# $(p,q)$-EQUATIONS WITH SINGULAR AND CONCAVE CONVEX NONLINEARITIES

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ABSTRACT. We consider a nonlinear Dirichlet problem driven by the $(p,q)$-Laplacian with $1<q<p$. The reaction is parametric and exhibits the competing effects of a singular term and of concave and convex nonlinearities. We are looking for positive solutions and prove a bifurcation-type theorem describing in a precise way the set of positive solutions as the parameter varies. Moreover, we show the existence of a minimal positive solution and we study it as a function of the parameter.

## 1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial\Omega$. In this paper, we study the following parametric Dirichlet $(p,q)$-equation

$$\begin{aligned} &-\Delta_p u-\Delta_q u=\lambda\left[u^{-\eta}+a(x)u^{\tau-1}\right]+f(x,u) \quad \text{in } \Omega\\ &u\big|_{\partial\Omega}=0, \quad u>0, \quad \lambda>0, \quad 1<\tau<q<p, \quad 0<\eta<1. \end{aligned} \tag{P$_\lambda$}$$

For $r\in(1,\infty)$ we denote by $\Delta_r$ the $r$-Laplace differential operator defined by

$$\Delta_r u=\operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right) \quad \text{for all } u\in W^{1,r}_0(\Omega).$$

The perturbation in problem (P$_\lambda$), namely $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$, is a Carathéodory function, that is, $f$ is measurable in the first argument and continuous in the second one. We suppose that $f(x,\cdot)$ is $(p-1)$-superlinear near $+\infty$ but it does not satisfy the well-known Ambrosetti-Rabinowitz condition which we will write AR-condition for short. Hence, we have in problem (P$_\lambda$) the combined effects of singular terms (the function $s\to\lambda s^{-\eta}$), of sublinear (concave) terms (the function $s\to\lambda s^{\tau-1}$ since $1<\tau<q<p$) and of superlinear (convex) terms (the function $s\to f(x,s)$). For the precise conditions on $f$ we refer to hypotheses H$(f)$ in Section 2. Consider the following two functions (for the sake of simplicity we drop the $x$-dependence)

$$f_1(s)=\left(s^+\right)^{r-1}, \quad p<r<p^*, \qquad f_2(s)=\begin{cases}\left(s^+\right)^l & \text{if } s\leq 1,\\ s^{p-1}\ln(s)+1 & \text{if } 1<s,\end{cases} \quad q<l.$$

Both functions satisfy our hypotheses H$(f)$ but only $f_1$ satisfies the AR-condition.

We are looking for positive solutions and we establish the precise dependence of the set of positive solutions of (P$_\lambda$) on the parameter $\lambda>0$ as the latter varies. For the weight $a(\cdot)$ we suppose the following assumptions

H$(a)$: $a\in L^\infty(\Omega)$, $a(x)\geq a_0>0$ for a. a. $x\in\Omega$;

The main result in this paper is the following one.

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**Theorem 1.1.** *If hypotheses H(a) and H(f) hold, then there exists $\lambda^* \in (0, +\infty)$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$, problem ($P_\lambda$) has at least two positive solutions*
$$u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \text{ with } u_0 \leq \hat{u} \text{ and } u_0 \neq \hat{u};$$
(b) *for $\lambda = \lambda^*$, problem ($P_\lambda$) has at least one positive solution $u^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$;*
(c) *for $\lambda > \lambda^*$, problem ($P_\lambda$) has no positive solution;*
(d) *for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, problem ($P_\lambda$) has a smallest positive solution $u_\lambda^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$ is strictly increasing, that is, $0 < \mu < \lambda \leq \lambda^*$ implies $u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and it is left continuous.*

The study of elliptic problems with combined nonlinearities was initiated with the seminal paper of Ambrosetti-Brezis-Cerami [1] who studied semilinear Dirichlet equations driven by the Laplacian without any singular term. Their work has been extended to nonlinear problems driven by the $p$-Laplacian by García Azorero-Peral Alonso-Manfredi [5] and Guo-Zhang [11]. In both works there is no singular term and the reaction has the special form
$$x \to \lambda s^{\tau-1} + s^{r-1} \quad \text{for all } s \geq 0 \text{ with } 1 < \tau < p < r < p^*,$$
where $p^*$ is the critical Sobolev exponent to $p$ given by
$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$
More recently there have been generalizations involving more general nonlinear differential operators, more general concave and convex nonlinearities and different boundary conditions. We refer to the works of Papageorgiou-Rădulescu-Repovš [23] for Robin problems and Papageorgiou-Winkert [26], Leonardi-Papageorgiou [14] and Marano-Marino-Papageorgiou [16] for Dirichlet problems. None of these works involves a singular term. Singular equations driven by the $p$-Laplacian and with a superlinear perturbation were investigated by Papageorgiou-Winkert [27].

We mention that $(p, q)$-equations arise in many mathematical models of physical processes. We refer to Benci-D'Avenia-Fortunato-Pisani [2] for quantum physics and Cherfils-Il′yasov [3] for reaction diffusion systems.

Finally, we mention recent papers which are very close to our topic dealing with certain types of nonhomogeneous and/or singular problems. We refer to Papageorgiou-Rădulescu-Repovš [21, 22], Papageorgiou-Zhang [28] and Ragusa-Tachikawa [30].

## 2. Preliminaries and Hypotheses

We denote by $L^p(\Omega)$ $\left(\text{or } L^p\left(\Omega; \mathbb{R}^N\right)\right)$ and $W_0^{1,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_p$ and $\|\cdot\|$, respectively. By means of the Poincaré inequality we have
$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$
For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is known that
$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

Furthermore, we need the ordered Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

and its positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\},$$

where $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$. We will also use two more open cones. The first one is an open cone in the space $C^1(\overline{\Omega})$ and is defined by

$$D_+ = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

The second open cone is the interior of the order cone

$$K_+ = \left\{u \in C_0(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}$$

of the Banach space

$$C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}.$$

We know that

$$\operatorname{int} K_+ = \left\{u \in K_+ : c_u \hat{d} \leq u \text{ for some } c_u > 0\right\}$$

with $\hat{d}(\cdot) = d(\cdot, \partial\Omega)$. Let $\hat{u}_1$ denote the positive $L^p$-normalized, that is, $\|\hat{u}_1\|_p = 1$, eigenfunction of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$. We know that $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. From Papageorgiou-Rădulescu-Repovš [20] we have

$$c_u \hat{d} \leq u \text{ for some } c_u > 0 \quad \text{ if and only if } \quad \hat{c}_u \hat{u}_1 \leq u \text{ for some } \hat{c}_u > 0.$$

Given $u, v \in W_0^{1,p}(\Omega)$ with $u(x) \leq v(x)$ for a. a. $x \in \Omega$ we define

$$[u, v] = \left\{y \in W_0^{1,p}(\Omega) : u(x) \leq y(x) \leq v(x) \text{ for a. a. } x \in \Omega\right\},$$

$$\operatorname{int}_{C_0^1(\overline{\Omega})}[u, v] = \text{the interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v] \cap C_0^1(\overline{\Omega}),$$

$$[u) = \left\{y \in W_0^{1,p}(\Omega) : u(x) \leq y(x) \text{ for a. a. } x \in \Omega\right\}.$$

If $h, g \in L^\infty(\Omega)$, then we write $h \prec g$ if and only if for every compact set $K \subseteq \Omega$, there exists $c_K > 0$ such that $c_K \leq g(x) - h(x)$ for a. a. $x \in K$. Note that if $h, g \in C(\Omega)$ and $h(x) < g(x)$ for all $x \in \Omega$, then $h \prec g$.

If $X$ is a Banach space and $\varphi \in C^1(X)$, then we denote by $K_\varphi$ the critical set of $\varphi$, that is,

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\}.$$

Moreover, we say that $\varphi$ satisfies the "Cerami condition", C-condition for short, if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|_X)\, \varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

For every $r \in (1,\infty)$, let $A_r\colon W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ with $\frac{1}{r}+\frac{1}{r'}=1$ be defined by

$$\langle A_r(u),h\rangle = \int_\Omega |\nabla u|^{r-2}\nabla u \cdot \nabla h\,dx \quad \text{for all } u,h \in W_0^{1,r}(\Omega).$$

This operator has the following properties, see Gasiński-Papageorgiou [6, p. 279].

**Proposition 2.1.** *The map $A_r\colon W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone (so maximal monotone) and of type* $(\mathrm{S})_+$*, that is,*

$$u_n \overset{\mathrm{w}}{\to} u \text{ in } W_0^{1,r}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_r(u_n),u_n-u\rangle \leq 0$$

*imply*

$$u_n \to u \quad \text{in } W_0^{1,r}(\Omega).$$

The hypotheses on the function $f(\cdot)$ are the following ones:

H($f$): $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

(i)

$$0 \leq f(x,s) \leq c_1\left[1+s^{r-1}\right]$$

for a. a. $x\in\Omega$, for all $s\geq 0$ with $c_1>0$ and $r\in(p,p^*)$;

(ii) if $F(x,s)=\int_0^s f(x,t)\,dt$, then

$$\lim_{s\to+\infty}\frac{F(x,s)}{s^p}=+\infty \quad \text{uniformly for a. a. } x\in\Omega;$$

(iii) there exists $\mu\in\left((r-p)\max\left\{1,\frac{N}{p}\right\},p^*\right)$ with $\mu>\tau$ such that

$$0<c_2\leq \liminf_{s\to+\infty}\frac{f(x,s)s-pF(x,s)}{s^\mu} \quad \text{uniformly for a. a. } x\in\Omega;$$

(iv)

$$\lim_{s\to 0^+}\frac{f(x,s)}{s^{q-1}}=0 \quad \text{uniformly for a. a. } x\in\Omega;$$

(v) for every $\rho>0$ there exists $\hat{\xi}_\rho>0$ such that the function

$$s\mapsto f(x,s)+\hat{\xi}_\rho s^{p-1}$$

is nondecreasing on $[0,\rho]$ for a. a. $x\in\Omega$.

**Remark 2.2.** *Since our aim is to produce positive solutions and all the hypotheses above concern the positive semiaxis $\mathbb{R}_+=[0,+\infty)$, we may assume, without any loss of generality, that*

$$f(x,s)=0 \quad \text{for a. a. } x\in\Omega \text{ and for all } s\leq 0. \tag{2.1}$$

Note that hypothesis H($f$)(iv) implies that $f(x,0)=0$ for a. a. $x\in\Omega$. From hypotheses H($f$)(ii), (iii) we infer that

$$\lim_{s\to+\infty}\frac{f(x,s)}{s^{p-1}}=+\infty \quad \text{uniformly for a. a. } x\in\Omega.$$

Therefore, the perturbation $f(x,\cdot)$ is $(p-1)$-superlinear for a. a. $x\in\Omega$. However, the superlinearity of $f(x,\cdot)$ is not expressed using the AR-condition which is common in the literature for superlinear problems. We recall that the AR-condition says that there exist $\beta > p$ and $M > 0$ such that

$$0 < \beta F(x,s) \le f(x,s)s \quad \text{for a. a. } x \in \Omega \text{ and for all } s \ge M, \tag{2.2}$$

$$0 < \operatorname*{ess\,inf}_{x\in\Omega} F(x,M). \tag{2.3}$$

In fact this is a unilateral version of the AR-condition due to (2.1). Integrating (2.2) and using (2.3) gives the weaker condition

$$c_3 s^\beta \le F(x,s) \quad \text{for a. a. } x \in \Omega, \text{ for all } x \ge M \text{ and for some } c_3 > 0,$$

which implies

$$c_3 s^{\beta-1} \le f(x,s) \quad \text{for a. a. } x \in \Omega \text{ and for all } s \ge M.$$

Hence, the AR-condition dictates that $f(x,\cdot)$ eventually has at least $(\beta-1)$-polynomial growth. In the present work we replace the AR-condition by hypothesis H($f$)(iii) which includes in our framework also superlinear nonlinearities with slower growth near $+\infty$.

Hypothesis H($f$)(v) is a one-sided Hölder condition. If $f(x,\cdot)$ is differentiable for a. a. $x \in \Omega$ and if for every $\rho > 0$ there exists $c_\rho > 0$ such that

$$f'_s(x,s)s \ge -c_\rho s^{p-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } 0 \le s \le \rho,$$

then hypothesis H($f$)(v) is satisfied.

We introduce the following sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ admits a positive solution}\},$$
$$\mathcal{S}_\lambda = \{u : u \text{ is a positive solution of } (\mathrm{P}_\lambda)\}.$$

Moreover, we consider the following auxiliary Dirichlet problem

$$\begin{aligned} &-\Delta_p u - \Delta_q u = \lambda a(x) u^{\tau-1} \quad \text{in } \Omega\\ &u\big|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 1 < \tau < q < p. \end{aligned} \tag{$\mathrm{Q}_\lambda$}$$

**Proposition 2.3.** *If hypothesis H(a) holds, then for every $\lambda > 0$ problem* ($\mathrm{Q}_\lambda$) *admits a unique solution $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

*Proof.* We consider the $C^1$-functional $\gamma_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega a(x)\left(u^+\right)^\tau dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $\tau < q < p$ it is clear that $\gamma_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive and by the Sobolev embedding theorem, we see that $\gamma_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous. Hence, there exists $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\gamma_\lambda(\tilde{u}_\lambda) = \min\left[\gamma_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{2.4}$$

If $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $t > 0$ then

$$\gamma_\lambda(tu) = \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^q}{q}\|\nabla u\|_q^q - \frac{\lambda t^\tau}{\tau}\int_\Omega a(x)u^2\,dx.$$

Since $\tau < q < p$, choosing $t \in (0,1)$ small enough, we have $\gamma_\lambda(tu) < 0$ and so,

$$\gamma_\lambda(\tilde{u}_\lambda) < 0 = \gamma_\lambda(0),$$

see (2.4), which shows that $\tilde{u}_\lambda \neq 0$. From (2.4) we know that $\gamma_\lambda'(\tilde{u}_\lambda) = 0$, that is,

$$\langle A_p(\tilde{u}_\lambda), h\rangle + \langle A_q(\tilde{u}_\lambda), h\rangle = \lambda \int_\Omega a(x) \left(\tilde{u}_\lambda^+\right)^{\tau-1} h\,dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{2.5}$$

Choosing $h = -\tilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$ in (2.5) gives

$$\left\|\nabla \tilde{u}_\lambda^-\right\|_p^p + \left\|\nabla \tilde{u}_\lambda^-\right\|_q^q = 0,$$

which shows that $\tilde{u}_\lambda \geq 0$ with $\tilde{u}_\lambda \neq 0$. Therefore, (2.5) becomes

$$-\Delta_p \tilde{u}_\lambda - \Delta_q \tilde{u}_\lambda = \lambda a(x) \tilde{u}_\lambda^{\tau-1} \quad \text{in } \Omega, \qquad \tilde{u}_\lambda\big|_{\partial\Omega} = 0.$$

We know that $\tilde{u}_\lambda \in L^\infty(\Omega)$, see, for example Marino-Winkert [17]. Then, from the nonlinear regularity theory of Lieberman [15] we have that $\tilde{u}_\lambda \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Moreover, the nonlinear maximum principle of Pucci-Serrin [29, pp. 111, 120] implies that $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

We still have to show that this positive solution is unique. Suppose that $\tilde{v}_\lambda \in W_0^{1,p}(\Omega)$ is another solution of ($Q_\lambda$). As before we can show that $\tilde{v}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We consider the integral functional $j\colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\left\|\nabla u^{\frac{1}{q}}\right\|_p^p + \frac{1}{q}\left\|\nabla u^{\frac{1}{q}}\right\|_q^q & \text{if } u \geq 0,\ u^{\frac{1}{q}} \in W_0^{1,p}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

From Díaz-Saá [4, Lemma 1] we see that $j$ is convex. Furthermore, applying Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [18, p. 274], we obtain that

$$\frac{\tilde{u}_\lambda}{\tilde{v}_\lambda},\ \frac{\tilde{v}_\lambda}{\tilde{u}_\lambda} \in L^\infty(\Omega).$$

We denote by

$$\operatorname{dom} j = \left\{u \in L^1(\Omega) : j(u) < +\infty\right\}$$

the effective domain of $j$ and set $h = \tilde{u}_\lambda^q - \tilde{v}_\lambda^q$. One gets

$$\tilde{u}_\lambda^q - th \in \operatorname{dom} j \quad \text{and} \quad \tilde{v}_\lambda^q + th \in \operatorname{dom} j \quad \text{for all } t \in [0,1].$$

Note that the functional $j : L^1(\Omega) \to \overline{\mathbb{R}}$ is Gateaux differentiable at $\tilde{u}_\lambda^q$ and at $\tilde{v}_\lambda^q$ in the direction $h$. Using the nonlinear Green's identity, see Papageorgiou-Rădulescu-Repovš [18, Corollary 1.5.16, p. 34], we obtain

$$\begin{aligned} j'\left(\tilde{u}_\lambda^q\right)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p \tilde{u}_\lambda - \Delta_q \tilde{u}_\lambda}{\tilde{u}_\lambda^{q-1}} h\,dx = \frac{\lambda}{q}\int_\Omega \frac{a(x)}{\tilde{u}_\lambda^{q-\tau}} h\,dx,\\ j'\left(\tilde{v}_\lambda^q\right)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p \tilde{v}_\lambda - \Delta_q \tilde{v}_\lambda}{\tilde{v}_\lambda^{q-1}} h\,dx = \frac{\lambda}{q}\int_\Omega \frac{a(x)}{\tilde{v}_\lambda^{q-\tau}} h\,dx. \end{aligned}$$

The convexity of $j\colon L^1(\Omega) \to \overline{\mathbb{R}}$ implies the monotonicity of $j'$. Hence

$$0 \leq \frac{\lambda}{q}\int_\Omega a(x)\left[\frac{1}{\tilde{u}_\lambda^{q-\tau}} - \frac{1}{\tilde{v}_\lambda^{q-\tau}}\right]\left[\tilde{u}_\lambda^q - \tilde{v}_\lambda^q\right] dx \leq 0,$$

which implies $\tilde{u}_\lambda = \tilde{v}_\lambda$. Therefore, $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique positive solution of the auxiliary problem ($Q_\lambda$). □

This solution will provide a useful lower bound for the elements of the set of positive solutions $\mathcal{S}_\lambda$.

## 3. Positive Solutions

Let $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be the unique positive solution of ($\mathrm{Q}_\lambda$), see Proposition 2.3. Let $s > N$. Then $\tilde{u}_\lambda^s \in \operatorname{int} K_+$ and so there exists $c_4 > 0$ such that

$$\hat{u}_1 \leq c_4 \tilde{u}_\lambda^s,$$

see Section 2. Hence

$$\tilde{u}_\lambda^{-\eta} \leq c_5 \hat{u}_1^{-\frac{\eta}{s}} \quad \text{for some } c_5 > 0.$$

Applying the Lemma of Lazer-McKenna [13] we have

$$\hat{u}_1^{-\frac{\eta}{s}} \in L^s(\Omega)$$

and thus

$$\tilde{u}_\lambda^{-\eta} \in L^s(\Omega). \tag{3.1}$$

We introduce the following modification of problem ($\mathrm{P}_\lambda$) in which we have neutralized the singular term

$$\begin{aligned} & -\Delta_p u - \Delta_q u = \lambda \tilde{u}_\lambda^{-\eta} + \lambda a(x) u^{\tau-1} + f(x,u) \quad \text{in } \Omega \\ & u\big|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 1 < \tau < q < p, \quad 0 < \eta < 1. \end{aligned} \tag{P$_\lambda$'}$$

Let $\psi_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the Euler energy functional of problem ($\mathrm{P}_\lambda$') defined by

$$\begin{aligned} \psi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega \tilde{u}_\lambda^{-\eta} u \, dx \\ &\quad - \frac{\lambda}{\tau} \int_\Omega a(x) \left(u^+\right)^\tau \, dx - \int_\Omega F(x,u^+) \, dx \end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$, see (3.1). It is clear that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

**Proposition 3.1.** *If hypotheses H(a) and H(f) hold and if $\lambda > 0$, then $\psi_\lambda$ satisfies the C-condition.*

*Proof.* Let $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\psi_\lambda(u_n)| \leq c_6 \quad \text{for all } n \in \mathbb{N} \text{ and for some } c_6 > 0, \tag{3.2}$$

$$(1 + \|u_n\|)\psi_\lambda'(u_n) \to 0 \quad \text{in } W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.3}$$

From (3.3) we have

$$\begin{aligned} &\bigg| \langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle - \lambda \int_\Omega \tilde{u}_\lambda^{-\eta} h \, dx - \lambda \int_\Omega a(x) \left(u_n^+\right)^{\tau-1} h \, dx \\ &\quad - \int_\Omega f\left(x, u_n^+\right) h \, dx \bigg| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{3.4}$$

Choosing $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.4) leads to

$$\left\|\nabla u_n^-\right\|_p^p \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

which implies

$$u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.5}$$

Combining (3.2) and (3.5) gives

$$\begin{aligned}&\left\|\nabla u_n^+\right\|_p^p+\frac{p}{q}\left\|\nabla u_n^+\right\|_q^q-\lambda p\int_\Omega \tilde{u}_\lambda^{-\eta}u_n^+\,dx-\frac{\lambda p}{\tau}\int_\Omega a(x)\left(u_n^+\right)^\tau\,dx\\&-\int_\Omega pF\left(x,u_n^+\right)\,dx\le c_7\quad\text{for all } n\in\mathbb{N} \text{ and for some } c_7>0.\end{aligned}\tag{3.6}$$

On the other hand, if we choose $h=u_n^+\in W_0^{1,p}(\Omega)$ in (3.4), we obtain

$$\begin{aligned}&-\left\|\nabla u_n^+\right\|_p^p-\left\|\nabla u_n^+\right\|_q^q+\lambda\int_\Omega \tilde{u}_\lambda^{-\eta}u_n^+\,dx+\lambda\int_\Omega a(x)\left(u_n^+\right)^\tau\,dx\\&+\int_\Omega f\left(x,u_n^+\right)u_n^+\,dx\le\varepsilon_n\quad\text{for all } n\in\mathbb{N}.\end{aligned}\tag{3.7}$$

Adding (3.6) and (3.7) yields

$$\begin{aligned}&\int_\Omega\left[f\left(x,u_n^+\right)u_n^+-pF\left(x,u_n^+\right)\right]\,dx\\&\le\lambda(p-1)\int_\Omega \tilde{u}_\lambda^{-\eta}u_n^+\,dx+\lambda\left[\frac{p}{\tau}-1\right]\int_\Omega a(x)\left(u_n^+\right)^\tau\,dx.\end{aligned}\tag{3.8}$$

By hypotheses H$(f)$(i), (iii) we can find $c_8>0$ such that

$$\frac{c_2}{2}s^\mu-c_8\le f(x,s)s-pF(x,s)\quad\text{for a. a. } x\in\Omega \text{ and for all } s\ge 0.$$

This implies

$$\frac{c_2}{2}s^\mu\left\|u_n^+\right\|_\mu^\mu-c_9\le\int_\Omega\left[f\left(x,u_n^+\right)u_n^+-pF\left(x,u_n^+\right)\right]\,dx\tag{3.9}$$

for some $c_9>0$ and for all $n\in\mathbb{N}$.

Since $s>N$ we have $s'<N'\le p^*$. Hence, $u_n^+\in L^{s'}(\Omega)$. Then, taking (3.1) along with Hölder's inequality into account, we get

$$\lambda[p-1]\int_\Omega \tilde{u}_\lambda^{-\eta}u_n^+\,dx\le c_{10}\left\|\tilde{u}_\lambda^{-\eta}\right\|_s\left\|u_n^+\right\|_{s'}\tag{3.10}$$

for some $c_{10}=c_{10}(\lambda)>0$ and for all $n\in\mathbb{N}$. Moreover, by hypothesis H$(a)$, we have

$$\lambda\left[\frac{p}{\tau}-1\right]\int_\Omega a(x)\left(u_n^+\right)^\tau\,dx\le c_{11}\left\|u_n^+\right\|_\tau^\tau\tag{3.11}$$

for some $c_{11}=c_{11}(\lambda)>0$ and for all $n\in\mathbb{N}$.

Now we choose $s>N$ large enough such that $s'<\mu$. Returning to (3.8), using (3.9), (3.10) as well as (3.11) and using the fact that $s',\tau<\mu$ by hypothesis H$(f)$(iii) leads to

$$\left\|u_n^+\right\|_\mu^\mu\le c_{12}\left[\left\|u_n^+\right\|_\mu+\left\|u_n^+\right\|_\mu^\tau+1\right]$$

for some $c_{12}>0$ and for all $n\in\mathbb{N}$. Since $\tau<\mu$ we obtain

$$\left\{u_n^+\right\}_{n\ge1}\subseteq L^\mu(\Omega)\text{ is bounded.}\tag{3.12}$$

Assume that $N\neq p$. From hypothesis H$(f)$(iii) it is clear that we may assume $\mu<r<p^*$. Then there exists $t\in(0,1)$ such that

$$\frac{1}{r}=\frac{1-t}{\mu}+\frac{t}{p^*}.$$

Taking the interpolation inequality into account, see Papageorgiou-Winkert [25, Proposition 2.3.17, p. 116], we have

$$\left\|u_n^+\right\|_r \leq \left\|u_n^+\right\|_\mu^{1-t} \left\|u_n^+\right\|_{p^*}^t,$$

which by (3.12) implies that

$$\left\|u_n^+\right\|_r^r \leq c_{13} \left\|u_n^+\right\|^{tr} \tag{3.13}$$

for some $c_{13} > 0$ and for all $n \in \mathbb{N}$.

From hypothesis H$(f)$(i) we know that

$$f(x,s)s \leq c_{14} \left[1 + s^r\right] \tag{3.14}$$

for a. a. $x \in \Omega$, for all $s \geq 0$ and for some $c_{14} > 0$. We choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.4), that is,

$$\begin{aligned}\left\|\nabla u_n^+\right\|_p^p + \left\|\nabla u_n^+\right\|_q^q - \lambda \int_\Omega \tilde{u}_\lambda^{-\eta} u_n^+ \, dx - \lambda \int_\Omega a(x) \left(u_n^+\right)^\tau \, dx& \\ - \int_\Omega f\left(x, u_n^+\right) u_n^+ \, dx \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}.&\end{aligned}$$

From this it follows by using (3.13), (3.14) and $1 < \tau < p < r$

$$\left\|u_n^+\right\|^p \leq c_{15} \left[1 + \left\|u_n^+\right\|^{tr}\right] \tag{3.15}$$

for some $c_{15} > 0$ and for all $n \in \mathbb{N}$. The condition on $\mu$, see hypothesis H$(f)$(iii), implies that $tr < p$. Then from (3.15) we infer

$$\left\{u_n^+\right\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.16}$$

If $N = p$, then we have by definition $p^* = \infty$. The Sobolev embedding theorem ensures that $W_0^{1,p}(\Omega) \hookrightarrow L^\vartheta(\Omega)$ for all $1 \leq \vartheta < \infty$. So, in order to apply the previous arguments we need to replace $p^*$ by $\vartheta > r > \mu$ and choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{\vartheta},$$

which implies

$$tr = \frac{\vartheta(r-\mu)}{\vartheta - \mu}.$$

Note that $\frac{\vartheta(r-\mu)}{\vartheta-\mu} \to r - \mu < p$ as $\vartheta \to +\infty$. So, for $\vartheta > r$ large enough, we see that $tr < p$ and again (3.16) holds.

From (3.5) and (3.16) we infer that

$$\left\{u_n\right\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \overset{\mathrm{w}}{\to} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega). \tag{3.17}$$

We choose $h = u_n - u \in W_0^{1,p}(\Omega)$ in (3.4), pass to the limit as $n \to \infty$ and use the convergence properties in (3.17). This gives

$$\lim_{n \to \infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle\right] = 0$$

and since $A_q$ is monotone we obtain

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle\right] \le 0.$$

By (3.16) we then conclude that

$$\lim_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \le 0.$$

Applying Proposition 2.1 shows that $u_n \to u$ in $W_0^{1,p}(\Omega)$ and so we conclude that $\psi_\lambda$ satisfies the C-condition. □

**Proposition 3.2.** *If hypotheses H(a) and H(f) hold, then there exists $\hat{\lambda} > 0$ such that for every $\lambda \in \left(0, \hat{\lambda}\right)$ we can find $\rho_\lambda > 0$ for which we have*

$$\psi_\lambda(0) = 0 < \inf\left[\psi_\lambda(u) : \|u\| = \rho_\lambda\right] = m_\lambda.$$

*Proof.* Hypotheses H(f)(i), (iv) imply that for a given $\varepsilon > 0$ we can find $c_{16} = c_{16}(\varepsilon) > 0$ such that

$$F(x,s) \le \frac{\varepsilon}{q} s^q + c_{16} s^r \quad \text{for a. a. } x \in \Omega \text{ and for all } s \ge 0. \tag{3.18}$$

Recall that $\tilde{u}_\lambda^{-\eta} \in L^s(\Omega)$ with $s > N$, see (3.1). We choose $s > N$ large enough such that $s' < p^*$. Then, by Hölder's inequality, we have

$$\lambda \int_\Omega \tilde{u}_\lambda^{-\eta} u \, dx \le \lambda c_{17} \|u\| \quad \text{for some } c_{17} > 0. \tag{3.19}$$

Moreover, one gets

$$\frac{\lambda}{\tau} \int_\Omega a(x)|u|^\tau \, dx \le \frac{\lambda \|a\|_\infty}{\tau} \|u\|^\tau. \tag{3.20}$$

Applying (3.18), (3.19) and (3.20) leads to

$$\psi_\lambda(u) \ge \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\left[\|\nabla u\|_q^q - \varepsilon \|u\|_q^q\right] - c_{18}\left[\|u\|^r + \lambda\left(\|u\| + \|u\|^\tau\right)\right] \tag{3.21}$$

for some $c_{18} > 0$. Let $\hat{\lambda}_1(q) > 0$ be the principal eigenvalue of $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$. Then, from the variational characterization of $\hat{\lambda}_1(q)$, see Gasiński-Papageorgiou [8, p. 732], we obtain

$$\frac{1}{q}\left[\|\nabla u\|_q^q - \varepsilon \|u\|_q^q\right] \ge \frac{1}{q}\left[1 - \frac{\varepsilon}{\hat{\lambda}_1(q)}\right] \|\nabla u\|_q^q.$$

Choosing $\varepsilon \in \left(0, \hat{\lambda}_1(q)\right)$ we infer that

$$\frac{1}{q}\left[\|\nabla u\|_q^q - \varepsilon \|u\|_q^q\right] > 0. \tag{3.22}$$

Since $1 < \tau < r$, it holds

$$\|u\|^\tau \le \|u\| + \|u\|^r. \tag{3.23}$$

Applying (3.22) and (3.23) to (3.21) gives

$$\begin{aligned}
\psi_\lambda(u) &\ge \frac{1}{p}\|u\|^p - c_{18}\left[2\lambda\|u\| + (\lambda+1)\|u\|^r\right] \\
&\ge \left[\frac{1}{p} - c_{18}\left(2\lambda\|u\|^{1-p} + (\lambda+1)\|u\|^{r-p}\right)\right]\|u\|^p.
\end{aligned} \tag{3.24}$$

We consider now the function

$$k_\lambda(t) = 2\lambda t^{1-p} + (\lambda+1)t^{r-p} \quad \text{for all } t > 0.$$

It is clear that $k_\lambda \in C^1(0,\infty)$ and since $1 < p < r$ we see that

$$k_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Hence, there exists $t_0 > 0$ such that

$$k_\lambda(t_0) = \min\left[k_\lambda(t) : t > 0\right],$$

which implies that $k'_\lambda(t_0) = 0$. Therefore,

$$2\lambda(p-1)t_0^{-p} = (r-p)(\lambda+1)t_0^{r-p-1}.$$

From this we deduce that

$$t_0 = t_0(\lambda) = \left[\frac{2\lambda(p-1)}{(r-p)(\lambda+1)}\right]^{\frac{1}{r-1}}.$$

We have

$$k_\lambda(t_0) = 2\lambda\frac{(r-p)(\lambda+1)^{\frac{p-1}{r-1}}}{(2\lambda(p-1))^{\frac{p-1}{r-1}}} + (\lambda+1)\frac{(2\lambda(p-1))^{\frac{r-p}{r-1}}}{((r-p)(\lambda+1))^{\frac{r-p}{r-1}}}.$$

Since $1 < p < r$ we see that

$$k_\lambda(t_0) \to 0 \quad \text{as } \lambda \to 0^+.$$

Therefore, we can find $\hat{\lambda} > 0$ such that

$$k_\lambda(t_0) < \frac{1}{pc_{18}} \quad \text{for all } \lambda \in \left(0, \hat{\lambda}\right).$$

Then, by (3.24) we see that

$$\psi_\lambda(u) > 0 = \psi_\lambda(0) \quad \text{for all } \|u\| = t_0(\lambda) = \rho_\lambda \text{ and for all } \lambda \in \left(0, \hat{\lambda}\right).$$

□

From hypothesis H$(f)$(ii) we see that for every $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ we have

$$\psi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.25}$$

**Proposition 3.3.** *If hypotheses H(a) and H(f) hold and if $\lambda \in \left(0, \hat{\lambda}\right)$, then problem* ($P_\lambda$') *admits a solution $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

*Proof.* Propositions 3.1, 3.2 and (3.25) permit the use of the mountain pass theorem. So, we can find $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\overline{u}_\lambda \in K_{\psi_\lambda} \quad \text{and} \quad \psi_\lambda(0) = 0 < m_\lambda \le \psi_\lambda(\overline{u}_\lambda). \tag{3.26}$$

From (3.26) we see that $\overline{u}_\lambda \neq 0$ and $\psi'_\lambda(\overline{u}_\lambda) = 0$, that is,

$$\begin{aligned} &\langle A_p(\overline{u}_\lambda), h\rangle + \langle A_q(\overline{u}_\lambda), h\rangle \\ &= \lambda\int_\Omega \tilde{u}_\lambda^{-\eta} h\,dx + \lambda\int_\Omega a(x)\left(\overline{u}_\lambda^+\right)^{\tau-1} h\,dx + \int_\Omega f\left(x, \overline{u}_\lambda^+\right) h\,dx \end{aligned} \tag{3.27}$$

for all $h \in W_0^{1,p}(\Omega)$. We choose $h = -\overline{u}_\lambda^- \in W_0^{1,p}(\Omega)$ in (3.27) which shows that

$$\left\|\overline{u}_\lambda^-\right\|^p \le 0.$$

Thus, $\overline{u}_\lambda \ge 0$ with $\overline{u}_\lambda \neq 0$.

From (3.27) we know that $\overline{u}_\lambda$ is a positive solution of ($P_\lambda$') with $\lambda \in \left(0, \hat{\lambda}\right)$. This means

$$-\Delta_p \overline{u}_\lambda - \Delta_q \overline{u}_\lambda = \lambda \tilde{u}_\lambda^{-\eta} + \lambda a(x) \overline{u}_\lambda^{\tau-1} + f(x, \overline{u}_\lambda) \quad \text{in } \Omega, \quad \overline{u}_\lambda\big|_{\partial\Omega} = 0.$$

As before, see the proof of Proposition 2.3, using the nonlinear regularity theory, we have $\overline{u}_\lambda \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. The nonlinear maximum principle, see Pucci-Serrin [29, pp. 111, 120] implies that $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

**Proposition 3.4.** *If hypotheses H(a) and H(f) hold and if $\lambda \in \left(0, \hat{\lambda}\right)$, then $\tilde{u}_\lambda \leq \overline{u}_\lambda$.*

*Proof.* We introduce the Carathéodory function $g_\lambda \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_\lambda(x,s) = \begin{cases} \lambda a(x) \left(s^+\right)^{\tau-1} & \text{if } s \leq \overline{u}_\lambda(x), \\ \lambda a(x) \overline{u}_\lambda(x)^{\tau-1} & \text{if } \overline{u}_\lambda(x) < s. \end{cases} \tag{3.28}$$

We set $G_\lambda(x,s) = \int_0^s g_\lambda(x,t)\,dt$ and consider the $C^1$-functional $\sigma_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_\lambda(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.28) it is clear that $\sigma_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive. Moreover, by the Sobolev embedding, we have that $\sigma_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous. Then, by the Weierstraß-Tonelli theorem, we can find $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\lambda\left(\hat{u}_\lambda\right) = \min\left[\sigma_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.29}$$

Since $\tau < q < p$, we have $\sigma_\lambda\left(\hat{u}_\lambda\right) < 0 = \sigma_\lambda(0)$ which implies $\hat{u}_\lambda \neq 0$.

From (3.29) we have $\sigma_\lambda'\left(\hat{u}_\lambda\right) = 0$, that is,

$$\left\langle A_p\left(\hat{u}_\lambda\right), h\right\rangle + \left\langle A_q\left(\hat{u}_\lambda\right), h\right\rangle = \int_\Omega g_\lambda\left(x, \hat{u}_\lambda\right) h\,dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.30}$$

First, we choose $h = -\hat{u}_\lambda^- \in W_0^{1,p}(\Omega)$ in (3.30). Then, by the definition of the truncation in (3.28) we easily see that $\|\hat{u}_\lambda^-\|^p \leq 0$ and so, $\hat{u}_\lambda \geq 0$ with $\hat{u}_\lambda \neq 0$.

Next, we choose $h = \left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+ \in W_0^{1,p}(\Omega)$ in (3.30) which gives, due to (3.28) and $f \geq 0$,

$$\begin{aligned} &\left\langle A_p\left(\hat{u}_\lambda\right), \left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\right\rangle + \left\langle A_q\left(\hat{u}_\lambda\right), \left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\right\rangle \\ &= \int_\Omega \lambda a(x) \overline{u}_\lambda^{\tau-1}\left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\,dx \\ &\leq \int_\Omega \left[\lambda \tilde{u}_\lambda^{-\eta} + \lambda a(x)\overline{u}_\lambda^{\tau-1} + f\left(x, \overline{u}_\lambda\right)\right]\left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\,dx \\ &= \left\langle A_p\left(\overline{u}_\lambda\right), \left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\right\rangle + \left\langle A_q\left(\overline{u}_\lambda\right), \left(\hat{u}_\lambda - \overline{u}_\lambda\right)^+\right\rangle. \end{aligned}$$

This shows that $\hat{u}_\lambda \leq \overline{u}_\lambda$. We have proved that

$$\hat{u}_\lambda \in \left[0, \overline{u}_\lambda\right], \ \hat{u}_\lambda \neq 0.$$

Hence, $\hat{u}_\lambda$ is a positive solution of ($Q_\lambda$) and due to Proposition 2.3 we know that $\hat{u}_\lambda = \tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Therefore, $\tilde{u}_\lambda \leq \overline{u}_\lambda$ for all $\lambda \in \left(0, \hat{\lambda}\right)$. □

Now we are able to establish the nonemptiness of the set $\mathcal{L}$ (being the set of all admissible parameters) determine the regularity of the elements in the solution set $\mathcal{S}_\lambda$.

**Proposition 3.5.** *If hypotheses H(a) and H(f) hold, then* $\mathcal{L} \neq \emptyset$ *and, for every* $\lambda > 0$, $\mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

*Proof.* Let $\lambda \in \left(0, \hat{\lambda}\right)$. From Proposition 3.4 we know that $\tilde{u}_\lambda \leq \overline{u}_\lambda$. So we can define the truncation $e_\lambda \colon \Omega \times \mathbb{R} \to \mathbb{R}$ of the reaction of problem ($\mathrm{P}_\lambda$)

$$
\begin{aligned}
&e_\lambda(x,s)\\
&= \begin{cases} \lambda\left[\tilde{u}_\lambda(x)^{-\eta} + a(x)\tilde{u}_\lambda(x)^{\tau-1}\right] + f\left(x, \tilde{u}_\lambda(x)\right) & \text{if } s < \tilde{u}_\lambda(x),\\ \lambda\left[s^{-\eta} + a(x)s^{\tau-1}\right] + f(x,s) & \text{if } \tilde{u}_\lambda(x) \leq s \leq \overline{u}_\lambda(x),\\ \lambda\left[\overline{u}_\lambda(x)^{-\eta} + a(x)\overline{u}_\lambda(x)^{\tau-1}\right] + f\left(x, \overline{u}_\lambda(x)\right) & \text{if } \overline{u}_\lambda(x) < s. \end{cases}
\end{aligned} \tag{3.31}
$$

This is a Carathéodory function. We set $E_\lambda(x,s) = \int_0^s e_\lambda(x,t)\,dt$ and consider the $C^1$-functional $J_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
J_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega E_\lambda(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).
$$

From (3.31) we see that $J_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive and the Sobolev embedding theorem implies that $J$ is also sequentially weakly lower semicontinuous. Hence, its global minimizer $u_\lambda \in W_0^{1,p}(\Omega)$ exists, that is,

$$
J_\lambda(u_\lambda) = \min\left[J_\lambda(u) : u \in W_0^{1,p}(\Omega)\right].
$$

Hence, $J_\lambda'(u_\lambda) = 0$ which means that

$$
\langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle = \int_\Omega e_\lambda(x, u_\lambda)\, h\,dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.32}
$$

We choose $h = (u_\lambda - \overline{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.32). Then, by using (3.31) and Propositions 3.4 and 3.3 we obtain

$$
\begin{aligned}
&\left\langle A_p(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+\right\rangle + \left\langle A_q(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+\right\rangle\\
&= \int_\Omega \left(\lambda\left[\overline{u}_\lambda^{-\eta} + a(x)\overline{u}_\lambda^{\tau-1}\right] + f(x, \overline{u}_\lambda)\right)(u_\lambda - \overline{u}_\lambda)^+\,dx\\
&\leq \int_\Omega \left(\lambda\left[\tilde{u}_\lambda^{-\eta} + a(x)\overline{u}_\lambda^{\tau-1}\right] + f(x, \overline{u}_\lambda)\right)(u_\lambda - \overline{u}_\lambda)^+\,dx\\
&= \left\langle A_p(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\right\rangle + \left\langle A_q(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\right\rangle.
\end{aligned}
$$

This shows that $u_\lambda \leq \overline{u}_\lambda$.

Next, we choose $h = (\tilde{u}_\lambda - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.32). Then, by (3.31) and hypotheses H($a$) as well as H($f$)(i) it follows

$$\begin{aligned}
&\left\langle A_p(u_\lambda), (\tilde{u}_\lambda - u_\lambda)^+ \right\rangle + \left\langle A_q(u_\lambda), (\tilde{u}_\lambda - u_\lambda)^+ \right\rangle \\
&= \int_\Omega \left(\lambda \left[\tilde{u}^{-\eta} + a(x)\tilde{u}_\lambda^{\tau-1}\right] + f(x, \tilde{u}_\lambda)\right)(\tilde{u}_\lambda - u_\lambda)^+ \, dx \\
&\geq \int_\Omega \lambda \tilde{u}_\lambda^{-\eta} (\tilde{u}_\lambda - u_\lambda)^+ \, dx \\
&= \left\langle A_p(\tilde{u}_\lambda), (\tilde{u}_\lambda - u_\lambda)^+ \right\rangle + \left\langle A_q(\tilde{u}_\lambda), (\tilde{u}_\lambda - u_\lambda)^+ \right\rangle.
\end{aligned}$$

Hence, $\tilde{u}_\lambda \leq u_\lambda$ and so we have proved that $u_\lambda \in [\tilde{u}_\lambda, \overline{u}_\lambda]$. Then, with view to (3.31) and (3.32), we see that $u_\lambda$ is a positive solution of ($\mathrm{P}_\lambda$) for $\lambda \in \left(0, \hat{\lambda}\right)$. In particular, we have

$$-\Delta_p u_\lambda(x) - \Delta_q u_\lambda(x) = \lambda u_\lambda(x)^{-\eta} + a_\lambda(x) u_\lambda(x)^{\tau-1} + f(x, u_\lambda(x)) \quad \text{for a. a. } x \in \Omega.$$

The nonlinear regularity theory, see Lieberman [15], and the nonlinear maximum principle, see Pucci-Serrin [29, pp. 111 and 120], imply that $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Concluding we can say that $\left(0, \hat{\lambda}\right) \subseteq \mathcal{L}$ which means that $\mathcal{L}$ is nonempty. Moreover, for all $\lambda > 0$, $\mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

Reasoning as in the proof of Proposition 3.4 with $\overline{u}_\lambda$ replaced by $u \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we obtain the following result.

**Proposition 3.6.** *If hypotheses H(a) and H(f) hold and if $\lambda \in \mathcal{L}$, then $\tilde{u}_\lambda \leq u$ for all $u \in \mathcal{S}_\lambda$.*

Moreover, the map $\lambda \to \tilde{u}_\lambda$ from $(0, +\infty)$ into $C_0^1(\overline{\Omega})$ exhibits a strong monotonicity property which we will use in the sequel.

**Proposition 3.7.** *If hypotheses H(a) holds and if $0 < \lambda < \lambda'$, then $\tilde{u}_{\lambda'} - \tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

*Proof.* Following the proof of Proposition 3.4 we can show that

$$\tilde{u}_\lambda \leq \tilde{u}_{\lambda'}. \tag{3.33}$$

From (3.33) we have

$$\begin{aligned}
-\Delta_p \tilde{u}_\lambda - \Delta_q \tilde{u}_\lambda &= \lambda a(x) \tilde{u}_\lambda^{\tau-1} \\
&= \lambda' a(x) \tilde{u}_\lambda^{\tau-1} - (\lambda' - \lambda)\, \tilde{u}_\lambda^{\tau-1} \\
&\leq \lambda' a(x) \tilde{u}_{\lambda'}^{\tau-1} \\
&= -\Delta_p \tilde{u}_{\lambda'} - \Delta_q \tilde{u}_{\lambda'}.
\end{aligned} \tag{3.34}$$

Note that $0 \prec (\lambda' - \lambda)\, \tilde{u}_\lambda^{\tau-1}$. So, from (3.34) and Gasiński-Papageorgiou [9, Proposition 3.2] we have

$$\tilde{u}_{\lambda'} - \tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

□

Next we are going to show that $\mathcal{L}$ is an interval.

**Proposition 3.8.** *If hypotheses $H(a)$ and $H(f)$ hold and if $\lambda \in \mathcal{L}$ and $\mu \in (0,\lambda)$, then $\mu \in \mathcal{L}$.*

*Proof.* Since $\lambda \in \mathcal{L}$ there exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Proposition 3.5. From Propositions 3.4 and 3.7 we have

$$\tilde{u}_\mu \leq u_\lambda.$$

We introduce the truncation function $\hat{k}_\mu \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{k}_\mu(x,s) = \begin{cases} \mu\left[\tilde{u}_\mu(x)^{-\eta} + a(x)u_\mu(x)^{\tau-1}\right] + f\left(x,u_\mu(x)\right) & \text{if } s < \tilde{u}_\mu(x), \\ \mu\left[s^{-\eta} + a(x)s^{\tau-1}\right] + f\left(x,s\right) & \text{if } \tilde{u}_\mu(x) \leq s \leq u_\lambda(x), \\ \mu\left[u_\lambda(x)^{-\eta} + a(x)u_\lambda(x)^{\tau-1}\right] + f\left(x,u_\lambda(x)\right) & \text{if } u_\lambda(x) < s, \end{cases} \tag{3.35}$$

which is a Carathéodory function. We set $\hat{K}_\mu(x,s) = \int_0^s \hat{k}_\mu(x,t)\,dt$ and consider the $C^1$-functional $\hat{\sigma}_\mu \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \hat{K}_\mu(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

This functional is coercive because of (3.35) and sequentially weakly lower semicontinuous due to the Sobolev embedding theorem. Hence, there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_\mu(u_\mu) = \inf\left[\hat{\sigma}_\mu(u) : W_0^{1,p}(\Omega)\right].$$

Therefore, $\hat{\sigma}_\mu'(u_\mu) = 0$ and so

$$\langle A_p\left(u_\mu\right), h\rangle + \langle A_q\left(u_\mu\right), h\rangle = \int_\Omega \hat{k}_\mu\left(x,u_\mu\right) h\,dx \tag{3.36}$$

for all $h \in W_0^{1,p}(\Omega)$. We first choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.36). Then, by (3.35), $\mu < \lambda$ and since $u_\lambda \in \mathcal{S}_\lambda$, we obtain

$$\begin{aligned} &\left\langle A_p\left(u_\mu\right), (u_\mu - u_\lambda)^+\right\rangle + \left\langle A_q\left(u_\mu\right), (u_\mu - u_\lambda)^+\right\rangle \\ &= \int_\Omega \left[\mu\left(u_\mu^{-\eta} + a(x)u_\lambda^{\tau-1}\right) + f\left(x,u_\lambda\right)\right]\left(u_\mu - u_\lambda\right)^+\,dx \\ &\leq \int_\Omega \left[\lambda\left(u_\lambda^{-\eta} + a(x)u_\lambda^{\tau-1}\right) + f\left(x,u_\lambda\right)\right]\left(u_\mu - u_\lambda\right)^+\,dx \\ &= \left\langle A_p\left(u_\lambda\right), (u_\mu - u_\lambda)^+\right\rangle + \left\langle A_q\left(u_\lambda\right), (u_\mu - u_\lambda)^+\right\rangle. \end{aligned}$$

Hence, $u_\mu \leq v_\lambda$. In the same way, choosing $h = (\tilde{u}_\mu - u_\mu)^+ \in W_0^{1,p}(\Omega)$, we get from (3.35), hypotheses H$(a)$, H$(f)$(i) and Proposition 2.3 that

$$\begin{aligned} &\left\langle A_p\left(u_\mu\right), (\tilde{u}_\mu - u_\mu)^+\right\rangle + \left\langle A_q\left(u_\mu\right), (\tilde{u}_\mu - u_\mu)^+\right\rangle \\ &= \int_\Omega \left[\mu\left(\tilde{u}_\mu^{-\eta} + a(x)\tilde{u}_\mu^{\tau-1}\right) + f\left(x,\tilde{u}_\mu\right)\right]\left(\tilde{u}_\mu - u_\mu\right)^+\,dx \\ &\geq \int_\Omega \mu\tilde{u}_\mu^{-\eta}\left(\tilde{u}_\mu - u_\mu\right)^+\,dx \\ &= \left\langle A_p\left(\tilde{u}_\mu\right), (\tilde{u}_\mu - u_\mu)^+\right\rangle + \left\langle A_q\left(\tilde{u}_\mu\right), (\tilde{u}_\mu - u_\mu)^+\right\rangle. \end{aligned}$$

Thus, $\tilde{u}_\mu \leq u_\mu$. We have proved that

$$u_\mu \in [\tilde{u}_\mu, u_\lambda]. \tag{3.37}$$

From (3.37), (3.35) and (3.36) it follows that

$$u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \text{ and so } \mu \in \mathcal{L}.$$

□

Now we are going to prove that the solution multifunction $\lambda \to \mathcal{S}_\lambda$ has a kind of weak monotonicity property.

**Proposition 3.9.** *If hypotheses H(a) and H(f) hold and if $\lambda \in \mathcal{L}, u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\mu \in (0,\lambda)$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that*

$$u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

*Proof.* From Proposition 3.8 and its proof we know that $\mu \in \mathcal{L}$ and that we can find $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\mu \leq v_\lambda$. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H$(f)$(v). Using $u_\mu \in \mathcal{S}_\mu$, hypotheses H$(a)$, H$(f)$(v) and recalling that $\mu < \lambda$ we obtain

$$\begin{aligned}
& -\Delta_p u_\mu - \Delta_q u_\mu + \hat{\xi}_\rho u_\mu^{p-1} - \mu u_\mu^{-\eta} \\
& = \mu a(x) u_\mu^{\tau-1} + f(x,u_\mu) + \hat{\xi}_\rho u_\mu^{p-1} \\
& = \lambda a(x) u_\mu^{\tau-1} + f(x,u_\mu) + \hat{\xi}_\rho u_\mu^{p-1} - (\lambda-\mu) a(x) u_\mu^{\tau-1} \\
& \leq \lambda a(x) u_\lambda^{\tau-1} + f(x,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \\
& \leq -\Delta_p u_\lambda - \Delta_q u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \mu u_\lambda^{-\eta}.
\end{aligned} \tag{3.38}$$

We have

$$0 \prec (\lambda-\mu) a(x) u_\mu^{\tau-1}.$$

Therefore, from (3.38) and Papageorgiou-Smyrlis [24, Proposition 4], see also Proposition 7 in Papageorgiou-Rădulescu-Repovš [19], we have

$$u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

□

Let $\lambda^* = \sup \mathcal{L}$.

**Proposition 3.10.** *If hypotheses H(a) and H(f) hold, then $\lambda^* < \infty$.*

*Proof.* From hypotheses H$(a)$ and H$(f)$ we can find $\tilde{\lambda} > 0$ such that

$$\tilde{\lambda} a(x) s^{\tau-1} + f(x,s) \geq s^{p-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.39}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Consider a domain $\Omega_0 \subset\subset \Omega$, that is, $\Omega_0 \subseteq \Omega$ and $\overline{\Omega}_0 \subseteq \Omega$, with a $C^2$-boundary $\partial\Omega_0$ and let $m_0 = \min_{\overline{\Omega}_0} u_\lambda > 0$. We set

$$m_0^\delta = m_0 + \delta \quad \text{with} \quad \delta \in (0,1].$$

Let $\rho = \max\left\{\|u_\lambda\|_\infty, m_0^1\right\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H$(f)$(v). Applying (3.39), hypothesis H$(f)$(v) and recalling that $u_\lambda \in \mathcal{S}_\lambda$ as well as $\tilde{\lambda} < \lambda$, we obtain

$$\begin{aligned}
& -\Delta_p m_0^\delta - \Delta_q m_0^\delta + \hat{\xi}_\rho \left(m_0^\delta\right)^{p-1} - \tilde{\lambda}\left(m_0^\delta\right)^{-\eta} \\
& \leq \hat{\xi}_\rho m_0^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
& \leq \left[\hat{\xi}_\rho + 1\right] m_0^{p-1} + \chi(\delta) \\
& \leq \tilde{\lambda} a(x) m_0^{\tau-1} + f(x,u_0) + \hat{\xi}_\rho m_0^{p-1} + \chi(\delta) \\
& = \lambda a(x) m_0^{\tau-1} + f(x,m_0) + \hat{\xi}_\rho m_0^{p-1} - \left(\lambda - \tilde{\lambda}\right) m_0^{\tau-1} + \chi(\delta) \\
& \leq \lambda a(x) m_0^{\tau-1} + f(x,m_0) + \hat{\xi}_\rho m_0^{p-1} \quad \text{for } \delta \in (0,1] \text{ small enough} \\
& \leq \lambda a(x) u_\lambda^{\tau-1} + f(x,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \\
& = -\Delta_p u_\lambda - \Delta_q u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \lambda u_\lambda^{-\eta} \\
& \leq -\Delta_p u_\lambda - \Delta_q u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \tilde{\lambda} u_\lambda^{-\eta} \quad \text{for a. a. } x \in \Omega_0.
\end{aligned} \tag{3.40}$$

From (3.40) and Papageorgiou-Rădulescu-Repovš [19, Proposition 6] we know that

$$u_\lambda - m_0^\delta \in D_+ \quad \text{for } \delta \in (0,1] \text{ small enough},$$

a contradiction. Therefore, $\lambda^* \leq \tilde{\lambda} < \infty$. □

**Proposition 3.11.** *If hypotheses H(a) and H(f) hold and if $\lambda \in (0,\lambda^*)$, then problem (P$_\lambda$) has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \text{ with } u_0 \leq \hat{u} \text{ and } u_0 \neq \hat{u}.$$

*Proof.* Let $\vartheta \in (\lambda,\lambda^*)$. According to Proposition 3.9 we can find $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $u_0 \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$u_\vartheta - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

Recall that $\tilde{u}_\lambda \leq u_0$, see Proposition 3.4. Hence $u_0^{-\eta} \in L^s(\Omega)$ for all $s > N$, see (3.1).

We introduce the Carathéodory function $i_\lambda\colon \Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$i_\lambda(x,s) = \begin{cases} \lambda\left[u_0(x)^{-\eta} + a(x)u_0(x)^{\tau-1}\right] + f(x,u_0(x)) & \text{if } s \leq u_0(x), \\ \lambda\left[s^{-\eta} + a(x)s^{\tau-1}\right] + f(x,s) & \text{if } u_0(x) < s. \end{cases} \tag{3.41}$$

We set $I_\lambda(x,s) = \int_0^s i_\lambda(x,t)\,dt$ and consider the $C^1$-functional $w_\lambda\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$w_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega I_\lambda(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (3.41) and the nonlinear regularity theory along with the nonlinear maximum principle we can easily check that

$$K_{w_\lambda} \subseteq [u_0) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.42}$$

Then, from (3.41) and (3.42) it follows that, without any loss of generality, we may assume

$$K_{w_\lambda} \cap [u_0,u_\vartheta] = \{u_0\}. \tag{3.43}$$

Otherwise, on account of (3.41) and (3.42), we see that we already have a second positive smooth solution of ($\mathrm{P}_\lambda$) distinct and larger than $u_0$.

We introduce the following truncation of $i_\lambda(x,\cdot)$, namely, $\hat{i}_\lambda\colon \Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$\hat{i}_\lambda(x,s)=\begin{cases} i_\lambda(x,s) & \text{if } s\leq u_\vartheta(x),\\ i_\lambda(x,u_\vartheta(x)) & \text{if } u_\vartheta(x)<s, \end{cases} \tag{3.44}$$

which is a Carathéodory function. We set $\hat{I}_\lambda(x,s)=\int_0^s \hat{i}_\lambda(x,t)\,dt$ and consider the $C^1$-functional $\hat{w}_\lambda\colon W^{1,p}_0(\Omega)\to\mathbb{R}$ defined by

$$\hat{w}_\lambda(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega \hat{I}_\lambda(x,u)\,dx \quad \text{for all } u\in W^{1,p}_0(\Omega).$$

From (3.41) and (3.44) it is clear that $\hat{w}_\lambda$ is coercive and due to the Sobolev embedding theorem we know that $\hat{w}_\lambda$ is also sequentially weakly lower semicontinuous. Hence, we find $\hat{u}_0\in W^{1,p}_0(\Omega)$ such that

$$\hat{w}_\lambda\left(\hat{u}_0\right)=\min\left[\hat{w}_\lambda(u): u\in W^{1,p}_0(\Omega)\right]. \tag{3.45}$$

It is easy to see, using (3.44), that

$$K_{\hat{w}_\lambda}\subseteq[u_0,u_\vartheta]\cap \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right) \tag{3.46}$$

and

$$\hat{w}_\lambda\big|_{[0,u_\vartheta]}=w_\lambda\big|_{[0,u_\vartheta]},\quad \hat{w}'_\lambda\big|_{[0,u_\vartheta]}=w'_\lambda\big|_{[0,u_\vartheta]}. \tag{3.47}$$

From (3.45) we have $\hat{u}_0\in K_{\hat{w}'_\lambda}$ which by (3.43), (3.46) and (3.47) implies that $\hat{u}_0=u_0$.

Recall that $u_\vartheta-u_0\in \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)$. So, on account of (3.47), we have that $u_0$ is a local $C^1_0(\overline{\Omega})$-minimizer of $w_\lambda$ and then $u_0$ is also a local $W^{1,p}_0(\Omega)$-minimizer of $w_\lambda$, see, for example Gasiński-Papageorgiou [7].

We may assume that $K_{w_\lambda}$ is finite, otherwise, we see from (3.42) that we already have an infinite number of positive smooth solutions of ($\mathrm{P}_\lambda$) larger than $u_0$ and so we are done. From Papageorgiou-Rădulescu-Repovš [18, Theorem 5.7.6, p. 449] we find $\rho\in(0,1)$ small enough such that

$$w_\lambda(u_0)<\inf\left[w_\lambda(u): \|u-u_0\|=\rho\right]=m_\lambda. \tag{3.48}$$

If $u\in \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)$, then by hypothesis H$(f)$(ii) we have

$$w_\lambda(tu)\to-\infty \quad \text{as } t\to+\infty. \tag{3.49}$$

Moreover, reasoning as in the proof of Proposition 3.1, we show that

$$w_\lambda \text{ satisfies the C-condition,} \tag{3.50}$$

see also (3.41). Then, (3.48), (3.49) and (3.50) permit the use of the mountain pass theorem. So we can find $\hat{u}\in W^{1,p}_0(\Omega)$ such that

$$\hat{u}\in K_{w_\lambda}\subseteq[u_0)\cap \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right),\quad m_\lambda\leq w_\lambda\left(\hat{u}\right). \tag{3.51}$$

From (3.51), (3.48) and (3.41) it follows that

$$\hat{u}\in\mathcal{S}_\lambda,\quad u_0\leq\hat{u},\quad u_0\neq\hat{u}.$$

□

**Remark 3.12.** *If $1 < q = 2 \leq \lambda < p$, then, using the tangency principle of Pucci-Serrin [29, p. 35], we can say that $\hat{u} - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

**Proposition 3.13.** *If hypotheses $H(a)$ and $H(f)$ hold, then $\lambda^* \in \mathcal{L}$.*

*Proof.* Let $\lambda_n \nearrow \lambda^*$. With $\hat{u}_{n+1} \in \mathcal{S}_{\lambda_{n+1}} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ we introduce the following Carathéodory function (recall that $\tilde{u}_{\lambda_1} \leq \tilde{u}_{\lambda_n} \leq u$ for all $u \in \mathcal{S}_{\lambda_n}$ and for all $n \in \mathbb{N}$, see Propositions 3.4 and 3.7)

$$\tilde{t}_n(x,s) =$$
$$\begin{cases} \lambda_n \left[\tilde{u}_{\lambda_1}(x)^{-\eta} + a(x)\tilde{u}_{\lambda_1}(x)^{\tau-1}\right] + f\left(x, \tilde{u}_{\lambda_1}(x)\right) & \text{if } s < \tilde{u}_{\lambda_1}(x) \\ \lambda_n \left[s^{-\eta} + a(x)s^{\tau-1}\right] + f\left(x,s\right) & \text{if } \tilde{u}_{\lambda_1}(x) \leq s \leq \hat{u}_{n+1}(x) \\ \lambda_n \left[\hat{u}_{n+1}(x)^{-\eta} + a(x)\hat{u}_{n+1}(x)^{\tau-1}\right] + f\left(x, \hat{u}_{n+1}(x)\right) & \text{if } \hat{u}_{n+1}(x) < s. \end{cases}$$

Let $\tilde{T}_n(x,s) = \int_0^s \tilde{t}_n(x,t)\,dt$ and consider the $C^1$-functional $\tilde{I}_n\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{I}_n(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \tilde{T}_n(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Applying the direct method of the calculus of variations, see the definition of the truncation $\tilde{t}_n\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we can find $u_n \in W_0^{1,p}(\Omega)$ such that

$$\tilde{I}_n(u_n) = \min\left[\tilde{I}_n(u) : u \in W_0^{1,p}(\Omega)\right].$$

Hence, $\tilde{I}_n'(u_n) = 0$ and so $u_n \in [\tilde{u}_{\lambda_1}, \hat{u}_{n+1}] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see the definition of $\tilde{t}_n$. Moreover, $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

From Proposition 2.3 we know that

$$\tilde{I}_n(u_n) \leq \tilde{I}_n\left(\tilde{u}_{\lambda_1}\right) < 0.$$

Now we introduce the truncation function $\hat{t}_n\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{t}_n(x,s) = \begin{cases} \lambda_n \left[\tilde{u}_{\lambda_1}(x)^{-\eta} + a(x)\tilde{u}_{\lambda_1}(x)^{\tau-1}\right] + f\left(x, \tilde{u}_{\lambda_1}(x)\right) & \text{if } s \leq \tilde{u}_{\lambda_1}(x), \\ \lambda_n \left[s^{-\eta} + a(x)s^{\tau-1}\right] + f(x,s) & \text{if } \tilde{u}_{\lambda_1}(x) < s. \end{cases} \tag{3.52}$$

We set $\hat{T}_n(x,s) = \int_0^s \hat{t}_n(x,t)\,dt$ and consider the $C^1$-functional $\hat{I}_n\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{I}_n(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \hat{T}_n(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from the definition of the truncation $\tilde{t}_n\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and (3.52) that

$$\hat{I}_n\big|_{[0,\hat{u}_{n+1}]} = \tilde{I}_n\big|_{[0,\hat{u}_{n+1}]} \quad \text{and} \quad \hat{I}_n'\big|_{[0,\hat{u}_{n+1}]} = \tilde{I}_n'\big|_{[0,\hat{u}_{n+1}]}.$$

Then from the first part of the proof, we see that we can find a sequence $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, $n \in \mathbb{N}$, such that

$$\hat{I}_n(u_n) < 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.53}$$

Moreover we have

$$\left\langle \hat{I}_n'(u_n), h \right\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and for all } n \in \mathbb{N}. \tag{3.54}$$

From (3.53) and (3.54), reasoning as in the proof of Proposition 3.1, we show that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So we may assume that

$$u_n \xrightarrow{\text{w}} u^* \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^* \text{ in } L^r(\Omega).$$

As before, see the proof of Proposition 3.1, using Proposition 2.1 we show that

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega).$$

Then $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, recall that $\tilde{u}_{\lambda_1} \leq u_n$ for all $n \in \mathbb{N}$. This shows that $\lambda^* \in \mathcal{L}$. □

According to Proposition 3.13 we have

$$\mathcal{L} = (0, \lambda^*].$$

The set $\mathcal{S}_\lambda$ is downward directed, see Papageorgiou-Rădulescu-Repovš [19, Proposition 18], that is, if $u, \hat{u} \in \mathcal{S}_\lambda$, we can find $\tilde{u} \in \mathcal{S}_\lambda$ such that $\tilde{u} \leq u$ and $\tilde{u} \leq \hat{u}$. Using this fact we can show that, for every $\lambda \in \mathcal{L}$, problem ($\mathrm{P}_\lambda$) has a smallest positive solution.

**Proposition 3.14.** *If hypotheses H(a) and H(f) hold and if $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($\mathrm{P}_\lambda$) has a smallest positive solution $u_\lambda^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

*Proof.* Applying Lemma 3.10 of Hu-Papageorgiou [12, p. 178] we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_\lambda$ such that

$$\inf_{n\geq 1} u_n = \inf \mathcal{S}_\lambda.$$

It is clear that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Then, applying Proposition 2.1, we obtain

$$u_n \to u_\lambda^* \text{ in } W_0^{1,p}(\Omega).$$

Since $\tilde{u}_\lambda \leq u_n$ for all $n \in \mathbb{N}$ it holds $u_\lambda^* \in \mathcal{S}_\lambda$ and $u_\lambda^* = \inf \mathcal{S}_\lambda$. □

We examine the map $\lambda \to u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$.

**Proposition 3.15.** *If hypotheses H(a) and H(f) hold, then the map $\lambda \to u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$ is*

- *(a) strictly increasing, that is, $0 < \mu < \lambda \leq \lambda^*$ implies $u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$;*
- *(b) left continuous.*

*Proof.* (a) Let $0 < \mu < \lambda \leq \lambda^*$ and let $u_\lambda^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be the minimal positive solution of problem ($\mathrm{P}_\lambda$), see Proposition 3.14. According to Proposition 3.9 we can find $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Since $u_\mu^* \leq u_\mu$ we have $u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so, we have proved that $\lambda \to u_\lambda^*$ is strictly increasing.

(b) Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{L} = (0, \lambda^*]$ be such that $\lambda_n \nearrow \lambda$ as $n \to \infty$. We have

$$\tilde{u}_{\lambda_1} \leq u_{\lambda_1}^* \leq u_{\lambda_n}^* \leq u_{\lambda^*}^* \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$\left\{u_{\lambda_n}^*\right\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}$$

and so

$$\left\{u_{\lambda_n}^*\right\}_{n\geq 1} \subseteq L^\infty(\Omega) \text{ is bounded},$$

see Guedda-Véron [10, Proposition 1.3]. Therefore, we can find $\beta \in (0,1)$ and $c_{19} > 0$ such that

$$u^*_{\lambda_n} \in C^{1,\beta}_0(\overline{\Omega}) \quad \text{and} \quad \left\|u^*_{\lambda_n}\right\|_{C^{1,\beta}_0(\overline{\Omega})} \leq c_{19} \quad \text{for all } n \in \mathbb{N},$$

see Lieberman [15]. The compact embedding of $C^{1,\beta}_0(\overline{\Omega})$ into $C^1_0(\overline{\Omega})$ and the monotonicity of $\left\{u^*_{\lambda_n}\right\}_{n\geq 1}$, see part (a), imply that

$$u^*_{\lambda_n} \to \hat{u}^*_\lambda \quad \text{in } C^1_0(\overline{\Omega}). \tag{3.55}$$

If $\hat{u}^*_\lambda \neq u^*_\lambda$, then there exists $x_0 \in \Omega$ such that

$$u^*_\lambda(x_0) < \hat{u}^*_\lambda(x_0) \quad \text{for all } n \in \mathbb{N}.$$

From (3.55) we then conclude that

$$u^*_\lambda(x_0) < \hat{u}^*_{\lambda_n}(x_0) \quad \text{for all } n \in \mathbb{N},$$

which contradicts part (a). Therefore, $\hat{u}^*_\lambda = u^*_\lambda$ and so we have proved the left continuity of $\lambda \to u^*_\lambda$. □

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