t)(u_n + t\varphi)) &\le \frac{1}{n}\|f_{n,\varphi}(t)(u_n + t\varphi) - u_n\| \\ &\le \frac{1}{n}\left(t f_{n,\varphi}(t)\|\varphi\| + \|u_n\| \cdot |f_{n,\varphi}(t) - 1|\right). \end{aligned}$$

Dividing the last inequality by $t > 0$ and passing to the limit, we get

$$\liminf_{t\to 0^+} \frac{1}{t}\left(I(u_n) - I(f_{n,\varphi}(t)(u_n + t\varphi))\right) \le \frac{1}{n}\left(\|\varphi\| + \|u_n\| \cdot |f'_{n,\varphi}(0)|\right). \tag{3.13}$$

On the other hand, it holds

$$\begin{aligned} &I(u_n) - I(f_{n,\varphi}(t)(u_n + t\varphi)) \\ &= \frac{1}{p}\left(\|\nabla u_n\|_p^p - f_{n,\varphi}(t)^p \|\nabla(u_n + t\varphi)\|_p^p\right) \\ &\quad + \frac{1}{q}\left(\|\nabla u_n\|_{q,\mu}^q - f_{n,\varphi}(t)^q \|\nabla(u_n + t\varphi)\|_{q,\mu}^q\right) \\ &\quad - \frac{1}{1-r}\int_\Omega \left(h(x)u_n^{1-r} - h(x) f_{n,\varphi}(t)^{1-r}(u_n + t\varphi)^{1-r}\right) \mathrm{d}x \\ &= \frac{1}{p}\left(\|\nabla u_n\|_p^p - \|\nabla(u_n + t\varphi)\|_p^p\right) - \left(f_{n,\varphi}(t)^p - 1\right)\frac{1}{p}\|\nabla(u_n + t\varphi)\|_p^p \\ &\quad + \frac{1}{q}\left(\|\nabla u_n\|_{q,\mu}^q - \|\nabla(u_n + t\varphi)\|_{q,\mu}^q\right) - \left(f_{n,\varphi}(t)^q - 1\right)\frac{1}{q}\|\nabla(u_n + t\varphi)\|_{q,\mu}^q \\ &\quad - \frac{1}{1-r}\int_\Omega \left(h(x)u_n^{1-r} - h(x)(u_n + t\varphi)^{1-r}\right) \mathrm{d}x \\ &\quad - \frac{1}{1-r}\left(1 - f_{n,\varphi}(t)^{1-r}\right)\int_\Omega h(x)(u_n + t\varphi)^{1-r}\,\mathrm{d}x. \end{aligned}$$

Dividing by $t>0$ and passing to the limit as $t\to 0^+$, from Fatou's Lemma and the fact that $u_n\in\mathcal{N}$, we get

$$\begin{aligned}
&\liminf_{t\to 0^+}\frac{1}{t}\left(I(u_n)-I(f_{n,\varphi}(t)(u_n+t\varphi))\right)\\
&\geq -\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad - f_{n,\varphi}'(0)\|\nabla u_n\|_p^p - f_{n,\varphi}'(0)\|\nabla u_n\|_{q,\mu}^q+\int_\Omega h(x)u_n^{-r}\varphi\,\mathrm{d}x\\
&\quad + f_{n,\varphi}'(0)\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x\\
&= -\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad +\int_\Omega h(x)u_n^{-r}\varphi\,\mathrm{d}x.
\end{aligned}\tag{3.14}$$

From (3.13) and (3.14), it follows that

$$\begin{aligned}
&\frac{1}{n}\left(\|\varphi\|+\|u_n\|\cdot|f_{n,\varphi}'(0)|\right)\\
&\geq -\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad +\int_\Omega h(x)u_n^{-r}\varphi\,\mathrm{d}x.
\end{aligned}\tag{3.15}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$ and from (3.12), calculating the $\liminf$ as $n\to+\infty$ in (3.15), we have from Fatou's Lemma that (3.5) holds true. In particular, $u_0\in\mathcal{M}$ and we can proceed as in (3.4) and show that, also in this case, $t_{u_0}=1$ and $u_0\in\mathcal{N}$.

Now, it remains to prove that (3.12) holds. Since $u_n, f_{n,\varphi}(t)(u_n+t\varphi)\in\mathcal{N}$, we have that

$$\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,\mu}^q=\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x \tag{3.16}$$

and

$$\begin{aligned}
&f_{n,\varphi}(t)^p\|\nabla(u_n+t\varphi)\|_p^p+f_{n,\varphi}(t)^q\|\nabla(u_n+t\varphi)\|_{q,\mu}^q\\
&=f_{n,\varphi}(t)^{1-r}\int_\Omega h(x)(u_n+t\varphi)^{1-r}\,\mathrm{d}x.
\end{aligned}\tag{3.17}$$

Subtracting (3.16) from (3.17), we get

$$\begin{aligned}
0&=\left(f_{n,\varphi}(t)^p-1\right)\|\nabla(u_n+t\varphi)\|_p^p+\left(\|\nabla(u_n+t\varphi)\|_p^p-\|\nabla u_n\|_p^p\right)\\
&\quad+\left(f_{n,\varphi}(t)^q-1\right)\|\nabla(u_n+t\varphi)\|_{q,\mu}^q+\left(\|\nabla(u_n+t\varphi)\|_{q,\mu}^q-\|\nabla u_n\|_{q,\mu}^q\right)\\
&\quad-\left(f_{n,\varphi}(t)^{1-r}-1\right)\int_\Omega h(x)(u_n+t\varphi)^{1-r}\,\mathrm{d}x\\
&\quad-\int_\Omega\left(h(x)(u_n+t\varphi)^{1-r}-h(x)u_n^{1-r}\right)\,\mathrm{d}x\\
&\geq\left(f_{n,\varphi}(t)^p-1\right)\|\nabla(u_n+t\varphi)\|_p^p+\left(\|\nabla(u_n+t\varphi)\|_p^p-\|\nabla u_n\|_p^p\right)\\
&\quad+\left(f_{n,\varphi}(t)^q-1\right)\|\nabla(u_n+t\varphi)\|_{q,\mu}^q+\left(\|\nabla(u_n+t\varphi)\|_{q,\mu}^q-\|\nabla u_n\|_{q,\mu}^q\right)\\
&\quad-\left(f_{n,\varphi}(t)^{1-r}-1\right)\int_\Omega h(x)(u_n+t\varphi)^{1-r}\,\mathrm{d}x.
\end{aligned}\tag{3.18}$$

Then, we divide (3.18) by $t > 0$ and let $t \to 0^+$. This gives, by taking (3.16) into account, that

$$
\begin{aligned}
0 &\geq p f'_{n,\varphi}(0)\|\nabla u_n\|_p^p + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad + q f'_{n,\varphi}(0)\|\nabla u_n\|_{q,\mu}^q + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad - (1-r) f'_{n,\varphi}(0)\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x \\
&= f'_{n,\varphi}(0)\left(p\|\nabla u_n\|_p^p + q\|\nabla u_n\|_{q,\mu}^q - (1-r)\int_\Omega h(x)u_n^{1-r}\,\mathrm{d}x\right) \\
&\quad + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&= f'_{n,\varphi}(0)\left(p\|\nabla u_n\|_p^p - (1-r)\|\nabla u_n\|_p^p\right) + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad + f'_{n,\varphi}(0)\left(q\|\nabla u_n\|_{q,\mu}^q - (1-r)\|\nabla u_n\|_{q,\mu}^q\right) \\
&\quad + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&\geq (p+r-1)\|\nabla u_n\|_p^p f'_{n,\varphi}(0) + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad + (p+r-1)\|\nabla u_n\|_{q,\mu}^q f'_{n,\varphi}(0) + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x.
\end{aligned}
\tag{3.19}
$$

Then, from Lemma 3.3, Proposition 2.1 (iii) and (iv), (3.19) and Hölder's inequality, it follows that

$$
\begin{aligned}
f'_{n,\varphi}(0) &\leq \frac{-\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x}{(p+r-1)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q)} \\
&\leq \frac{\|\nabla u_n\|_p^{p-1}\|\nabla\varphi\|_p + \|\nabla u_n\|_{q,\mu}^{q-1}\|\nabla\varphi\|_{q,\mu}}{(p+r-1)\min\{\delta^p,\delta^q\}} \\
&\leq C \quad \text{for all } n\in\mathbb{N},
\end{aligned}
$$

where we have used that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ is bounded.

Now, let us start proving that there exists $C > 0$ such that

$$
-C \leq f'_{n,\varphi}(0) \quad \text{for all } n\in\mathbb{N}.
$$

From (3.2) we know that

$$
\begin{aligned}
I(u_n) - I(f_{n,\varphi}(t)(u_n+t\varphi)) &\leq \frac{1}{n}\|f_{n,\varphi}(t)(u_n+t\varphi) - u_n\| \\
&\leq \frac{1}{n}\left(t|f_{n,\varphi}(t)|\|\varphi\| + \|u_n\||f_{n,\varphi}(t)-1|\right).
\end{aligned}
\tag{3.20}
$$

Moreover, since $u_n, f_{n,\varphi}(t)(u_n+t\varphi) \in \mathcal{N}$, it follow that

$$\begin{aligned}
&I(u_n) - I(f_{n,\varphi}(t)(u_n+t\varphi))\\
&= \left(\frac{1}{p} - \frac{1}{1-r}\right)\left(\|\nabla u_n\|_p^p - \|\nabla(u_n+t\varphi)\|_p^p\right)\\
&\quad + \left(\frac{1}{q} - \frac{1}{1-r}\right)\left(\|\nabla u_n\|_{q,\mu}^q - \|\nabla(u_n+t\varphi)\|_{q,\mu}^q\right)\\
&\quad - \left(\frac{1}{p} - \frac{1}{1-r}\right)\left(f_{n,\varphi}(t)^p - 1\right)\|\nabla(u_n+t\varphi)\|_p^p\\
&\quad - \left(\frac{1}{q} - \frac{1}{1-r}\right)\left(f_{n,\varphi}(t)^q - 1\right)\|\nabla(u_n+t\varphi)\|_{q,\mu}^q.
\end{aligned} \tag{3.21}$$

Hence, from (3.20) and (3.21), we have that

$$\begin{aligned}
&\frac{t}{n}\|\varphi\||f_{n,\varphi}(t)| + \left(\frac{1}{p} - \frac{1}{1-r}\right)\left(\|\nabla(u_n+t\varphi)\|_p^p - \|\nabla u_n\|_p^p\right)\\
&+ \left(\frac{1}{q} - \frac{1}{1-r}\right)\left(\|\nabla(u_n+t\varphi)\|_{q,\mu}^q - \|\nabla u_n\|_{q,\mu}^q\right)\\
&\geq -\frac{1}{n}|f_{n,\varphi}(t)-1|\|u_n\|\\
&\quad - \left(\frac{1}{p} - \frac{1}{1-r}\right)\left(f_{n,\varphi}(t)^p - 1\right)\|\nabla(u_n+t\varphi)\|_p^p\\
&\quad - \left(\frac{1}{q} - \frac{1}{1-r}\right)\left(f_{n,\varphi}(t)^q - 1\right)\|\nabla(u_n+t\varphi)\|_{q,\mu}^q\\
&\geq (f_{n,\varphi}(t)-1)\left[-\frac{\|u_n\|}{n}\operatorname{sgn}(f_{n,\varphi}(t)-1)\right.\\
&\qquad\qquad - \left(\frac{1}{p} - \frac{1}{1-r}\right)\frac{f_{n,\varphi}(t)^p-1}{f_{n,\varphi}(t)-1}\|\nabla(u_n+t\varphi)\|_p^p\\
&\qquad\qquad \left.- \left(\frac{1}{p} - \frac{1}{1-r}\right)\frac{f_{n,\varphi}(t)^q-1}{f_{n,\varphi}(t)-1}\|\nabla(u_n+t\varphi)\|_{q,\mu}^q\right].
\end{aligned} \tag{3.22}$$

Note that

$$\frac{f_{n,\varphi}(t)^p - 1}{f_{n,\varphi}(t)-1} = f_{n,\varphi}(t)^{p-1} + f_{n,\varphi}(t)^{p-2} + \ldots + 1 \tag{3.23}$$

and analogously for q. Then, dividing (3.22) by $t > 0$, taking the limit as $t \to 0^+$ and taking (3.11), (3.23) and Lemma 3.3 as well as Proposition 2.1 (iii), (iv) into account, we have that

$$\left(\left(\frac{1}{p} - \frac{1}{1-r}\right) q \min\{\delta^p, \delta^q\} \pm \frac{\|u_n\|}{n}\right) f'_{n,\varphi}(0) \geq C.$$

Hence, we see that for $n \in \mathbb{N}$ large enough,

$$f'_{n,\varphi}(0) \geq C$$

and so we are done. □

Before we can state the proof of Theorem 1.1 we need another auxiliary result.

Lemma 3.5. *If $w \in \mathcal{N}$ is such that*

$$\begin{aligned}
&\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla w|^{q-2}\nabla w \cdot \nabla\varphi\,\mathrm{d}x\\
&-\int_\Omega h(x)(w^+)^{-r}\varphi\,\mathrm{d}x \geq 0
\end{aligned} \tag{3.24}$$

for all $\varphi \in W^{1,\mathcal{H}}_0(\Omega)$ with $\varphi \geq 0$, then

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla w|^{q-2}\nabla w \cdot \nabla\varphi\,\mathrm{d}x - \int_\Omega h(x)(w^+)^{-r}\varphi\,\mathrm{d}x = 0$$

for all $\varphi \in W^{1,\mathcal{H}}_0(\Omega)$.

Proof. Let us fix $\psi \in W^{1,\mathcal{H}}_0(\Omega)$. For $\varepsilon > 0$, let us consider $\varphi = (w+\varepsilon\psi)^+ \in W^{1,\mathcal{H}}_0(\Omega)$. Then, since $w \in \mathcal{N}$ satisfying (3.24), we have

$$\begin{aligned}
0 &\leq \int_\Omega \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega h(x)(w^+)^{-r}\varphi\,\mathrm{d}x\\
&= \int_{\{w+\varepsilon\psi\geq 0\}} \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla(w+\varepsilon\psi)\,\mathrm{d}x\\
&\quad - \int_{\{w+\varepsilon\psi\geq 0\}} h(x)(w^+)^{-r}(w+\varepsilon\psi)\,\mathrm{d}x\\
&= \int_\Omega \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla(w+\varepsilon\psi)\,\mathrm{d}x\\
&\quad - \int_{\{w+\varepsilon\psi< 0\}} \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla(w+\varepsilon\psi)\,\mathrm{d}x\\
&\quad - \int_\Omega h(x)(w^+)^{-r}(w+\varepsilon\psi)\,\mathrm{d}x + \int_{\{w+\varepsilon\psi< 0\}} h(x)(w^+)^{-r}(w+\varepsilon\psi)\,\mathrm{d}x\\
&= \|\nabla w\|_p^p + \|\nabla w\|_{q,\mu}^q - \int_\Omega h(x)w^{1-r}\,\mathrm{d}x\\
&\quad + \varepsilon\int_\Omega \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla\psi\,\mathrm{d}x - \varepsilon\int_\Omega h(x)(w^+)^{-r}\psi\,\mathrm{d}x\\
&\quad - \int_{\{w+\varepsilon\psi< 0\}} \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla(w+\varepsilon\psi)\,\mathrm{d}x\\
&\quad + \int_{\{w+\varepsilon\psi< 0\}} h(x)(w^+)^{-r}(w+\varepsilon\psi)\,\mathrm{d}x\\
&= \varepsilon\int_\Omega \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla\psi\,\mathrm{d}x - \varepsilon\int_\Omega h(x)(w^+)^{-r}\psi\,\mathrm{d}x\\
&\quad - \int_{\{w+\varepsilon\psi< 0\}} \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla(w+\varepsilon\psi)\,\mathrm{d}x\\
&\quad + \int_{\{w+\varepsilon\psi< 0\}} h(x)(w^+)^{-r}(w+\varepsilon\psi)\,\mathrm{d}x\\
&\leq \varepsilon\int_\Omega \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla\psi\,\mathrm{d}x - \varepsilon\int_\Omega h(x)(w^+)^{-r}\psi\,\mathrm{d}x\\
&\quad - \varepsilon\int_{\{w+\varepsilon\psi< 0\}} \left(|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla\psi\,\mathrm{d}x.
\end{aligned}$$

Dividing the last inequality by $\varepsilon > 0$, letting $\varepsilon \to 0^+$ and using the fact that $|\{w + \varepsilon\psi < 0\}|_N \to 0$ as $\varepsilon \to 0^+$, we see that

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla \psi \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla w|^{q-2}\nabla w \cdot \nabla \psi \,\mathrm{d}x - \int_\Omega h(x)(w^+)^{-r}\psi \geq 0 \quad (3.25)$$

for all $\psi \in W_0^{1,\mathcal{H}}(\Omega)$. Substituting ψ by $-\psi$ in (3.25), the result follows. □

Finally, we are able to present a complete proof of Theorem 1.1. Indeed, by Proposition 3.4, $u_0 \in \mathcal{N}$ and

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla \varphi \,\mathrm{d}x \geq \int_\Omega h(x)u_0^{-r}\varphi \,\mathrm{d}x$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$. Hence, by Lemma 3.5, u_0 is such that

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla \varphi \,\mathrm{d}x - \int_\Omega h(x)u_0^{-r}\varphi \,\mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$. Therefore, u_0 is a positive solution to problem (1.4).

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(M.T.O. Pimenta) Departamento de Matemática e Computação, Universidade Estadual Paulista - Unesp, CEP: 19060-900 , Presidente Prudente - SP, Brazil

Email address: marcos.pimenta@unesp.br

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany

Email address: winkert@math.tu-berlin.de