ELLIPTIC NEUMANN PROBLEMS WITH HIGHLY DISCONTINUOUS NONLINEARITIES

GIUSEPPINA D'AGUÌ, VALERIA MORABITO, AND PATRICK WINKERT

Abstract. This paper investigates nonlinear differential problems involving the p-Laplace operator and subject to Neumann boundary value conditions whereby the right-hand side consists of a nonlinearity which is highly discontinuous. Using variational methods suitable for nonsmooth functionals, we prove the existence of at least two nontrivial weak solutions of such problems.

1. Introduction

In this paper we study a class of elliptic problems driven by the p-Laplacian and with Neumann boundary conditions. Such boundary conditions, which specify the derivative of the solution at the boundary of the domain, are important in scenarios where the flow across the boundary is either controlled or prescribed. The main difficulty in our research is the presence of a highly discontinuous nonlinear term on the right-hand side of the partial differential equation under consideration. Such discontinuities often occur in models that exhibit significant changes, as observed in phase transitions, population dynamics, and certain fluid mechanics problems. The introduction of such a discontinuous nonlinear term leads to an additional complexity of the theoretical analysis. For a comprehensive overview of nonsmooth analysis, we refer the readers to the book by Motreanu and Panagiotopoulos [9, Chapter 3] and the references therein. The theory for locally Lipschitz functionals discussed in the previous book is developed from the framework established in the seminal work of Chang [4], which is based on the results in Nonsmooth Analysis by Clarke [5]. This theory extends the study of variational inequalities explored by Szulkin [14]. In this context, it is also worth mentioning the works of Marano and Motreanu [7, 8], where the authors have established multiple critical points theorems, which extend the results previously obtained by Ricceri [12, 13] for differentiable functionals to nonsmooth functionals. In this direction, we also mention the papers by Bonanno [1] and Bonanno and Candito [2].

In 2019, Bonanno, D'Aguì and Winkert [3] developed an abstract two critical points theorem for general nonsmooth functions based on which they established the existence of at least two positive weak solutions for the following elliptic Dirichlet differential problem

$$-\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, is a bounded domain, $1 < p < N$, λ is a real positive parameter and f is a highly discontinuous function. Using their abstract theorem,

2020 *Mathematics Subject Classification.* 34B15, 34A36, 35B30, 49J52.

Key words and phrases. Discontinuous nonlinearity, multiple solution, Neumann problem, nonsmooth analysis.

our aim is to establish the existence of two distinct nontrivial weak solutions for the following highly discontinuous elliptic PDE involving the p-Laplacian operator under Neumann boundary conditions:

$$-\Delta_p u + \delta(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a C^1-boundary $\partial\Omega$ and $\nu(x)$ is the outer unit normal of Ω at $x \in \partial\Omega$. We deal with the case $p < N$ and assume that $\delta \in L^\infty(\Omega)$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} \delta(x) > 0. \tag{1.2}$$

Moreover, the nonlinear function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first variable and is locally essentially bounded in the second variable. The function $t \to f(x,t)$ may exhibit discontinuities for a.a. $x \in \Omega$. In addition, since $t \to f(x,t)$ is locally essentially bounded for a.a. $x \in \Omega$, the function $F(x,\xi) = \int_0^\xi f(x,t)\,\mathrm{d}t$ is locally Lipschitz in the second variable. Thus, both its generalized directional derivative $F^\circ(x,\cdot)$ and its generalized gradient $\partial F(x,\cdot)$ in the sense of Clarke are well-defined. More details about nonsmooth analysis can be found in Section 2.

Traditional differential equation methods often fail to deal with problems involving such discontinuities or complex nonlinearities. Here, differential inclusions provide a broader framework for defining solutions in nonlinear and discontinuous problems. This means that our problem can be written as an differential inclusion, which expresses the problem as an inclusion between the differential operator and a nonlinear term, often using the subdifferential of a functional. Therefore, our problem can be reformulated as:

$$-\Delta_p u \in \lambda \partial F(x,u) - \delta(x)|u|^{p-2}u \quad \text{in } W^{1,p}(\Omega)^*.$$

The structure of the paper is as follows: Section 2 is devoted to present the relevant mathematical background of the theory of generalized differentiation for locally Lipschitz functions. Moreover, we discuss the variational framework and the associated functional setting. In Section 3 we present the main result, see Theorem 3.1, and a consequence under a particular behavior of the nonlinearity near zero, see Corollary 3.2.

2. Preliminaries and variational framework

In order to define the variational framework needed for our problem, we recall some basic preliminaries on nonsmooth analysis developed by Clarke [5]. To this end, let X be a Banach space with norm $\|\cdot\|_X$ and denote by X^* its topological dual and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. A function $f\colon X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exists a neighborhood U_x of x and a constant $L_x > 0$ such that

$$|f(y) - f(z)| \leq L_x \|y - z\|_X \quad \text{for all } y,z \in U_x.$$

The constant L_x in the previous inequality is called the Lipschitz constant of f near x. Moreover, if $f\colon X \to \mathbb{R}$ is a locally Lipschitz function on a Banach space X, the generalized directional derivative (in the sense of Clarke) of f at the point $x \in X$ in the direction $y \in X$ is defined by

$$f^\circ(x;y) := \limsup_{z\to x,\, t\to 0^+} \frac{f(z+ty) - f(z)}{t}.$$

It is worth noting that the generalized directional derivative extends the classical directional derivative and, in the case of a strictly differentiable function, the conventional directional derivative and the generalized directional derivative are identical. The (Clarke) subdifferential or the generalized gradient in the sense of Clarke of a locally Lipschitz function $f\colon X \to \mathbb{R}$ at $x \in X$ is the subset of the dual space X^* given by

$$\partial f(x) = \{z \in X^*\colon \langle z, y\rangle \ \leq f^\circ(x; y)\ \forall\, y \in X\}.$$

Based on the Hahn-Banach theorem we easily verify that $\partial f(x)$ is non-empty.

The following proposition outlines some of the properties of the generalized directional derivative and generalized gradient that will be useful for further consideration. For the proofs can be found in the book by Clarke [5].

Proposition 2.1. *Let $f, g\colon X \to \mathbb{R}$ be two locally Lipschitz functions. Then, for every $x, y \in X$ the following conditions hold:*

- (p_1) $f^\circ(x; cy) = cf^\circ(x; y)$ *with* $c > 0$;
- (p_2) $f^\circ(x; y_1 + y_2) \leq f^\circ(x; y_1) + f^\circ(x; y_2)$ *for all* $y_1, y_2 \in X$;
- (p_3) $(f + g)^\circ(x; y) \leq f^\circ(x; y) + g^\circ(x; y)$ *or, equivalently,* $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$;
- (p_4) $(-f)^\circ(x; y) = f^\circ(x; -y)$;
- (p_5) *the function* $(x, y) \longmapsto f^\circ(x; y)$ *is upper semicontinuous;*
- (p_6) *if X is finite-dimensional, ∂f is upper semicontinuous in X;*
- (p_7) $f^\circ(x; y) = \max\{< \xi, y >,\ \xi \in \partial f(x)\}$.

Finally, we say that $x \in X$ is a critical point of f when $0 \in \partial f(x)$, namely $f^\circ(x; y) \geq 0 \quad$ for all $y \in X$. We refer to the books by Motreanu and Panagiotopoulos [9] and Motreanu and Rădulescu [10] for more information on this topic.

Let $\Phi, \Psi\colon X \to R$ be two locally Lipschitz continuous functions. We put $I = \Phi - \Psi$. Next, we recall an appropriate version of the Palais-Smale condition for a class of nonsmooth functionals.

Definition 2.2. *We say that $I\colon X \to \mathbb{R}$ verifies the Palais-Smale condition ((PS)-condition for short) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that*

- *$I(u_n)$ is bounded*
- *there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$ such that $I^\circ(u_n; v) \geq -\varepsilon_n \|v\|_X$ for all $v \in X$*

has a strongly convergent subsequence in X.

The main tool for our investigation is the following abstract critical point theorem, useful to establish the existence of at least two weak solutions for problem (1.1), see Bonanno, D'Aguì and Winkert [3, Theorem 2.10].

Theorem 2.3. *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two locally Lipschitz functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Suppose that there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.1}$$

and for all $\lambda \in \Lambda_{r,\tilde{u}}$*, where* $\Lambda_{r,\tilde{u}}$ *is given by*

$$\Lambda_{r,\tilde{u}} = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies* (PS)*-condition and it is unbounded from below. Then, for each* $\lambda \in \Lambda_{r,\tilde{u}}$*, the functional* $\Phi - \lambda\Psi$ *admits at least two nontrivial critical points* $u_{\lambda,1}, u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$*.*

The variational framework and notation to be used in this paper are now introduced. For any $1 \leq p \leq +\infty$ we denote by $L^p(\Omega)$ the usual Lebesgue space endowed with the usual norm $\|\cdot\|_p$ and for any $1 \leq p < +\infty$ we denote by $W^{1,p}(\Omega)$ the corresponding Sobolev space endowed with the usual norm $\|\cdot\|_{1,p}$. We note that, with the previous norms, $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. Due to hypothesis (1.2), we can equip $W^{1,p}(\Omega)$ with the equivalent norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega \delta(x)|u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

In the following proposition we mention the embedding of the space $W^{1,p}(\Omega)$ into suitable $L^r(\Omega)$-spaces, see, for example, Papageorgiou-Winkert [11, Theorem 4.5.25 (a)].

Proposition 2.4. *Let* $p < N$ *and* $p^* = \dfrac{Np}{N-p}$ *the critical Sobolev exponent of* p *in* Ω*. Then, the following embedding holds:* $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1, p^*[$*.*

In the sequel, we denote by h_q the best constant for which we have $\|u\|_q \leq h_q\|u\|$ for all $1 \leq q < p^*$. Now we define two functionals $\Phi, \Psi \colon W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = \int_\Omega F(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega).$$

Standard arguments show that Φ is a C^1 functional on $W^{1,p}(\Omega)$ with derivative

$$\langle \Phi'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla v + \delta(x)|u|^{p-2}uv\right)\mathrm{d}x.$$

In order to introduce additional hypotheses useful for the variational framework by specifying the type of discontinuity allowed on the nonlinear term f, we denote by $\mathcal{H}$ the family of highly discontinuous functions, that is the family of all functions $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that

- (h_1) the function $x \mapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$;
- (h_2) there exists a set $\Omega_0 \subset \Omega$ with $m(\Omega_0) = 0$ such that the set
$$D_f := \cup_{x\in\Omega\setminus\Omega_0}\{t \in \mathbb{R}\colon f(x,\cdot) \text{ is discontinuous at t}\}$$
has measure zero, where $m(A)$ means the Lebesgue measure of a set $A \subset \mathbb{R}^N$;
- (h_3) the function $t \mapsto f(x,t)$ is locally essentially bounded for a.a. $x \in \Omega$;
- (h_4) the functions
$$f^-(x,t) := \lim_{\xi\to 0^+} \operatorname*{ess\,inf}_{|t-z|<\xi} f(x,z), \qquad f^+(x,t) := \lim_{\xi\to 0^+} \operatorname*{ess\,sup}_{|t-z|<\xi} f(x,z)$$

are superpositionally measurable, i.e.,$f^-(x,u(x))$ and $f^+(x,u(x))$ are measurable for all measurable functions $u\colon \Omega \to \mathbb{R}$.

Now we are able to give the precise assumptions on the nonlinearity f:

(H_f) Let $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a function such that:

(f_1) f belongs to $\mathcal{H}$;

(f_2) there exist $s\in[1,p[$ and $q\in]p,p^*[$, such that

$$|f(x,t)|\le a_s|t|^{s-1}+a_q|t|^{q-1}$$

for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$, where a_s and a_q are two positive constants;

(f_3) for each $\lambda>0$, for a.a. $x\in\Omega$ and for all $t\in D_f$

$$\lambda f^-(x,t)\le \delta(x)|t|^{p-2}t\le \lambda f^+(x,t) \quad \text{implies} \quad \lambda f(x,t)=\delta(x)|t|^{p-2}t;$$

(AR) there exist $\eta>p$, $m>0$ such that

$$0<\eta F(x,t)\le t f(x,t)$$

for a.a. $x\in\Omega$ and for all $|t|\ge m$.

In accordance with these assumptions, in particular due to the growth condition (f_2), it can be shown that the functional Ψ is locally Lipschitz on $W^{1,p}(\Omega)$. Consequently, the energy functional associated with the problem (1.1), given by $I_\lambda=\Phi-\lambda\Psi$ is also locally Lipschitz. In this context, a weak solution of problem (1.1) is any $u\in W^{1,p}(\Omega)$ such that

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u\cdot\nabla v+\delta(x)|u|^{p-2}uv\big)\mathrm{d}x=\lambda\int_\Omega f(x,u)v\,\mathrm{d}x. \tag{2.2}$$

is satisfied for all $v\in W^{1,p}(\Omega)$.

The next lemma shows that any function u satisfying (2.2) is a critical point of the energy functional I_λ.

Lemma 2.5. *Let f be a function which satisfies hypotheses (f_1),(f_2) and (f_3). Then, for each $\lambda>0$, the critical points of the functional I_λ are weak solutions of problem (1.1).*

Proof. Fix $\lambda>0$ and let $u_0\in W^{1,p}(\Omega)$ be a critical point of the energy functional I_λ, namely

$$(\Phi-\lambda\Psi)^\circ(u_0;v)\ge 0 \quad \text{for all } v\in W^{1,p}(\Omega).$$

In particular, by using (p_3) of Proposition 2.1, one has

$$0\le \Phi^\circ(u_0;v)+(-\lambda\Psi)^\circ(u_0;v)=\Phi'(u_0,v)+(-\lambda\Psi)^\circ(u_0;v).$$

Replacing the derivative of Φ, we obtain

$$-\int_\Omega \big(|\nabla u|^{p-2}\nabla u\cdot\nabla v+\delta(x)|u|^{p-2}uv\big)\mathrm{d}x\le(-\lambda\Psi)^\circ(u_0;v),$$

that is,

$$\Phi'(u_0)\in\partial(\lambda\Psi)(u_0). \tag{2.3}$$

Since $W^{1,p}(\Omega)$ is embedded and dense in $L^p(\Omega)$, taking Theorem 2.2 by Chang [4] into account, one has

$$\partial(-\lambda\Psi)(u_0)\subseteq\partial(-\lambda\Psi)|_{L^p(\Omega)}(u_0).$$

Therefore, if we define

$$T^*(w) = -\left[\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla w \,\mathrm{d}x + \int_\Omega \delta(x)|u_0|^{p-2}u_0 w \,\mathrm{d}x\right]$$

for all $w \in W^{1,p}(\Omega)$, one has that T^* is a linear and continuous operator on $W^{1,p}(\Omega)$ such that $T^* \in \lambda\partial(-\Psi)(u_0)$ and, additionally, it is a linear and continuous operator on $L^p(\Omega)$. Moreover, since f satisfies hypothesis (f_2) and $-\lambda f \in \mathcal{H}$, we can apply Theorem 2.1 by Chang [4] and obtain

$$\partial(\lambda\Psi)(u_0)_{L^p(\Omega)} \subseteq \lambda[f^-(x,u_0(x)), f^+(x,u_0(x))]|_{L^q(\Omega)}$$

where p and q are conjugated exponents, i.e. $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}&[f^-(x,u_0(x)), f^+(x,u_0(x))]|_{L^q(\Omega)}\\ &= \{w \in L^q(\Omega)\colon w(x) \in [f^-(x,u_0(x)), f^+(x,u_0(x))]\}.\end{aligned}$$

From (2.3), one has

$$\Phi'(u_0) \in \lambda[f^-(x,u_0(x)), f^+(x,u_0(x))]|_{L^q(\Omega)},$$

then

$$\begin{aligned}-\Delta_p u \in [&-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^-(\cdot,u_0(\cdot)),\\ &-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^+(\cdot,u_0(\cdot))]|_{L^q(\Omega)}.\end{aligned}$$

Thus, there exists a unique

$$\begin{aligned}w_0 \in [&-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^-(\cdot,u_0(\cdot)),\\ &-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^+(\cdot,u_0(\cdot))]|_{L^q(\Omega)},\end{aligned}$$

such that $-\Delta_p u_0 = w_0$, that is

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v \,\mathrm{d}x = \int_\Omega w_0 v \,\mathrm{d}x \tag{2.4}$$

for all $v \in L^p(\Omega)$. Now, we show that

$$w_0(x) = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x,u_0(x)) \quad \text{a.e. in } \Omega. \tag{2.5}$$

Let $\Omega_1 \subseteq \Omega$ be a set such that $m(\Omega_1)=0$ and for all $x \in \Omega\setminus\Omega_1$, one has

$$\begin{aligned}w_0(x) \in [&-\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^-(x,u_0(x)),\\ &-\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^+(x,u_0(x))].\end{aligned}$$

We denote by $\Omega_f := \{x \in \Omega\colon u_0(x) \in D_f\}$. Clearly $\Omega_f = u_0^{-1}(D_f)$. Formula (2.4) ensures that u_0 is a weak solution of the problem

$$-\Delta_p u = w_0(x) \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

Using Lemma 1 by De Giorgi, Buttazzo and Dal Maso [6, Lemma 1], one has $-\Delta_p u_0(x) = 0$ for a.a. $x \in \Omega_f$, so that $w_0 = 0$ for a.a. $x \in \Omega_f$.

Let $\Omega_3 \subseteq \Omega_f$ be a set of measure zero such that $w_0 = 0$ for all $x \in \Omega_f \setminus \Omega_3$. Taking into account assumption (f_3), there exists $\Omega_2 \subseteq \Omega$ with $m(\Omega_2)=0$ such that for all $x \in \Omega\setminus\Omega_2$ and for all $t \in D_f$ one has

$$\lambda f(x,t) = \delta(x)|t|^{p-2}t. \tag{2.6}$$

We put $\Omega^* = \bigcup_{i=0}^3 \Omega_i$ and prove (2.5) for each $x \in \Omega\setminus\Omega^*$. We observe that $m(\Omega^*)=0$. Fix $x \in \Omega\setminus\Omega^*$. There are two possible cases: if $u_0(x) \notin D_f$, in particular $x \in \Omega \setminus \Omega_0$ and the definition of D_f implies the continuity of the function $f(\cdot, u_0(\cdot))$ at the point x. Then

$$w_0(x) = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x, u_0(x)).$$

If, on the other hand $u_0(x) \in D_f$, one has that $x \in (\Omega \setminus \Omega_1) \cap (\Omega \setminus \Omega_3)$ and

$$\begin{aligned} 0 = w_0(x) \in [&-\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^-(x, u_0(x)), \\ &- \delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^+(x, u_0(x))], \end{aligned}$$

that is

$$\lambda f^-(x, u_0(x)) \leq \delta(x)|u_0(x)|^{p-2}u_0(x) < x \leq \lambda f^+(x, u_0(x)).$$

Note that $x \in \Omega \setminus \Omega_2$, from (2.6) one has

$$\lambda f(x, u_0(x)) = \delta(x)|u_0(x)|^{p-2}u_0(x).$$

Therefore

$$w_0(x) = 0 = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x, u_0(x))$$

and this completes the proof. □

3. Main results

This section is devoted to apply Theorem 2.3 to our problem. The main result reads as follows.

Theorem 3.1. *Let f be a function satisfying assumptions (H_f). Suppose that there exist $r, d > 0$ such that*

(k$_1$) $d < \left(\dfrac{pr}{\|\delta\|_1}\right)^{\frac{1}{p}}$,

(k$_2$) $\dfrac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \dfrac{a_q}{q} h_q^q (pr)^{\frac{q}{p}} < \displaystyle\int_\Omega F(x, d)\,\mathrm{d}x$,

then for each $\lambda \in \Lambda$, with

$$\Lambda := \left] \frac{d^p \|\delta\|_1}{p \displaystyle\int_\Omega F(x, d)\,\mathrm{d}x}, \frac{r}{\frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}}} \right[,$$

problem (1.1) admits at least two nontrivial weak solutions with opposite energy sign.

Proof. Our aim is to apply Theorem 2.3. Note that the Ambrosetti-Rabinowitz condition stated in (AR) implies that the functional I_λ is unbounded from below and satisfies the Palais-Smale condition. So, we only have to verify that condition (2.1) is satisfied. Fix $\lambda \in \Lambda$ and let $\tilde{u} \in W^{1,p}(\Omega)$ be such that $\tilde{u}(x) = d$ for all $x \in \Omega$. It easy to verify that $0 < \phi(\tilde{u}) < r$. Indeed

$$0 < \Phi(\tilde{u}) = \frac{1}{p}\|\tilde{u}\|^p = \frac{1}{p}\int_\Omega \delta(x) d^p\,\mathrm{d}x = \frac{d^p}{p}\|\delta\|_1 < r, \tag{3.1}$$

from hypothesis (k$_1$). Moreover,

$$\Psi(\tilde{u}) = \int_\Omega F(x, d)\,\mathrm{d}x. \tag{3.2}$$

Then, we consider $u \in X$ such that $\Phi(u) < r$. We observe that

$$\Phi^{-1}(]-\infty, r]) = \left\{ u \in W^{1,p}(\Omega) \colon \frac{1}{p}\|u\|^p < r \right\} = \left\{ u \in W^{1,p}(\Omega) \colon \|u\| < (pr)^{\frac{1}{p}} \right\}.$$

Therefore, exploiting the Sobolev embedding provided in Proposition 2.4 and the growth condition (f$_2$), which implies that

$$F(x,t) \leq \frac{a_s}{s}|t|^s + \frac{a_q}{q}|t|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R},$$

we obtain

$$\begin{aligned}
&\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \\
&= \sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega F(x, u(x))\,\mathrm{d}x \leq \sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega \left(\frac{a_s}{s}|u|^s + \frac{a_q}{q}|u|^q \right) \mathrm{d}x \\
&= \sup_{u \in \Phi^{-1}(]-\infty, r])} \left(\frac{a_s}{s}\|u\|_s^s + \frac{a_q}{q}\|u\|_q^q \right) \leq \sup_{u \in \Phi^{-1}(]-\infty, r])} \left(\frac{a_s}{s}h_s^s\|u\|^s + \frac{a_q}{q}h_q^q\|u\|^q \right) \\
&\leq \frac{a_s}{s}h_s^s(pr)^{\frac{s}{p}} + \frac{a_q}{q}h_q^q(pr)^{\frac{q}{p}}.
\end{aligned}$$

Hence, combining the previous inequality, (3.1), (3.2) and taking into account the choice of λ one has

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Then, hypothesis (2.1) is verified and Theorem 2.3 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$. □

A direct consequence of Theorem 3.1 is the following corollary.

Corollary 3.2. *Let hypothesis* (H$_f$) *be satisfied and suppose that*

$$\limsup_{t \to 0^+} \frac{F(x,t)}{t^p} = +\infty \quad \textit{uniformly for a.a. } x \in \Omega. \tag{3.3}$$

Then, for each $\lambda \in]0, \lambda^*[$, *where*

$$\lambda^* = \sup_{r > 0} \frac{r}{\frac{a_s}{s}h_s^s(pr)^{\frac{s}{p}} + \frac{a_q}{q}h_q^q(pr)^{\frac{q}{p}}},$$

problem (1.1) *admits at least two nontrivial weak solutions.*

Proof. Fix $\lambda \in]0, \lambda^*[$. Then, there exists $r > 0$ such that

$$\lambda < \frac{r}{\frac{a_s}{s}h_s^s(pr)^{\frac{s}{p}} + \frac{a_q}{q}h_q^q(pr)^{\frac{q}{p}}}. \tag{3.4}$$

On the other hand, from (3.3), it follows that

$$\limsup_{d \to 0^+} \frac{\|\delta\|_1}{p} \frac{d^p}{F(x,d)} = 0,$$

so there exist $d > 0$ small enough such that

$$\frac{\|\delta\|_1}{p} \frac{d^p}{F(x,d)} < \lambda,$$

which together with (3.4) implies (k_2). Then, the assertion follows. □

Acknowledgment

The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This paper is partially supported by PRIN 2022 "Nonlinear differential problems with applications to real phenomena" (2022ZXZTN2). The third author thanks the University of Messina for the kind hospitality during a research stay in March 2024.

References

[1] G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012), no. 5, 2992–3007.
[2] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), no. 12, 3031–3059.
[3] G. Bonanno, G. D'Aguì, P. Winkert, *A two critical points theorem for non-differentiable functions and applications to highly discontinuous PDE's*, Pure Appl. Funct. Anal. **4** (2019), no. 4, 709–725.
[4] K.C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), no. 1, 102–129.
[5] F.H. Clarke, "Optimization and Nonsmooth Analysis", Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.
[6] E. De Giorgi, G. Buttazzo, G. Dal Maso, *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **74** (1983), no. 5, 274–282.
[7] S.A. Marano, D. Motreanu, *Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian*, J. Differential Equations **182** (2002), no. 1, 108–120.
[8] S.A. Marano, D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Anal. **48** (2002), no. 1, 37–52.
[9] D. Motreanu, P.D. Panagiotopoulos, "Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities", Kluwer Academic Publishers, Dordrecht, 1999.
[10] D. Motreanu, V.D. Rădulescu, "Variational and Non-Variational Methods in Nonlinear Analysis and Boundary Value Problems", Kluwer Academic Publishers, Dordrecht, 2003.
[11] N.S. Papageorgiou, P. Winkert, "Applied Nonlinear Functional Analysis. An Introduction", 2nd edition, De Gruyter, Berlin, 2024.
[12] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113** (2000), no. 1-2, 401–410.
[13] B. Ricceri, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), no. 3, 220–226.
[14] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 2, 77–109.

(G. D'Aguì) Department of Engineering, University of Messina, 98166 Messina, Italy
Email address: `dagui@unime.it`

(V. Morabito) Department of Engineering, University of Messina, 98166 Messina, Italy
Email address: `valeria.morabito2@studenti.unime.it`

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: `winkert@math.tu-berlin.de`