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Parabolic double phase obstacle problems

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ABSTRACT

We prove existence results for the parabolic double phase obstacle problem: Find $u \in K \subset X_0$ with $u(\cdot, 0) = 0$ satisfying

$$0 \in u_t + Au + F(u) + \partial I_K(u) \quad \text{in } X_0^*,$$

where $A : X_0 \to X_0^*$ given by

$$Au := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \quad \text{for } u \in X_0,$$

is the double phase operator acting on $X_0 = L^p(0, \tau; W_0^{1,\mathcal{H}}(\Omega))$ with $W_0^{1,\mathcal{H}}(\Omega)$ denoting the associated Musielak–Orlicz Sobolev space with generalized homogeneous boundary values. The obstacle is represented by the closed convex set K with the obstacle function ψ through

$$K = \{v \in X_0 \ : \ v(x,t) \le \psi(x,t) \text{ for a.a. } (x,t) \in Q = \Omega \times (0,\tau)\}$$

and I_K is the indicator function related to K with ∂I_K denoting its subdifferential in the sense of convex analysis.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and $Q = \Omega \times (0, \tau)$, $\tau > 0$, the space–time cylindrical domain. In this paper we consider the following parabolic obstacle problem: Find $u \in K \subset X_0$ with $u(\cdot, 0) = 0$ such that

$$0 \in u_t + Au + F(u) + \partial I_K(u) \quad \text{in } X_0^*, \tag{1.1}$$

where $A : X_0 \to X_0^*$ given by

$$Au := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \quad \text{for } u \in X_0,$$

is the double phase operator acting on $X_0 = L^p(0, \tau; W_0^{1,\mathcal{H}}(\Omega))$ with $W_0^{1,\mathcal{H}}(\Omega)$ denoting the associated Musielak–Orlicz Sobolev space with generalized homogeneous boundary values which is a closed subspace of the Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ whose precise definition is provided in Section 2. The obstacle is represented by the closed convex set K with the obstacle function ψ through

$$K = \{v \in X_0 \ : \ v(x,t) \le \psi(x,t) \text{ for a.a. } (x,t) \in Q\}$$

and I_K is the indicator function related to K with ∂I_K denoting its subdifferential in the sense of convex analysis. The time derivative $u_t := \frac{\mathrm{d}u(\cdot,t)}{\mathrm{d}t}$ is understood as the distributional time-derivative of the vector space-valued function $t \mapsto u(\cdot, t) \in W_0^{1,\mathcal{H}}(\Omega)$ with $t \in (0, \tau)$.

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By definition of the subdifferential ∂I_K, the obstacle problem (1.1) is equivalent to the following parabolic variational inequality: Find $u \in K$ with $u(\cdot,0)=0$ such that

$$\langle u_t + Au + F(u), v-u\rangle \geq 0 \quad \text{for all } v \in K, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X_0 and its dual X_0^*, and F is the Nemytskij operator generated by the nonlinearity $f: Q\times\mathbb{R}\to\mathbb{R}$ through $F(u)(x,t)=f(x,t,u(x,t))$. Let us introduce the function spaces Y_0 and Y by means of X_0 and $X = L^p(0,\tau;W^{1,\mathcal{H}}(\Omega))$, respectively, as follows

$$Y_0 = \left\{u \in X_0 \,:\, u_t \in X_0^*\right\} \quad \text{and} \quad Y = \left\{u \in X \,:\, u_t \in X^*\right\}.$$

It will be seen that Y_0 and Y are separable, uniformly convex Banach spaces equipped with the norms

$$\|u\|_{Y_0} = \|u\|_{X_0} + \|u_t\|_{X_0^*} \quad \text{and} \quad \|u\|_Y = \|u\|_X + \|u_t\|_{X^*},$$

where $\|\cdot\|_{X_0}$, $\|\cdot\|_X$ are defined by

$$\|u\|_{X_0} = \left(\int_0^\tau \|u(\cdot,t)\|_{V_0}^p\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad \|u\|_X = \left(\int_0^\tau \|u(\cdot,t)\|_V^p\,\mathrm{d}t\right)^{\frac{1}{p}},$$

with $\|\cdot\|_{V_0}$, $\|\cdot\|_V$ being the norms in $V_0 := W_0^{1,\mathcal{H}}(\Omega)$, $V := W^{1,\mathcal{H}}(\Omega)$, respectively, and $X_0^* = L^{p'}(0,\tau;(W_0^{1,\mathcal{H}}(\Omega))^*)$ and $X^* = L^{p'}(0,\tau;(W^{1,\mathcal{H}}(\Omega))^*)$ are equipped with the norms $\|\cdot\|_{X_0^*}$ and $\|\cdot\|_{X^*}$, given by

$$\|u\|_{X_0^*} = \left(\int_0^\tau \|u(\cdot,t)\|_{V_0^*}^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}}, \quad \|u\|_X = \left(\int_0^\tau \|u(\cdot,t)\|_{V^*}^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}},$$

with $\frac{1}{p}+\frac{1}{p'}=1$. We assume the following hypotheses on the data of the variational inequality (1.2):

(H1) $2 \leq p < N$, $p<q<p^* = \frac{Np}{N-p}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;

(H2) the obstacle function $\psi: Q \to \mathbb{R}$ is supposed to satisfy

$$\begin{aligned} &\psi \in Y, \quad \psi(\cdot,0)\geq 0 \quad \text{in } \Omega, \quad \psi|_\Sigma \geq 0,\\ &\langle \psi_t + A\psi, \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X_0 \text{ with } \varphi \geq 0, \end{aligned}$$

with Σ being the lateral boundary of Q;

(H3) $f: Q\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, that is, $(x,t)\mapsto f(x,t,s)$ is measurable in Q for all $s\in\mathbb{R}$ and $s\mapsto f(x,t,s)$ is continuous in $\mathbb{R}$ for a.a. $(x,t)\in Q$. Further, f satisfies the following growth condition

$$|f(x,t,s)| \leq \alpha(x,t) + \beta|s|^{p-1} \tag{1.3}$$

for a.a. $(x,t)\in Q$ and for all $s \in \mathbb{R}$ where $\alpha \in L^{p'}(Q)$ as well as $\beta \geq 0$.

Definition 1.1. A function $u \in Y_0 \cap K$ is called a solution of the obstacle problem (1.1) (resp. (1.2)) if $u(\cdot,0)=0$ and the variational inequality

$$\begin{aligned} &\langle u_t, v-u\rangle + \int_Q \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot(\nabla v - \nabla u)\,\mathrm{d}x\mathrm{d}t\\ &+ \int_Q F(u)(v-u)\,\mathrm{d}x\mathrm{d}t \geq 0 \quad \text{for all } v\in K, \end{aligned}$$

is satisfied.

The main result in this paper reads as follows.

Theorem 1.2. *Assume hypotheses* (H1)–(H3) *and suppose that*

$$\frac{1}{\|u\|_{X_0}}\langle Au + F(u), u\rangle \to \infty \quad \text{as } \|u\|_{X_0}\to\infty \tag{1.4}$$

is satisfied. Then the parabolic obstacle problem (1.1) *(resp.* (1.2)*) admits at least one solution* $u \in Y_0$.

The study of variational inequalities dates its origins back to the calculus of variations, however its systematic development only began in the 1960s, initiated by the works of Fichera [1] and Stampacchia [2,3], which was motivated by problems in mechanics, like obstacle problems in elasticity. Following the groundbreaking contributions of Lions-Stampacchia [4], the exploration of variational inequalities gained strong development, evolving into a significant domain within nonlinear analysis, calculus of variations, optimization theory, and various branches of mechanics, mathematical physics and engineering.

In recent years elliptic as well as parabolic problems governed by the double phase operator

$$Au := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$

and variants of it have gained increasing importance due to its applications in various fields of applied sciences such as nonlinear elasticity theory of composite materials (see e.g. Zhikov [5,6]), on transonic flows (see e.g. Bahrouni-Rădulescu-Repovš [7]), on quantum physics (see e.g. Benci-D'Avenia-Fortunato-Pisani [8]), on stationary reaction diffusion systems (see e.g. Cherfils-Il'yasov [9]), image noise removal and image processing (see e.g. Kbiri Alaoui-Nabil-Altanji [10], Charkaoui-Ben-Loghfyry-Zeng [11], Chen-Levine [12], Harjulehto-Hästö [13], Harjulehto-Hästö-Latvala-Toivanen [14] and Li-Li-Pi [15]), or heat diffusion in materials with heterogeneous thermal properties (see e.g. Arora-Shmarev [16]). A comprehensive review of the current state of the theory concerning elliptic variational problems with nonstandard growth conditions including double phase problems is given in Mingione-Rădulescu [17]. The ability to capture different behaviors of the specific medium in different regions makes the double phase operator a powerful tool. The origin of the double phase operator is related with the following 'energy functional'

$$\Phi(u)=\int_\Omega \left(|\nabla u|^p+\mu(x)|\nabla u|^q\right)\,\mathrm{d}x,\quad u\in W_0^{1,\mathcal{H}}(\Omega),$$

which was first introduced by Zhikov (see [5,6]) to describe models for strongly anisotropic materials. It describes the phenomenon that the energy density changes its ellipticity and growth properties according to the point $x\in\Omega$. In nonlinear elasticity theory the functional Φ can be used as a model for composite media with different hardening exponents p and q. The geometry of the mixture of the two materials is then described by the zero set $\{x\in\Omega:\mu(x)=0\}$ of the modulating coefficient $x\mapsto\mu(x)$, where the transition from q-growth to p-growth takes place. That is why Φ is called double phase functional.

In order to get an idea of how the double phase integral is related to the double phase operator, let Ω be some composite material, which undergoes deformation by exerting an external force f, which may nonlinearly depend on the deformation $u(x)$, that is $x\mapsto f(x,u(x))$. Then the total energy E stored is mathematically described by

$$E(u)=\int_\Omega\left(\frac{1}{p}|\nabla u|^p+\frac{1}{q}\mu(x)|\nabla u|^q\right)\mathrm{d}x+\int_\Omega G(x,u)\,\mathrm{d}x,\quad u\in W_0^{1,\mathcal{H}}(\Omega), \tag{1.5}$$

where $G(x,u)=\int_0^u f(x,s)\,\mathrm{d}s$ is the primitive of f. Note that the first integral on the right-hand side of (1.5) is basically a multiple of Φ, which is immaterial with respect to the energy balance. According to fundamental physical principle, the deformation u we are looking for is obtained by minimizing the energy functional, that is, we need to solve the minimization problem: Find $u\in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$E(u)=\inf_{v\in W_0^{1,\mathcal{H}}(\Omega)} E(v).$$

Since $E: W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ is a C^1-functional, which – under certain growth restrictions on $s\mapsto f(x,s)$ – is bounded below, coercive, and weakly lower semicontinuous, such that a global minimizer exists. Therefore, a necessary condition is $E'(u)=0$, where E' denotes the Frechet derivative of E, that is,

$$\langle E'(u),\varphi\rangle=0\quad\text{for all }\varphi\in W_0^{1,\mathcal{H}}(\Omega), \tag{1.6}$$

where here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space. Calculating the Frechet derivative (1.6) becomes

$$\int_\Omega\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x+\int_\Omega f(x,u)\varphi\,\mathrm{d}x=0$$

for all $\varphi\in W_0^{1,\mathcal{H}}(\Omega)$, which is nothing but the weak formulation of the following Dirichlet problem

$$Au+f(x,u)=0\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,$$

where A is the double phase operator.

Now consider the problem of deformation of the composite material under an additional constraint given by an obstacle K of the form

$$K=\left\{v\in W_0^{1,\mathcal{H}}(\Omega): v(x)\le\psi(x)\text{ for }x\in\Omega\right\},$$

which means, we are looking for deformations u that, in addition, satisfy $u(x)\le\psi(x)$. In this situation the deformation is obtained as the solution of the following minimization problem under constraint: Find $u\in K$ such that

$$E(u)=\inf_{v\in K}E(v)=\inf_{v\in W_0^{1,\mathcal{H}}(\Omega)}[E(v)+I_K(v)], \tag{1.7}$$

where I_K is the indicator function related to K. By standard variational calculus, a necessary condition for finding u satisfying (1.7) is the following variational inequality: Find $u\in K$ such that

$$\langle E'(u),v-u\rangle\ge 0\quad\text{for all }v\in K,$$

which is equivalent to the following elliptic double phase variational inequality

$$\begin{aligned}&\int_\Omega\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x\\&+\int_\Omega f(x,u)(v-u)\,\mathrm{d}x\ge 0\quad\text{for all }v\in K.\end{aligned} \tag{1.8}$$

In this paper we consider the evolutionary counterpart of the elliptic double phase variational inequality (1.8), which is the parabolic double phase variational inequality (1.2). Consider first the following Cauchy–Dirichlet problem for the parabolic double phase equation

$$u_t + Au + F(u) = 0 \quad \text{in } Q, \quad u = 0 \quad \text{on } \partial\Omega \times (0,\tau), \quad u(x,0) = u_0(x) \quad \text{in } \Omega.$$

Existence results for the Cauchy–Dirichlet problem of the parabolic double phase equation under homogeneous Dirichlet data on the lateral boundary $\partial\Omega\times(0,\tau)$ have been obtained e.g. by Arora-Shmarev [16] and Bögelein-Duzaar-Marcellini [18]. Similaraly as in the stationary case considered before, this kind of problems may be used e.g. as a mathematical model describing the thermal diffusion in heterogeneous media, where the heat conduction in composite materials with varying thermal properties is described through the double phase operator A. In this context, the parabolic double phase variational inequality (1.2) stands for a heat conduction model in some composite materials with varying thermal properties under the additional constraint given by K.

Unlike in the treatment of an elliptic double phase obstacle problem, in the treatment of the evolutionary obstacle problem considered here an additional difficulty arises, which is due to the subdifferential of the indicator function ∂I_K in (1.1). Because of this no growth condition can be assumed on ∂I_K, and thus, in general there is no growth estimate of the time derivative u_t in the dual space X_0^* available, which would be needed for proving existence of solutions. In case that K admits a nonempty interior, that is, $\operatorname{int}(K) \neq \emptyset$, this difficulty can be overcome by applying Rockafellar's theorem about the sums of maximal monotone operators, which allows to study evolutionary variational inequalities by implementation of arguments and results for elliptic variational inequalities to evolutionary variational inequalities. Unfortunately, the interior of the constraint K we are dealing with is empty, i.e., $\operatorname{int}(K) = \emptyset$, and therefore a similar approach as for elliptic variational inequalities cannot be applied. Instead, we are dealing with this difficulty by using a penalty technique.

As far as we know this is the first work dealing with existence results for parabolic double phase obstacle problems. Our main goal is to establish a functional analytic framework and derive existence results for parabolic double phase obstacle problems (1.1), resp. (1.2).

In the case of parabolic double phase equations we refer to the work of Arora-Shmarev [16] who considered parabolic equations of the form

$$u_t - \operatorname{div}\left(|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u\right) = F(x,u,\nabla u) \quad \text{in } Q_T = \Omega \times (0,T)$$

for $z = (x,t) \in Q$ with

$$\frac{2N}{N+2} < p^- \leq p(z) \leq q(z) < p(z) + \frac{r^*}{2}, \quad r^* = r^*(p^-,N), \quad p^- = \min_{\overline{Q}_T} p(z).$$

Under certain conditions on the right-hand side the existence of a unique solution has been shown. The same authors [19] obtained the unique strong solution with a certain kind of regularity to the parabolic equation

$$u_t - \operatorname{div}\left(a(z)|\nabla u|^{p(z)-2}\nabla u + b(z)|\nabla u|^{q(z)-2}\nabla u\right) = f$$

for $z = (x,t) \in Q_T = \Omega \times (0,T)$. Yuan-Ge-Cao-Zhang [20] proved the existence of solutions for parabolic problems with the limiting case of double phase flux given by

$$\begin{cases} u_t - \operatorname{div}\left(\frac{Du}{|Du|} + \mu(x)\frac{Du}{|Du|}\right) = f(x,u) & \text{in } \Omega \times (0,+\infty),\\ u = 0 & \text{in } \partial\Omega \times (0,+\infty),\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

with a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ that fulfills the Ambrosetti–Rabinowitz condition. Finally, we mention some works dealing with regularity results for local and nonlocal parabolic double phase type, see, for example, Buryachenko-Skrypnik [21] (local continuity and Harnack's inequality for parabolic double phase equations), Giacomoni-Kumar-Sreenadh [22] (Hölder regularity results for nonlocal parabolic double phase problems), Grimaldi-Ipocoana [23] (higher differentiability results), Kim-Kinnunen-Moring [24] (gradient higher integrability for degenerate parabolic double phase systems), Meng-Zhang [25] (asymptotic mean value properties), Prasad-Tewary [26] (local boundedness to nonlocal parabolic double phase equations), Savchenko-Skrypnik-Yevgenieva [27] (Harnack's inequality for degenerate parabolic double phase equations under the non-logarithmic Zhikov's condition), Shang-Zhang [28] (regularity for mixed local and nonlocal parabolic double phase equations) and the references therein. In the case of elliptic double phase obstacle problems there are some more references in the direction of existence results and regularity properties. We mention, for example, the papers of Byun-Cho-Oh [29], Byun-Liang-Zheng [30], Zeng-Bai-Gasiński-Winkert [31], Zeng-Bai-Rădulescu-Winkert [32], Zeng-Gasiński-Winkert-Bai [33], Zeng-Rădulescu-Winkert [34–36], Zhao-Zheng [37], see also the references therein.

The paper is organized as follows. In Section 2 we present the underlying function spaces and some auxiliary results about certain parabolic embeddings as well as properties of the penalty operator. Section 3 is concerned with the proof of Theorem 1.2 based on surjectivity results for pseudomonotone operators along with a penalty technique. Finally, in Section 4 we state the notion of sub-supersolutions to (1.1), give sufficient conditions for the existence of it (see Lemmas 4.3, 4.4) and prove an existence result for (1.1) for a given ordered pair of sub-supersolutions, see Theorem 4.5.

2. Preliminaries

In this section we introduce the Musielak–Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega)$ and summarize some of their relevant properties. Further, based on the Musielak–Orlicz spaces we are going to characterize the space–time function spaces X, X_0, Y, and Y_0.

Suppose (H1) and consider the nonlinear function $\mathcal{H} : \Omega \times [0,\infty) \to [0,\infty)$ given by

$$\mathcal{H}(x,s) = s^p + \mu(x)s^q,$$

then the Musielak–Orlicz spaces $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \lambda > 0 \ : \ \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \le 1 \right\}.$$

The space $L^{\mathcal{H}}(\Omega)$ is separable and uniformly convex and thus a reflexive Banach space. Further let us introduce the weighted space $L^q_\mu(\Omega)$ defined by

$$L^q_\mu(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega \mu(x)|u|^q\,\mathrm{d}x < +\infty \right\}$$

equipped with the seminorm $\|\cdot\|_{q,\mu}$ given by

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

Denote by $L^r(U)$ the usual Lebesgue spaces equipped with the norm $\|\cdot\|_{r,U}$, then one readily verifies the following continuous embeddings to hold true

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega).$$

The relation between the norm $\|\cdot\|_{\mathcal{H}}$ and the $\mathcal{H}$-modular $\rho_{\mathcal{H}}$ are summarized in the following proposition, see Liu-Dai [38, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [39, Proposition 2.13].

Proposition 2.1. *Under hypothesis* (H1) *the following relations hold true:*

(i) $\|u\|_{\mathcal{H}} = a \neq 0 \Longleftrightarrow \rho_{\mathcal{H}}\left(\frac{u}{a}\right) = 1$;
(ii) $\|u\|_{\mathcal{H}} < 1$ *(resp.* $\|u\|_{\mathcal{H}} = 1$, $\|u\|_{\mathcal{H}} > 1$*)* $\Longleftrightarrow \rho_{\mathcal{H}}(u) < 1$ *(resp.* $\rho_{\mathcal{H}}(u) = 1$, $\rho_{\mathcal{H}}(u) > 1$*)*;
(iii) $\|u\|_{\mathcal{H}} < 1 \Longrightarrow \|u\|^q_{\mathcal{H}} \le \rho_{\mathcal{H}}(u) \le \|u\|^p_{\mathcal{H}}$;
(iv) $\|u\|_{\mathcal{H}} > 1 \Longrightarrow \|u\|^p_{\mathcal{H}} \le \rho_{\mathcal{H}}(u) \le \|u\|^q_{\mathcal{H}}$;
(v) $\|u\|_{\mathcal{H}} \to 0 \Longleftrightarrow \rho_{\mathcal{H}}(u) \to 0$, $\|u\|_{\mathcal{H}} \to +\infty \Longleftrightarrow \rho_{\mathcal{H}}(u) \to +\infty$.

The Musielak–Orlicz Sobolev space $V := W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \ : \ |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_V = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}},$$

and its subspace $V_0 := W^{1,\mathcal{H}}_0(\Omega)$, whose functions have generalized homogeneous boundary values, is defined as the completion of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_V$, that is

$$V_0 := W^{1,\mathcal{H}}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_V}.$$

As $L^{\mathcal{H}}(\Omega)$ is separable and uniformly convex, the Musielak–Orlicz Sobolev spaces V and V_0 are separable and uniformly convex as well, and thus reflexive. Moreover, the following proposition holds, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [39, Propositions 2.16 and 2.18].

Proposition 2.2. *Suppose hypothesis* (H1) *is satisfied.*

(i) Continuous embedding: $V_0 \hookrightarrow L^r(\Omega)$ *for all* $r \in [1, p^*]$ *with* $p^* = \frac{Np}{N-p}$ *being the critical Sobolev exponent;*
(ii) Compact embedding: $V_0 \hookrightarrow\hookrightarrow L^r(\Omega)$ *for all* $r \in [1, p^*)$;
(iii) Equivalent norm in V_0: $\|u\|_{V_0} = \|\nabla u\|_{\mathcal{H}}$ *for all* $u \in V_0$.

The double phase operator $Au := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$ generates a mapping (again denoted by A) from V_0 to its dual V_0^* defined by

$$\langle Au, \varphi\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x \tag{2.1}$$

for all $u, \varphi \in V_0$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_0^* and V_0. From Theorem 3.3 and its proof in Crespo-Blanco-Gasiński-Harjulehto-Winkert [39] we deduce the following properties of A.

Proposition 2.3. *The double phase operator A defined by (2.1) satisfies:*

(i) $A : V_0 \to V_0^*$ *is continuous, bounded, and strictly monotone;*
(ii) A *is of type* (S_+)*, that is, if* $u_n \rightharpoonup u$ *(weakly) in* V_0 *and*

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0,$$

then $u_n \to u$ *(strongly) in* V_0*;*
(iii) $\|Au\|_{V_0^*} \le 2\|u\|_{V_0}$ *for all* $u \in V_0$.

With the notations $V := W^{1,\mathcal{H}}(\Omega)$ and $V_0 := W_0^{1,\mathcal{H}}(\Omega)$ and the properties of the Musielak–Orlicz Sobolev spaces V and V_0, the Lebesgue spaces $X = L^p(0,\tau;V)$, $X_0 = L^p(0,\tau;V_0)$ and Y and Y_0 as introduced already in Section 1 are separable, and uniformly convex, and thus reflexive Banach spaces, see e.g. Zeidler [40, Proposition 23.2, Proposition 23.7].

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$. Note that the Musielak–Orlicz Sobolev space spaces V_0 and V have lattice structure and are closed under max and min operators (see Crespo-Blanco-Gasiński-Harjulehto-Winkert [39, Proposition 2.17]), which implies that the corresponding Lebesgue space X_0 and X have lattice structure as well and are closed under max and min operators.

Lemma 2.4. *Assume hypothesis* (H1)*. Then the following assertions hold true:*

(i) $V_0 \hookrightarrow L^2(\Omega) \hookrightarrow V_0^*$ *forms an evolution triple;*
(ii) Continuous embedding: $Y_0 \hookrightarrow C([0,\tau];L^2(\Omega))$*;*
(iii) Let $u \in Y_0$*, then the following integration by parts formula is valid*

$$\int_0^\tau \langle u_t(\cdot,t), u(\cdot,t)\rangle\,\mathrm{d}t = \frac{1}{2}\left(\|u(\cdot,\tau)\|_{2,\Omega}^2 - \|u(\cdot,0)\|_{2,\Omega}^2\right);$$

(iv) Let $u \in Y_0$*, then it holds*

$$\int_0^\tau \langle u_t(\cdot,t), u(\cdot,t)^+\rangle\,\mathrm{d}t = \frac{1}{2}\left(\|u(\cdot,\tau)^+\|_{2,\Omega}^2 - \|u(\cdot,0)^+\|_{2,\Omega}^2\right).$$

Proof. (i) V_0 is a separable and reflexive (even uniformly convex) Banach space, which by Proposition 2.2 (ii) is compactly embedded in the Hilbert space $L^2(\Omega)$, and moreover V_0 is dense in $L^2(\Omega)$.

(ii) and (iii) are immediate consequences of Proposition 23.23 in Zeidler [40].

(iv) In a similar way as in the proof of Lemma 2.146 in Carl-Le-Motreanu [41] one obtains this formula by regularization and density arguments. □

Remark 2.5. Clearly, Lemma 2.4 remains true if V_0 is replaced by V and Y_0 is replaced by Y.

Lemma 2.6. *Assume hypothesis* (H1)*. Then the following compact embedding holds*

$$Y_0 \hookrightarrow\hookrightarrow L^p(0,\tau;L^p(\Omega)) = L^p(Q).$$

Proof. Since $V_0 \hookrightarrow\hookrightarrow L^p(\Omega)$ and $L^p(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow V_0^*$ (note that $p \ge 2$), and thus $V_0 \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow V_0^*$, we may apply the Lions-Aubin Theorem (see e.g. Carl-Le [42, Theorem 2.52]), which results in $Y_0 \hookrightarrow\hookrightarrow L^p(Q)$. □

By means of the obstacle function ψ we define an operator P through

$$\langle P(u), \varphi\rangle = \int_Q \left[(u-\psi)^+\right]^{p-1}\varphi\,\mathrm{d}x\mathrm{d}t \quad \text{for all } u, \varphi \in X_0. \tag{2.2}$$

Lemma 2.7. *The operator* $P : X_0 \to X_0^*$ *is bounded, continuous and monotone, and satisfies*

$$P(u) = 0 \quad \Longleftrightarrow \quad u \in K, \tag{2.3}$$

that is, P *is a penalty operator associated with* K.

Proof. We define the function

$$g(x,t,s) = \left[(s-\psi(x,t))^+\right]^{p-1},$$

which is readily seen to be a Carathéodory function and has the growth

$$|g(x,t,s)| \le (|\psi(x,t)| + |s|)^{p-1} \le k(x,t) + c|s|^{p-1}$$

for a.a. $(x,t) \in Q$, for all $s \in \mathbb{R}$ where $k \in L^{p'}(Q)$ and $c \ge 0$. Furthermore, $s \mapsto g(x,t,s)$ is monotone nondecreasing. Thus the associated Nemytskij operator $G(u)(x,t) = g(x,t,u(x,t))$ defines a monotone and continuous mapping $G : L^p(Q) \to L^{p'}(Q)$. From the continuous embedding $V_0 \hookrightarrow L^p(\Omega)$ it follows that $i : X_0 \hookrightarrow L^p(Q)$ as well as the adjoint $i^* : L^{p'}(Q) \hookrightarrow X_0^*$ are continuous too. Hence

$$P = i^* \circ G \circ i : X_0 \to X_0^*$$

is a bounded, continuous and monotone operator. Let us prove (2.3). If $u \in K$, then $(u-\psi)^+ = 0$ and thus $\langle P(u), \varphi\rangle = 0$ for all $\varphi \in X_0$, that is $P(u) = 0$. Conversely, let $P(u) = 0$, which yields

$$0 = \langle P(u), \varphi\rangle = \int_Q \left[(u-\psi)^+\right]^{p-1} \varphi \,\mathrm{d}x\mathrm{d}t \quad \text{for all } \varphi \in X_0.$$

In particular also for $\varphi = (u-\psi)^+ \in X_0$, which implies

$$\int_Q \left[(u-\psi)^+\right]^{p} \mathrm{d}x\mathrm{d}t = 0.$$

Therefore $(u-\psi)^+ = 0$, that is, $u \in K$. □

Lemma 2.8. *The penalty operator $P : X_0 \to X_0^*$ defined by (2.2) fulfills the inequality*

$$\langle P(u), (u-\psi)^+\rangle \ge d\, \|P(u)\|_{X_0^*} \|(u-\psi)^+\|_{p,Q}$$

with $d > 0$.

Proof. From (2.2) we get

$$\langle P(u), (u-\psi)^+\rangle = \|(u-\psi)^+\|_{p,Q}^p. \tag{2.4}$$

Applying Hölder's inequality and the continuous embedding $X_0 \hookrightarrow L^p(Q)$ we obtain

$$\begin{aligned}|\langle P(u), \varphi\rangle| &\le \int_Q \left[(u-\psi)^+\right]^{p-1} |\varphi| \,\mathrm{d}x\mathrm{d}t\\ &\le \|(u-\psi)^+\|_{p,Q}^{p-1} \|\varphi\|_{p,Q}\\ &\le c\|(u-\psi)^+\|_{p,Q}^{p-1} \|\varphi\|_{X_0} \quad \text{for all } \varphi \in X_0,\end{aligned}$$

and thus

$$\|P(u)\|_{X_0^*} \le c\|(u-\psi)^+\|_{p,Q}^{p-1}. \tag{2.5}$$

From (2.4) and (2.5) it follows

$$\langle P(u), (u-\psi)^+\rangle \ge \frac{1}{c} \|P(u)\|_{X_0^*} \|(u-\psi)^+\|_{p,Q},$$

which completes the proof. □

Lemma 2.9. *Assume* (H2). *Then for any $u \in Y_0$ with $u(\cdot,0) = 0$ one has*

$$\langle u_t + Au, (u-\psi)^+\rangle \ge 0.$$

Proof. First note that $u - \psi \in Y$ and $(u-\psi)^+(x,0) = 0$, which by applying the integration by parts formula (see Lemma 2.4 and Remark 2.5) yields

$$\langle (u-\psi)_t, (u-\psi)^+\rangle = \frac{1}{2}\|(u-\psi)^+(\cdot,\tau)\|_{2,\Omega}^2. \tag{2.6}$$

With (2.6) and taking into account that the double phase operator A is monotone we get

$$\begin{aligned}&\langle u_t + Au - (\psi_t + A\psi), (u-\psi)^+\rangle\\ &= \langle (u-\psi)_t, (u-\psi)^+\rangle + \langle Au - A\psi, (u-\psi)^+\rangle \ge 0,\end{aligned}$$

which by means of (H2) completes the proof. □

3. Proof of Theorem 1.2

Before proving our main result we start with general considerations on an abstract evolution equation. Even though such an evolution equation is treated within the framework of a general evolution triple $V \hookrightarrow H \hookrightarrow V^*$ and the associated Lebesgue spaces $X = L^p(0,\tau;V)$ and its dual space $X^* = L^{p'}(0,\tau;V^*)$ of vector valued functions, in what follows we already specify to our situation, that is, to the evolution triple

$$V_0 \hookrightarrow L^2(\Omega) \hookrightarrow V_0^*,$$

and the associated Lebesgue spaces X_0, X_0^*, Y_0 introduced in the preceding section. Let us consider the evolution equation

$$u \in Y_0: \quad u'(t) + \hat{T}(t)u(t) = f(t), \quad 0 < t < \tau, \quad u(0) = 0, \tag{3.1}$$

where $f \in X_0^*$ and $\hat{T}(t): V_0 \to V_0^*$, and $u'(t)(x) = u_t(x,t)$. Let $Lu := u' = u_t$ be the time derivative operator with domain

$$D(L) = \{u \in Y_0 \ : \ u(0) = 0\}.$$

Then by using Proposition 32.10 of Zeidler [43] we have the following result.

Lemma 3.1. *The operator $L: D(L) \to X_0^*$ is densely defined, closed and maximal monotone.*

Next, we define the operator T by means of $\hat{T}$ by

$$T(u)(t) := \hat{T}(t)u(t), \quad t \in [0,\tau].$$

Hence the evolution Eq. (3.1) can equivalently be written in the form

$$u \in D(L): \quad Lu + T(u) = f \quad \text{in } X_0^*.$$

In Berkovits–Mustonen [44] it is proved that certain properties of the operator $\hat{T}(t): V_0 \to V_0^*$ transfer to the operator $T: X_0 \to X_0^*$. To this end let $D(L)$ be equipped with its graph norm $\|u\|_L = \|u\|_{X_0} + \|Lu\|_{X_0^*}$.

Definition 3.2.

(i) $T: X_0 \to X_0^*$ is called pseudomonotone with respect to the graph norm topology of $D(L)$ (for short: pseudomonotone w.r.t. $D(L)$), if for any sequence $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in X_0, $Lu_n \rightharpoonup Lu$ in X_0^* and

$$\limsup_{n\to\infty} \langle T(u_n), u_n - u\rangle \le 0$$

implies $T(u_n) \rightharpoonup T(u)$ and $\langle T(u_n), u_n\rangle \to \langle T(u), u\rangle$;

(ii) $T: X_0 \to X_0^*$ has the (S_+)-property w.r.t. $D(L)$, if $u_n \rightharpoonup u$ in X_0, $Lu_n \rightharpoonup Lu$ in X_0^* and $\limsup_{n\to\infty}\langle T(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in X_0.

The following theorem can be found in Berkovits–Mustonen [44].

Theorem 3.3.

(i) If $\hat{T}(t): V_0 \to V_0^*$ is pseudomonotone for all $t \in [0,\tau]$, then $T: X_0 \to X_0^*$ is pseudomonotone w.r.t. $D(L)$;

(ii) If $\hat{T}(t): V_0 \to V_0^*$ has the (S_+)-property for all $t \in [0,\tau]$, then $T: X_0 \to X_0^*$ has the (S_+)-property w.r.t. $D(L)$;

(iii) If

$$\|\hat{T}(t)u\|_{V_0^*} \le c\left(k(t) + \|u\|_{V_0}^{p-1}\right), \ \forall u \in V_0, \ \forall t \in [0,\tau],$$

where $c > 0$, $k \in L^{p'}(0,\tau)$, and $t \mapsto \langle \hat{T}(t)u, v\rangle$ is measurable on $(0,\tau)$ for all $u, v \in V_0$, then $T: X_0 \to X_0^*$ is bounded.

The following surjectivity theorem plays an important role in the proof of our main result, see Berkovits–Mustonen [45] or Lions [46].

Theorem 3.4. *Let $T: X_0 \to X_0^*$ be bounded, demicontinuous, and pseudomonotone w.r.t. $D(L)$. If T is coercive, that is,*

$$\frac{1}{\|u\|_{X_0}}\langle Tu, u\rangle \to \infty \quad \text{as } \|u\|_{X_0} \to \infty,$$

then $L + T: D(L) \to X_0^*$ is surjective, that is, $(L+T)(D(L)) = X_0^*$.

Using the notations introduced above, the parabolic double phase obstacle problem (1.1) (resp. (1.2)) can be reformulated as follows: Find $u \in D(L) \cap K$ such that

$$\langle u_t + Au + F(u), v - u\rangle \ge 0 \quad \text{for all } v \in K.$$

Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof is based on a penalty approach and is carried out in several steps.

Step 1: Assumption (H2) implies that the obstacle function ψ is nonnegative, that is, $\psi \geq 0$, which is seen as follows. By (H2), $\psi \in Y$, and nonnegative on the parabolic boundary of Q, and satisfies

$$\langle \psi_t + A\psi, \varphi \rangle \geq 0 \quad \text{for all } \varphi \in X_0 \text{ with } \varphi \geq 0.$$

Testing the inequality with $\varphi = \psi^- \in X_0$ we get

$$\langle \psi_t, \psi^- \rangle + \int_Q \left(|\nabla \psi|^{p-2} \nabla \psi + \mu(x) |\nabla \psi|^{q-2} \nabla \psi \right) \cdot \nabla \psi^- \,\mathrm{d}x\mathrm{d}t \geq 0.$$

By means of the integration by parts formula and taking the assumption $\psi(\cdot, 0) \geq 0$ into account we get

$$\begin{aligned}
\langle \psi_t, \psi^- \rangle &= \langle \psi_t, \psi^+ - \psi \rangle = \langle \psi_t, \psi^+ \rangle - \langle \psi_t, \psi \rangle \\
&= \|\psi(\cdot, \tau)^+\|_{2,\Omega}^2 - \|\psi(\cdot, 0)^+\|_{2,\Omega}^2 - \left[\|\psi(\cdot, \tau)\|_{2,\Omega}^2 - \|\psi(\cdot, 0)\|_{2,\Omega}^2 \right] \\
&= \|\psi(\cdot, \tau)^+\|_{2,\Omega}^2 - \|\psi(\cdot, \tau)\|_{2,\Omega}^2 \leq 0,
\end{aligned}$$

which yields

$$\int_Q \left(|\nabla \psi|^{p-2} \nabla \psi + \mu(x) |\nabla \psi|^{q-2} \nabla \psi \right) \cdot \nabla \psi^- \,\mathrm{d}x\mathrm{d}t \geq 0.$$

Thus

$$-\int_Q \left(|\nabla \psi^-|^p + \mu(x) |\nabla \psi^-|^q \right) \mathrm{d}x\mathrm{d}t \geq 0,$$

which results in $\nabla \psi^- = 0$, and therefore $\psi^- = 0$, that is, $\psi \geq 0$.

Step 2: Penalty equations

For $\varepsilon > 0$, we consider the following penalty equation:

$$u \in D(L): \quad u_t + Au + F(u) + \frac{1}{\varepsilon} P(u) = 0 \quad \text{in } X_0^*, \tag{3.2}$$

where

$$\begin{aligned}
\langle Au, v \rangle &= \int_Q \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) \nabla v \,\mathrm{d}x\mathrm{d}t, \\
\langle F(u), v \rangle &= \int_Q f(x,t,u) v \,\mathrm{d}x\mathrm{d}t, \\
\langle P(u), v \rangle &= \int_Q \left[(u-\psi)^+ \right]^{p-1} v \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

with P being the penalty operator. From Proposition 2.3(i), (iii), we infer that $A: X_0 \to X_0^*$ is bounded, continuous and strictly monotone, which implies that, $A: X_0 \to X_0^*$, is pseudomonotone in the usual sense (see e.g. Zeidler [43, Proposition 27.6 (a)]), and thus, in particular, pseudomonotone w.r.t. $D(L)$. By (H3), the Nemytskij operator $F: L^p(Q) \to L^{p'}(Q)$ is continuous and bounded with the estimate

$$\begin{aligned}
|\langle F(u), v \rangle| &\leq \int_Q |f(x,t,u)| |v| \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_Q \left(\alpha(x,t) + \beta |u|^{p-1} \right) |v| \,\mathrm{d}x\mathrm{d}t \\
&\leq \left(\|\alpha\|_{p',Q} + \beta \|u\|_{p,Q}^{p-1} \right) \|v\|_{p,Q},
\end{aligned}$$

and

$$\|v\|_{p,Q} = \left(\int_0^\tau \int_\Omega |v(x,t)|^p \,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p}} \leq c \left(\int_0^\tau \|v(\cdot,t)\|_{V_0}^p \,\mathrm{d}t \right)^{\frac{1}{p}} \leq c \|v\|_{X_0},$$

and thus

$$|\langle F(u), v \rangle| \leq c \left(\|\alpha\|_{p',Q} + \beta \|u\|_{p,Q}^{p-1} \right) \|v\|_{X_0}. \tag{3.3}$$

In view of the continuous embeddings

$$X_0 \hookrightarrow L^p(Q) \hookrightarrow L^{p'}(Q) \hookrightarrow X_0^*$$

and taking into account (3.3) we see that $F: X_0 \to X_0^*$ is continuous and bounded, which along with the compact embedding $Y_0 \hookrightarrow\hookrightarrow L^p(Q)$ (see Lemma 2.6) implies that $F: X_0 \to X_0^*$ is pseudomonotone w.r.t. $D(L)$. Taking Lemma 2.7 into account, the same arguments apply to the penalty operator $P: X_0 \to X_0^*$, which is continuous, bounded and pseudomonotone w.r.t. $D(L)$ as well. Since the sum of operators that are pseudomonotone w.r.t. $D(L)$ is again pseudomonotone w.r.t. $D(L)$, we have that T given by

$$T := A + F + \frac{1}{\varepsilon} P : X_0 \to X_0^*$$

is bounded, continuous and pseudomonotone w.r.t. $D(L)$. In Step 1 we have seen that the obstacle function ψ is nonnegative, and thus $0 \in K$, which implies that $\langle P(u), u\rangle \geq 0$ for all $u \in X_0$, and by the coercivity assumption on $A+F : X_0 \to X_0^*$ it follows that $T : X_0 \to X_0^*$ is coercive for any $\varepsilon > 0$. Applying Theorem 3.4, for each $\varepsilon > 0$ there exist solutions u_ε of the penalty Eq. (3.2). Let (ε_n) with $\varepsilon_n \searrow 0$ and select an associated sequence of penalty solutions $(u_{\varepsilon_n}) := (u_n)$, that is,

$$u_n \in D(L): \quad u_{nt} + Au_n + F(u_n) + \frac{1}{\varepsilon_n} P(u_n) = 0 \quad \text{in } X_0^*. \tag{3.4}$$

Testing (3.4) with $\varphi = u_n$ we get

$$\langle Au_n + F(u_n), u_n\rangle = -\frac{1}{\varepsilon_n}\langle P(u_n), u_n\rangle - \langle u_{nt}, u_n\rangle \leq 0.$$

Hence it follows

$$\frac{1}{\|u_n\|_{X_0}}\langle Au_n + F(u_n), u_n\rangle \leq 0,$$

which in view of the coercivity assumption on $A+F$ implies that $(\|u_n\|_{X_0})$ is bounded, and therefore, (Au_n) and $(F(u_n))$ are bounded in X_0^*. Consider next the sequence $(\frac{1}{\varepsilon_n}P(u_n))$. By Lemma 2.8 we have

$$\langle P(u_n), (u_n-\psi)^+\rangle \geq d\,\|P(u_n)\|_{X_0^*}\|(u_n-\psi)^+\|_{p,Q} \tag{3.5}$$

with $d > 0$. Testing the penalty Eq. (3.4) with $\varphi = (u_n-\psi)^+$ we obtain

$$\langle u_{nt} + Au_n, (u_n-\psi)^+\rangle + \left\langle F(u_n) + \frac{1}{\varepsilon_n}P(u_n), (u_n-\psi)^+\right\rangle = 0. \tag{3.6}$$

With Lemma 2.9 and

$$|\langle F(u_n), (u_n-\psi)^+\rangle| \leq c\|(u_n-\psi)^+\|_{p,Q},$$

from (3.5) and (3.6) we obtain

$$\frac{d}{\varepsilon_n}\|P(u_n)\|_{X_0^*}\|(u_n-\psi)^+\|_{p,Q} \leq c\|(u_n-\psi)^+\|_{p,Q}.$$

Hence

$$\frac{1}{\varepsilon_n}\|P(u_n)\|_{X_0^*} \leq \frac{c}{d} \quad \text{for all } \varepsilon_n, \tag{3.7}$$

that is, the sequence $(\frac{1}{\varepsilon_n}P(u_n))$ is bounded in X_0^*. From the penalty equation we have

$$u_{nt} = -\left(Au_n + F(u_n) + \frac{1}{\varepsilon_n}P(u_n)\right),$$

which shows that the sequence (u_{nt}) is bounded in X_0^*, which together with the boundedness of (u_n) in X_0 yields that (u_n) is bounded in Y_0. Hence there exists a subsequence (again denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{in } X_0 \quad \text{and} \quad u_{nt} \rightharpoonup u_t \quad \text{in } X_0^* \tag{3.8}$$

as $n \to \infty$ and $\varepsilon_n \searrow 0$. Since $D(L)$ is closed in Y_0 and convex, it is weakly closed, and therefore $u \in D(L)$.

Step 3: u is a solution of the obstacle problem.

We are going to show that the weak limit u is in fact a solution of the parabolic double phase obstacle problem. To this end let us first show that $P(u) = 0$, and thus $u \in K$. From (3.7) we see that $P(u_n) \to 0$ in X_0^*. Since $P : X_0 \to X_0^*$ is monotone, we get $\langle P(v) - P(u_n), v - u_n\rangle \geq 0$ for all $v \in X_0$ and for all n, which by passing to the limit as $n \to \infty$ yields

$$\langle P(v), v-u\rangle \geq 0 \quad \text{for all } v \in X_0.$$

In particular, the last inequality holds for $v = u + \delta\varphi$ for any $\delta > 0$ and $\varphi \in X_0$, which results in

$$\langle P(u+\delta\varphi), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X_0.$$

Passing to the limit as $\delta \searrow 0$ we get

$$\langle P(u), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X_0,$$

which implies $P(u) = 0$, that is, $u \in K$.

Testing the penalty equation with $\varphi = u_n - u$ and using $\langle u_{nt} - u_t, u_n - u\rangle \geq 0$ one gets

$$\langle Au_n, u_n - u\rangle \leq -\langle u_t, u_n - u\rangle - \left\langle F(u_n) + \frac{1}{\varepsilon_n}P(u_n), u_n - u\right\rangle. \tag{3.9}$$

With (3.8) and the compact embedding $Y_0 \hookrightarrow\hookrightarrow L^p(Q)$ it follows that $u_n \to u$ in $L^p(Q)$, which yields by passing to the lim sup in (3.9)

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0. \tag{3.10}$$

From Proposition 2.3 (ii) in conjunction with Theorem 3.3 (ii) we conclude that $A : X_0 \to X_0^*$ has the (S_+)-property w.r.t. $D(L)$ (note that $\hat{A}(t) \equiv A$). Hence, from (3.8) and (3.10) it follows that $u_n \to u$ in X_0.

Let $v \in K$ be arbitrarily given. Testing the penalty equation with $\varphi = v - u_n$ we get

$$\langle u_{nt}, v - u_n \rangle + \langle A u_n + F(u_n), v - u_n \rangle = \frac{1}{\varepsilon_n} \langle P(v) - P(u_n), v - u_n \rangle \geq 0$$

for all n, which in view of $u_n \to u$ in X_0 and $u_{nt} \rightharpoonup u_t$ and passing to the limit yields

$$\langle u_t, v - u \rangle + \langle A u + F(u), v - u \rangle \geq 0 \quad \text{for all } v \in K,$$

that is, u is a solution of the parabolic double phase obstacle problem. □

Remark 3.5. The coercivity condition (1.4) in Theorem 1.2 is satisfied if either the coefficient $\beta > 0$ in the growth condition (1.3) on f is small enough, or else if condition (1.3) is replaced by the following one

$$|f(x,t,s)| \leq \alpha(x,t) + \beta |s|^{r-1} \quad \text{with } 1 < r < p, \tag{3.11}$$

for a.a. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, where $\alpha \in L^{p'}(Q)$ and $\beta > 0$ an arbitrary constant. To verify these claims, consider first $\langle Au, u \rangle$. In view of Proposition 2.1 (iii) and Proposition 2.2 (iii), we have the estimate for $\|u\|_{V_0}$ large

$$\|u\|_{V_0}^p = \|\nabla u\|_{\mathcal{H}}^p \leq \rho_{\mathcal{H}}(\nabla u) = \int_\Omega \left(|\nabla u|^p + \mu(x) |\nabla u|^q \right) \mathrm{d}x. \tag{3.12}$$

With (3.12) we estimate

$$\begin{aligned}
\langle Au, u \rangle &= \int_Q \left(|\nabla u|^p + \mu(x) |\nabla u|^q \right) \mathrm{d}x \mathrm{d}t \\
&= \int_0^\tau \left(\int_\Omega \left(|\nabla u(x,t)|^p + \mu(x) |\nabla u(x,t)|^q \right) \mathrm{d}x \right) \mathrm{d}t \\
&= \int_0^\tau \rho_{\mathcal{H}}(\nabla u(\cdot,t)) \, \mathrm{d}t \geq \int_0^\tau \|u(\cdot,t)\|_{V_0}^p \, \mathrm{d}t \geq \|u\|_{X_0}^p.
\end{aligned}$$

With the growth condition (1.3) we get

$$|\langle F(u), u \rangle| \leq \|\alpha\|_{p',Q} \|u\|_{p,Q} + \beta \|u\|_{p,Q}^p,$$

and thus due to $\|u\|_{p,Q}^p \leq c \|u\|_{X_0}^p$, one gets

$$|\langle F(u), u \rangle| \leq \|\alpha\|_{p',Q} \|u\|_{p,Q} + c\beta \|u\|_{X_0}^p.$$

Hence for $\|u\|_{X_0}$ large we obtain

$$\frac{1}{\|u\|_{X_0}} \langle Au + F(u), u \rangle \geq \|u\|_{X_0}^{p-1} - c\beta \|u\|_{X_0}^{p-1} - c \|\alpha\|_{p',Q},$$

which for $\beta > 0$ small such that $c\beta < 1$ implies coercivity.

In case (3.11) is assumed we have the estimate for any $\varepsilon > 0$

$$\begin{aligned}
|\langle F(u), u \rangle| &\leq \int_Q |f(\cdot,\cdot,u)| \, |u| \, \mathrm{d}x\mathrm{d}t \leq \int_Q \left(\alpha(x,t) + \beta |u|^{r-1} \right) |u| \, \mathrm{d}x\mathrm{d}t \\
&\leq \|\alpha\|_{p',Q} \|u\|_{p,Q} + \beta \int_Q |u|^r \, \mathrm{d}x\mathrm{d}t \\
&\leq \|\alpha\|_{p',Q} \|u\|_{p,Q} + \beta \int_Q \left(c(\varepsilon) + \varepsilon |u|^p \right) \mathrm{d}x\mathrm{d}t \\
&\leq \|\alpha\|_{p',Q} \|u\|_{p,Q} + \beta c(\varepsilon) |Q| + \beta \varepsilon c \|u\|_{X_0}^p.
\end{aligned}$$

By choosing ε small enough such that $\varepsilon < \frac{1}{c\beta}$, we readily see that

$$\frac{1}{\|u\|_{X_0}} \langle Au + F(u), u \rangle \to \infty \quad \text{as } \|u\|_{X_0} \to \infty,$$

which shows the coercivity in this case.

Remark 3.6. In case that neither the coercivity condition (1.4) nor the growth condition (1.3) is fulfilled, existence results for the obstacle problem (1.1) can be proved provided sub-supersolutions for the problem exist, which we are going to introduce in the following section.

4. Sub-supersolution

Let us introduce the notation $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$. The sub- and supersolutions of the obstacle problem (1.1) (resp. (1.2)) are defined as follows, see Carl-Le [42, Chap. 5].

Definition 4.1. A function $\underline{u} \in Y$ is called a subsolution of (1.1) if $F(\underline{u}) \in L^{p'}(Q)$ such that the following holds:

(i) $\underline{u} \vee K \subset K$, $\underline{u}(\cdot, 0) \leq 0$ in Ω;

(ii) $\langle \underline{u}_t + A\underline{u}, v - \underline{u} \rangle + \int_Q F(\underline{u})(v - \underline{u})\,\mathrm{d}x\mathrm{d}t \geq 0$ for all $v \in \underline{u} \wedge K$.

We have a similar definition for supersolutions of (1.1).

Definition 4.2. A function $\overline{u} \in Y$ is called a supersolution of (1.1) if $F(\overline{u}) \in L^{p'}(Q)$ such that the following holds:

(i) $\overline{u} \wedge K \subset K$, $\overline{u}(\cdot, 0) \geq 0$ in Ω;

(ii) $\langle \overline{u}_t + A\overline{u}, v - \overline{u} \rangle + \int_Q F(\overline{u})(v - \overline{u})\,\mathrm{d}x\mathrm{d}t \geq 0$ for all $v \in \overline{u} \vee K$.

The following lemmas provide sufficient conditions for sub- and supersolutions of the obstacle problem (1.1) (resp. (1.2)).

Lemma 4.3. *If $u \in Y$ satisfies $u(\cdot, 0) \leq 0$ in Ω, $u \leq 0$ on Σ, $F(u) \in L^{p'}(Q)$ and*

$$\langle u_t + Au + F(u), \varphi \rangle \leq 0 \quad \text{for all } \varphi \in X_0 \text{ with } \varphi \geq 0,$$

then u is a subsolution of the obstacle problem (1.1) in the sense of Definition 4.1 provided $u \leq \psi$.

Proof. We recall $K = \{v \in X_0 \,:\, v \leq \psi \text{ in } Q\}$. Clearly, u fulfills (i) of Definition 4.1. In order to check (ii), we note that $v \in u \wedge K$ has the representation $v = u - (u - \varphi)^+$ for any $\varphi \in K$, which yields

$$\begin{aligned}
&\langle u_t + Au, v - u \rangle + \int_Q F(u)(v - u)\,\mathrm{d}x\mathrm{d}t \\
&= -\left\langle u_t + Au, (u - \varphi)^+ \right\rangle - \int_Q F(u)(u - \varphi)^+\,\mathrm{d}x\mathrm{d}t \geq 0,
\end{aligned}$$

since $(u - \varphi)^+ \in X_0$ and $(u - \varphi)^+ \geq 0$. $\square$

Lemma 4.4. *If $u \in Y$ satisfies $u(\cdot, 0) \geq 0$ in Ω, $u \geq 0$ on Σ, $F(u) \in L^{p'}(Q)$ and*

$$\langle u_t + Au + F(u), \varphi \rangle \geq 0 \quad \text{for all } \varphi \in X_0 \text{ with } \varphi \geq 0,$$

then u is a supersolution of the obstacle problem (1.1) in the sense of Definition 4.2.

Proof. Clearly, u satisfies (i) of Definition 4.2. Let us prove (ii) of Definition 4.2. First, note that $v \in u \vee K$ has the representation $v = u + (\varphi - u)^+$ for any $\varphi \in K$. Hence we get

$$\begin{aligned}
&\langle u_t + Au, v - u \rangle + \int_Q F(u)(v - u)\,\mathrm{d}x\mathrm{d}t \\
&= \langle u_t + Au, (\varphi - u)^+ \rangle + \int_Q F(u)(\varphi - u)^+\,\mathrm{d}x\mathrm{d}t \geq 0,
\end{aligned}$$

because $(\varphi - u)^+ \in X_0$ and $(\varphi - u)^+ \geq 0$. $\square$

By means of an ordered pair of sub-and supersolutions we are going to prove an existence result for the obstacle problem (1.1) (resp. (1.2)) without requiring the coercivity assumption (1.4), but instead replacing assumption (H3) by the following local boundedness on f.

(H4) Assume an ordered pair of sub-and supersolutions $\underline{u} \leq \overline{u}$ of the obstacle problem (1.1) in the sense of Definitions 4.1 and 4.2. The Carathéodory function $f : Q \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy the following bound with respect to the ordered interval $[\underline{u}, \overline{u}]$

$$|f(x, t, s)| \leq k(x, t)$$

for a.a. $(x, t) \in Q$ and for all $s \in [\underline{u}(x, t), \overline{u}(x, t)]$, where $k \in L^{p'}(Q)$.

Based on the existence of an ordered pair of sub- and supersolution we have the following result.

Theorem 4.5. *Assume hypotheses* (H1), (H2) *and* (H4). *Then the parabolic obstacle problem (1.1) (resp. (1.2)) admits at least one solution $u \in Y_0$ with $\underline{u} \leq u \leq \overline{u}$.*

Proof. Consider the cut-off function $b : Q \times \mathbb{R} \to \mathbb{R}$ given by

$$b(x, t, s) = \begin{cases} [s - \overline{u}(x, t)]^{p-1} & \text{if } s > \overline{u}(x, t) \\ 0 & \text{if } \underline{u}(x, t) \leq s \leq \overline{u}(x, t) \\ -[\underline{u}(x, t) - s]^{p-1} & \text{if } s < \underline{u}(x, t), \end{cases}$$

for a.a. $(x,t) \in Q$ and for all $s \in \mathbb{R}$. Obviously, b is a Carathéodory function and satisfies the following growth condition

$$|b(x,t,s)| \le c_1(x,t) + c_2|s|^{p-1} \quad \text{for a.a. } (x,t) \in Q, \text{ for all } s \in \mathbb{R},$$

with $c_1 \in L^{p'}(Q)$ and $c_2 > 0$. Therefore, the Nemytskij operator $B : u \mapsto b(\cdot,\cdot,u)$ is a continuous and bounded mapping from $L^p(Q)$ to $L^{p'}(Q)$ and the composed operator $i^* \circ B \circ i : X_0 \to X_0^*$ again denoted by B given by

$$\langle B(u), v\rangle = \int_Q b(\cdot,\cdot,u)\, v \,\mathrm{d}x\mathrm{d}t \quad \text{for all } u, v \in X_0$$

is pseudomonotone w.r.t. $D(L)$ due to the compact embedding $Y_0 \hookrightarrow\hookrightarrow L^p(Q)$. Moreover, there are constants $c_3, c_4 > 0$ such that

$$\int_Q b(\cdot,\cdot,u)u \,\mathrm{d}x\mathrm{d}t \ge c_3\|u\|_{p,Q}^p - c_4 \quad \text{for all } u \in L^p(Q). \tag{4.1}$$

Next, we introduce the cut-off function $f_0 : Q \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(x,t,s) = \begin{cases} f(x,t,\underline{u}(x,t)) & \text{if } s < \underline{u}(x,t), \\ f(x,t,s) & \text{if } \underline{u}(x,t) \le s \le \overline{u}(x,t), \\ f(x,t,\overline{u}(x,t)) & \text{if } s > \overline{u}(x,t), \end{cases}$$

which is a Carathéodory function that in view of (H4) satisfies the growth condition

$$|f_0(x,t,s)| \le k(x,t) \quad \text{for a.a. } (x,t) \in Q \text{ and for all } s \in \mathbb{R}. \tag{4.2}$$

Thus its Nemytskij operator $F_0 : L^p(Q) \to L^{p'}(Q)$ is bounded and continuous, and the composed operator $i^* \circ F_0 \circ i : X_0 \to X_0^*$ given by

$$\langle F_0(u), v\rangle = \int_Q f_0(\cdot,\cdot,u)\, v \,\mathrm{d}x\mathrm{d}t \quad \text{for all } u, v \in X_0$$

is pseudomonotone w.r.t. $D(L)$ due to the compact embedding $Y_0 \hookrightarrow\hookrightarrow L^p(Q)$. Hence the operator $\hat{F} := B + F_0 : X_0 \to X_0^*$ is bounded, continuous and pseudomonotone w.r.t. $D(L)$. Consider the auxiliary parabolic obstacle problem: Find $u \in D(L) \cap K$ such that

$$\langle u_t + Au + \hat{F}(u), v - u\rangle \ge 0 \quad \text{for all } v \in K. \tag{4.3}$$

By means of (4.1) and (4.2) we get

$$\langle \hat{F}(u), u\rangle \ge c_3\|u\|_{p,Q}^p - c_4 - \|k\|_{p',Q}\|u\|_{p,Q},$$

which implies that $A + \hat{F} : X_0 \to X_0^*$ satisfies the coercivity condition

$$\frac{1}{\|u\|_{X_0}}\langle Au + \hat{F}(u), u\rangle \to \infty \quad \text{as } \|u\|_{X_0} \to \infty.$$

Thus, by applying Theorem 1.2 with F replaced by $\hat{F}$, there exist solutions of the auxiliary obstacle problem (4.3). Therefore, the proof of Theorem 4.5 is complete provided any solution u of the auxiliary obstacle problem (4.3) satisfies $\underline{u} \le u \le \overline{u}$, because then $B(u) = 0$ and $\hat{F}(u) = F(u)$.

Let us verify that $\underline{u} \le u$. Since $u \in K$, it follows that

$$u + (\underline{u} - u)^+ = \underline{u} \vee u \in K.$$

Using $v = u + (\underline{u} - u)^+$ in (4.3), one obtains

$$\left\langle u_t, (\underline{u} - u)^+\right\rangle + \left\langle Au + B(u) + F_0(u), (\underline{u} - u)^+\right\rangle \ge 0. \tag{4.4}$$

As $\underline{u}$ is a subsolution we get with $v = \underline{u} \wedge u \in \underline{u} \wedge K$ in Definition 4.1 (ii) the following inequality

$$-\left\langle \underline{u}_t + A\underline{u}, (\underline{u} - u)^+\right\rangle - \int_Q F(\underline{u})(\underline{u} - u)^+ \,\mathrm{d}x\mathrm{d}t \ge 0. \tag{4.5}$$

where we have used that $\underline{u} \wedge u = \underline{u} - (\underline{u} - u)^+$. Adding inequalities (4.4) and (4.5) results in

$$\begin{aligned} &\left\langle B(u), (\underline{u} - u)^+\right\rangle + \int_Q \left(F_0(u) - F(\underline{u})\right)(\underline{u} - u)^+ \,\mathrm{d}x\mathrm{d}t \\ &\ge \left\langle \underline{u}_t - u_t, (\underline{u} - u)^+\right\rangle + \left\langle A\underline{u} - Au, (\underline{u} - u)^+\right\rangle. \end{aligned} \tag{4.6}$$

The right-hand side of (4.6) is nonnegative, and by the definition of f_0 it follows that

$$\int_Q \left(F_0(u) - F(\underline{u})\right)(\underline{u} - u)^+ \,\mathrm{d}x\mathrm{d}t = 0.$$

Thus we obtain from (4.6) and taking into account the definition of b

$$0 \le \langle B(u), (\underline{u} - u)^+\rangle = -\int_Q [(\underline{u} - u)^+]^p \,\mathrm{d}x\mathrm{d}t \le 0,$$

which implies $(\underline{u} - u)^+ = 0$, that is, $\underline{u} \le u$. The inequality $u \le \overline{u}$ can be proved in a similar way, which completes the proof of the theorem. □

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