

Multiple bound states for a class of fractional critical Schrödinger–Poisson systems with critical frequency

Cite as: *J. Math. Phys.* **65**, 071502 (2024); doi: [10.1063/5.0174872](https://doi.org/10.1063/5.0174872)

Submitted: 3 September 2023 • Accepted: 2 June 2024 •

Published Online: 1 July 2024



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ABSTRACT

In this paper we study the fractional Schrödinger–Poisson system $\varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u$, $\varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}$, $x \in \mathbb{R}^3$, where $s \in (0, 1)$, $\varepsilon > 0$ is a small parameter, $2_s^* = \frac{6}{3-2s}$ is the critical Sobolev exponent and $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$ is a nonnegative function which may be zero in some regions of \mathbb{R}^3 , e.g., it is of the critical frequency case. By virtue of a new global compactness lemma, and the Lusternik–Schnirelmann category theory, we relate the number of bound state solutions with the topology of the zero set where V attains its minimum for small values of ε .

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I. INTRODUCTION AND MAIN RESULTS

This paper deals with the following fractional Schrödinger–Poisson system with doubly critical growth and critical frequency of the form

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$ and $2_s^* = \frac{6}{3-2s}$ denotes the critical Sobolev exponent. Here, the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s w(x) = C_s \text{P.V.} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dy, \quad w \in \mathcal{S}(\mathbb{R}^3),$$

where P.V. stands for the Cauchy principal value, C_s is a normalizing constant and $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying functions. In (1.1), the first equation is a nonlinear fractional Schrödinger equation introduced by Laskin¹ which appeared in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes, see Laskin,² in which the potential satisfies a nonlinear fractional Poisson equation. For this reason, system (1.1) is called a fractional Schrödinger–Poisson system, also known as the fractional Schrödinger–Maxwell system, as a model describing solitary waves for the nonlinear stationary fractional Schrödinger equation interacting with the electrostatic field, which arises in many mathematical physics context and which is not only a physically relevant generalization of the classical nonlinear Schrödinger (NLS) but also an important model in the study of fractional quantum mechanics. It is worth

pointing out that the fractional Laplace operator $(-\Delta)^s$ becomes the classical Laplace operator $-\Delta$ as $s \rightarrow 1^{-1}$, see Proposition 4.4 of Di Nezza–Palatucci–Valdinoci.³ From a probabilistic point of view, the fractional Laplace operator could be viewed as the infinitesimal generator of a Lévy process, see, for example, Applebaum⁴ and Bertoin.⁵ This operator arises in the description of various phenomena in applied sciences, such as plasma physics (see Kurihura⁶), flame propagation (see Caffarelli–Roquejoffre–Sire⁷), financial (see Cont–Tankov⁸), anomalous diffusion (see Metzler–Klafter⁹), obstacle problems (see Silvestre¹⁰), conformal geometry, and minimal surfaces (see Chang–González¹¹). For more details on fractional Laplacian operators we refer the readers to Di Nezza–Palatucci–Valdinoci,³ Molica Bisci–Rădulescu–Servadei¹² and the references therein.

Note that, even if $s = 1$, (1.1) reduces to the following classical Schrödinger–Poisson system with nonlocal critical term

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u - \phi|u|^3 u = f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

which was of interest to a number of authors. Indeed it can be used as a model to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. We refer to Li–Li–Shi^{13,14} for more details in the physics aspects. In 2020, Feng¹⁵ considered (1.2) with $f \in C(\mathbb{R}, \mathbb{R})$ being subcritical growth and $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfying the global condition

$$(V) \quad V_\infty = \liminf_{|x| \rightarrow \infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^3} V(x) > 0.$$

This condition was first introduced by Rabinowitz.¹⁶ By employing the concentration-compactness principle and the mountain pass theorem, Feng¹⁵ proved the existence of positive ground state solutions under condition (V) and under some appropriate subcritical growth conditions on the nonlinearity f . Almost at the same time, Feng¹⁷ revisited problem (1.2) (with $\varepsilon = 1$), when the potential V is neither coercive nor periodic and asymptotic periodic. Based on the modified concentration-compactness principle and the mountain pass theorem the existence of ground state solution has been shown.

There are also some papers concerning the Schrödinger–Poisson system with nonlocal critical term, but without the Sobolev critical term $|u|^4 u$. Li–Li–Shi^{13,14} studied the existence of positive solutions to the following Schrödinger–Poisson-type system with nonlocal critical term

$$\begin{cases} -\Delta u + bu + q\phi|u|^3 u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \quad (1.3)$$

whereby the proofs are using variational methods which do not require any compactness conditions. Moreover, Guo¹⁸ obtained two positive solutions of (1.3) if $b = 0$, $q = -1$ and f is a sublinear term of the form $\lambda Q(x)|u|^{p-2}u$ by applying the Nehari manifold and variational methods. In Ref. 19, Liu investigated the following Schrödinger–Poisson-type system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^5, & x \in \mathbb{R}^3, \end{cases} \quad (1.4)$$

and obtained the existence of positive solutions by using the mountain pass theorem and the concentration-compactness principle. Furthermore, Li–He²⁰ studied the existence and multiplicity of semiclassical state solutions for (1.4) by using variational methods along with the Lusternik–Schnirelmann category theory.

From another perspective, systems (1.2)–(1.4) can be rewritten as the following Choquard-type equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{-\alpha} (I_\alpha * F(u))f(u) + h(u), \quad x \in \mathbb{R}^N, \quad (1.5)$$

which was introduced as an approximation to the Hartree–Fock theory of one component plasma, where I_α stands for the Riesz potential. Such a class of equations was already proposed by Pekar in 1954 in Ref. 21 to study the quantum theory of a polaron at rest. Later, by using symmetry decreasing rearrangement inequalities, Lieb²² investigated the existence and uniqueness for a class of nonlinear Choquard equation. Recently, Cassani–Zhang²³ showed the existence and concentration behavior of ground state solutions for Eq. (1.5). Under appropriate conditions on α and f , Alves–Yang²⁴ presented a new concentration behavior of solutions for (1.5) through variational methods. By means of a new nonlocal penalization technique, Moroz–Van Schaftingen²⁵ proved that (1.5) admits a family of solutions concentrating to the local minimum of V .

Note that in the aforementioned Schrödinger–Poisson systems in (1.2)–(1.4), the second equation has a nonlocal critical growth in the sense of the Hardy–Littlewood–Sobolev inequality. On the other hand, the following classical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.6)$$

with the Poisson equation being square growth, has been studied extensively by many authors over the past decades. As a model describing the interaction of a charge particle with an electromagnetic field, it arises in many mathematical physics context, see, for example,

Benci–Fortunato.²⁶ The existence and multiplicity of solutions for the Schrödinger–Poisson system (1.6) under variant assumptions on V , K and f has been widely investigated by numerous authors, which have been developed many effective methods to study equations or systems with nonlocal terms. We refer the readers to works of Ambrosetti,²⁷ Ambrosetti–Ruiz,²⁸ Azzollini–d’Avenia–Vaira,²⁹ Cerami–Vaira,³⁰ Ruiz,³¹ Wang–Chen–Liao,³² Zhang–Ma–Xie³³ and the references cited therein.

For the case $s \in (0, 1)$, only few recent papers considered the fractional Schrödinger–Poisson system like (1.1). It is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes, see Caffarelli–Silvestre³⁴ and Laskin.^{1,2} Zhang–do Ó–Squassina³⁵ considered, by using a perturbation approach, the existence and the asymptotical behaviors of positive solutions to the fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \quad (1.7)$$

with $V(x) = 0$, a parameter $K(x) = \lambda > 0$, and a general subcritical or critical nonlinearity f . Teng³⁶ studied the existence of ground state solutions of (1.7) with $K(x) = 1$ and $f(x, u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u$, $q \in (2, 2_s^*)$, by using the Pohozaev–Nehari manifold, the monotonic trick and the global compactness lemma. For further results on the existence and multiplicity of semiclassical bound states of (1.7), we refer to Liu–Zhang,³⁷ Murcia–Siciliano³⁸ and Yang–Yu–Zhao,³⁹ see also the papers of Ambrosio,⁴⁰ Chen–Li–Peng,⁴¹ Ghimenti–Van Schaftingen,⁴² Guo–Li,⁴³ Zhang–Zhang⁴⁴ and the references therein.

Recently, Qu–He⁴⁵ studied the following fractional Schrödinger–Poisson system with doubly critical growth

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u + f(u), & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \quad (1.8)$$

and showed the existence and multiplicity of concentrating solutions by using the Nehari manifold and the Lusternik–Schnirelmann category theory. In 2021, He⁴⁶ obtained a mountain pass solution to (1.8) (if $\varepsilon = 1$) with the help of the concentration-compactness principle. Feng–Yeng⁴⁷ considered a class of fractional Schrödinger–Poisson type systems with doubly critical terms

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi|u|^{2_s^*-3}u = K(x)|u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \quad (1.9)$$

where $s \in (\frac{3}{4}, 1)$, $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$, $K \in L^\infty(\mathbb{R}^3)$ and $\inf_{x \in \mathbb{R}^3} K(x) = K_0 > 0$. By applying the concentration-compactness principle and variational methods, the authors derived the existence of one ground state solution to the system (1.9). We remark that the novelty of (1.8) and (1.9) is the fact that the second equation has nonlocal critical growth and is driven by the nonlocal operator $(-\Delta)^s$, which makes the study of problem (1.8) and (1.9) more interesting and challenging. As we already observed, the previous results on the existence and multiplicity of solutions for systems (1.8) and (1.9) were mainly focused on the existence of ground state solutions under the condition (V). A natural question arises spontaneously is that, whether or not it admits multiple high energy solutions for (1.9) with critical frequency, that is, $V(\cdot) \geq 0$, $V \not\equiv 0$ on \mathbb{R}^3 , and doubly critical growth. This is still an open problem and the purpose of this paper is to fill this gap. To be more precise, we would like to establish the existence and multiplicity of bound state solutions to (1.1) when V is of critical frequency case.

Throughout this paper, we suppose that V satisfies the following assumptions:

- (V₁) $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$ and $V(\cdot) \geq 0$ on \mathbb{R}^3 ;
- (V₂) the set $M = \{x \in \mathbb{R}^3 : V(x) = 0\}$ is nonempty and bounded.

Based on the assumptions on $V(\cdot)$ above, it is easy to verify that the following equation can be seen as the limit equation of (1.1):

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \quad (1.10)$$

In the sequel, for each $\varepsilon > 0$, we define the energy functionals associated to (1.1) and (1.10) by J_ε and $J_{\varepsilon, \infty}$, respectively,

$$J_\varepsilon(u) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \frac{\varepsilon^{-2s}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx,$$

and

$$J_{\varepsilon, \infty}(u) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\varepsilon^{-2s}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx,$$

where ϕ_u is given in (2.1). Furthermore, we introduce the Nehari manifold associated to $J_\varepsilon, J_{\varepsilon, \infty}$, respectively, as

$$\mathcal{N}_\varepsilon = \{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} : J'_\varepsilon(u)u = 0\}, \quad \mathcal{N}_\varepsilon^\infty = \{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} : J'_{\varepsilon, \infty}(u)u = 0\}.$$

For every $\varepsilon > 0$, we define the infimum

$$m_\varepsilon := \inf_{u \in N_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad m_\varepsilon^\infty := \inf_{u \in N_{\varepsilon, \infty}} J_{\varepsilon, \infty}(u).$$

If Y is a closed set of a topological space X , we denote by $\text{cat}_X(Y)$ the Lusternik–Schnirelmann category of Y in X , namely the least number of closed and contractible sets in X which cover Y . For any fixed $\mu > 0$, denote $M_\mu = \{x \in \mathbb{R}^3 : \text{dist}(x, M) \leq \mu\}$. Now, our main result can be formulated as follows.

Theorem 1.1. *Let $s \in (\frac{1}{2}, 1)$ and suppose that the conditions (V_1) , (V_2) are satisfied. Then, the following properties hold:*

- (i) $m_\varepsilon = m_\varepsilon^\infty$ and m_ε is not achieved, for each $\varepsilon > 0$.
- (ii) For any fixed $\mu > 0$, there exist $\varepsilon_\mu > 0$ such that for any $\varepsilon \in (0, \varepsilon_\mu)$, system (1.1) possesses at least $\text{cat}_{M_\mu}(M)$ bound states in $D^{s,2}(\mathbb{R}^3)$.

The Proof of Theorem 1.1 is of variational character. From a technical perspective, the characteristics of the double non-localities from nonlocal critical convolution term $\phi_u |u|^{2_s^*-3} u$ and the Sobolev critical term $|u|^{2_s^*-2} u$ in system (1.1) make it more complicated to deal with the convergence of (PS)-sequences, due to the lack of compactness for the embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$. Moreover, it is difficult to check the $(PS)_c$ -condition since the nodal solutions of (1.1) no longer has the double energy properties, as we know the energy doubling characteristic plays a key role in proving the main results, see, for example, Chen–Li–Peng,⁴¹ Qu–He⁴⁸ and Zhang–Zhang.⁴⁴ In order to overcome these difficulties, we shall employ some ideas from Qu–He⁴⁵ and establish a new global compactness lemma in the fractional case and some estimates become more subtle and delicate to be established. We shall construct two barycenter functions and use the Lusternik–Schnirelmann category theory to obtain the desired results. As far as we know, the multiplicity of high energy solutions for system (1.1) has not been studied in the literature.

The paper is organized as follows: In Sec. II we give some preliminary results while in Sec. III, we introduce the corresponding limit problem and prove some useful lemmas. In Sec. IV, we present a new global compactness lemma which describes the behavior of (PS)-sequences and we regain the compactness if the functional energy lies in a suitable interval. In Sec. V, we first define two barycenter functions and give some necessary estimates. Then, we give the Proof of Theorem 1.1 by means of the Lusternik–Schnirelmann category theory.

II. NOTATIONS AND PRELIMINARY RESULTS

To begin with, we present some notations which will be used throughout this paper. First, we denote by $C, C_i > 0, i = 1, 2, \dots$, different positive constants whose values may vary from line to line and are not essential to the proofs of our results. We denote by $L^p = L^p(\mathbb{R}^3)$ for $p \in (1, \infty]$ the Lebesgue spaces with the standard norm $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$.

We recall that the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx < \infty \right\},$$

equipped with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3-2s}} dx dy,$$

see Di Nezza–Palatucci–Valdinoci³ and Molica Bisci–Rădulescu–Servadei.¹² The Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous and the best constant S is given by

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}},$$

see Servadei–Valdinoci.⁴⁹ By the Lax–Milgram theorem, we know that for each $u \in D^{s,2}(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u \in D^{s,2}(\mathbb{R}^3)$ such that $(-\Delta)^s \phi_u = |u|^{2_s^*-1}$ in a weak sense. The function ϕ_u can be expressed explicitly as

$$\phi_u(x) = K_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x - y|^{3-2s}} dy, \quad \forall x \in \mathbb{R}^3, \tag{2.1}$$

where $K_s = \pi^{-\frac{3}{2}} 2^{-2s} \Gamma(3 - 2s) (\Gamma(s))^{-1}$, see He⁴⁶ for example. The function ϕ_u has the following characteristics, see Qu–He⁴⁵ for the proof.

Lemma 2.1. *Let $u \in D^{s,2}(\mathbb{R}^3)$, then the following hold:*

- (i) $\phi_{tu} = t^{2_s^*-1} \phi_u$ for all $t \geq 0$.

(ii) For any $u \in D^{s,2}(\mathbb{R}^3)$, we have $\|\phi_u\| \leq S^{-\frac{1}{2}} \|u\|_{2_s^*}^{2_s^*-1}$ and

$$\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \leq S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx \right)^{\frac{2_s^*-1}{2_s^*}} \|\phi_u\| \leq S^{-1} \|u\|_{2_s^*}^{2(2_s^*-1)}.$$

(iii) If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^3 , then $\phi_{u_n} \rightarrow \phi_u$ in $D^{s,2}(\mathbb{R}^3)$ and $\phi_{u_n} - \phi_{u_n-u} - \phi_u \rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$.

(iv) If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^3 , then

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - \int_{\mathbb{R}^3} \phi_{u_n-u} |u_n - u|^{2_s^*-1} dx - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \rightarrow 0,$$

and

$$\phi_{u_n} |u_n|^{2_s^*-3} u_n - \phi_{u_n-u} |u_n - u|^{2_s^*-3} (u_n - u) - \phi_u |u|^{2_s^*-3} u \rightarrow 0$$

in $(D^{s,2}(\mathbb{R}^3))^*$, where $(D^{s,2}(\mathbb{R}^3))^*$ denotes the dual space of $D^{s,2}(\mathbb{R}^3)$.

Lemma 2.2. If $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$, then the functional

$$G(u) = \int_{\mathbb{R}^3} V(x)u^2 dx$$

is weakly continuous in $D^{s,2}(\mathbb{R}^3)$.

Proof. The proof is analogous to that of Lemma 2.13 in Willem,⁵⁰ we sketch it here for convenience. First, the functional G is well defined by Hölder's inequality. Let $u_n \rightharpoonup u$ be weakly in $D^{s,2}(\mathbb{R}^3)$. By virtue of the continuity of the Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, we see that $u_n \rightharpoonup u$ weakly in $L^{2_s^*}(\mathbb{R}^3)$. Thus, $u_n^2 \rightharpoonup u^2$ weakly in $L^{\frac{3}{3-2s}}(\mathbb{R}^3)$. Notice that $V \in L^{\frac{3}{2s}}(\mathbb{R}^3) = (L^{\frac{3}{3-2s}}(\mathbb{R}^3))^*$. Consequently,

$$\int_{\mathbb{R}^3} V(x)u^2 dx \rightarrow \int_{\mathbb{R}^3} V(x)u^2 dx \quad \text{as } n \rightarrow \infty,$$

which completes the proof. □

At the end of this paragraph, we introduce the following version of the Hardy–Littlewood–Sobolev inequality, see Lieb–Loss.⁵¹

Proposition 2.3. Assume that $t, r > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(t, N, \alpha, r)$ independent of f, h satisfying

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\alpha} dx dy \leq C(t, N, \alpha, r) \|f\|_t \|h\|_r. \tag{2.2}$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\frac{N}{2}} \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(N-\frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-1 + \frac{\alpha}{N}}.$$

In this case there is equality in (2.2) if and only if $f \equiv Ch$ and

$$h(x) = A(\delta^2 + |x - x_0|^2)^{-\frac{2N-\alpha}{2}}$$

for some $A \in \mathbb{C}$, $\delta \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}^N$.

III. LIMIT PROBLEM

From the assumptions (V₁) and (V₂), the limit equation of (1.1) takes the form as

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u - \phi |u|^{2_s^*-3} u = |u|^{2_s^*-2} u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

Recall that for each $\varepsilon > 0$, we have the Nehari manifolds

$$\mathcal{N}_\varepsilon = \{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} : J'_\varepsilon(u)u = 0\}, \quad \mathcal{N}_\varepsilon^\infty = \{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} : J'_{\varepsilon,\infty}(u)u = 0\},$$

and for each $\varepsilon > 0$, we define the infimums

$$m_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad m_\varepsilon^\infty := \inf_{u \in \mathcal{N}_{\varepsilon, \infty}} J_{\varepsilon, \infty}(u).$$

Now, we study the correspondence between the ground state solutions of (3.1) with $\varepsilon = 1$, i.e.,

$$\begin{cases} (-\Delta)^s u - \phi |u|^{2_s^* - 3} u = |u|^{2_s^* - 2} u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^* - 1}, & x \in \mathbb{R}^3, \end{cases} \quad (3.2)$$

and the ground state solutions to the following equation

$$(-\Delta)^s u = |u|^{2_s^* - 2} u, \quad x \in \mathbb{R}^3. \quad (3.3)$$

The following observation is useful in the energy estimation of the corresponding functionals.

Lemma 3.1.

- (i) The ground state solutions of problem (3.2) and problem (3.3) are one-to-one correspondences.
- (ii) Assume that u, ϕ are positive solutions of (3.2), then we have

$$u(x) = \phi(x) = \tilde{U}_{\delta, x_0}(x) := \kappa U_{\delta, x_0}(x) \quad \text{for some } x_0 \in \mathbb{R}^3 \text{ and } \delta > 0,$$

where the function

$$U_{\delta, x_0}(x) := \frac{d_s \delta^{\frac{3-2s}{2}}}{(\delta^2 + |x - x_0|^2)^{\frac{3-2s}{2}}}$$

solves (3.3) with the numbers

$$d_s = \left(\frac{S^{3/(2s)} \Gamma(3)}{\pi^{3/2} \Gamma(3/2)} \right)^{\frac{3-2s}{6}}, \quad \kappa = \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{4s}}.$$

Moreover, U_{δ, x_0} satisfies

$$\|U_{\delta, x_0}\|_{D^{s,2}}^2 = \|U_{\delta, x_0}\|_{2_s^*}^{2_s^*} = S^{\frac{3}{2s}}.$$

Proof.

- (i) Assume that $w_1(x)$ is a solution of problem (3.3), then

$$w_2(x) = D_{w_1} w_1(x),$$

where D_{w_1} is given by

$$D_{w_1}^{2_s^* - 2} = \frac{1}{2} \left(1 + \sqrt{1 + 4 \frac{\int_{\mathbb{R}^3} \phi w_1 |w_1|^{2_s^* - 1} dx}{\int_{\mathbb{R}^3} |w_1|^{2_s^*} dx}} \right) \quad (3.4)$$

solving problem (3.2). For any solutions $w_{11}(x)$ and $w_{12}(x)$ of problem (3.3), if $D_{w_{11}} w_{11}(x) = D_{w_{12}} w_{12}(x)$ holds, then we infer to

$$w_{11}(x) = \frac{D_{w_{12}}}{D_{w_{11}}} w_{12}(x).$$

Since both $w_{11}(x)$ and $w_{12}(x)$ are solutions of problem (3.3), we have $D_{w_{12}}/D_{w_{11}} = 1$ and hence $w_{11}(x) = w_{12}(x)$. On the other hand, assume that $w_2(x)$ is a solution of problem (3.2), then

$$w_1(x) = T_{w_2} w_2(x)$$

solves problem (3.3), where $T_{w_2} > 0$ fulfills

$$T_{w_2}^{2_s^* - 2} = \frac{\int_{\mathbb{R}^3} \phi w_2 |w_2|^{2_s^* - 1} dx + \int_{\mathbb{R}^3} |w_2|^{2_s^*} dx}{\int_{\mathbb{R}^3} |w_2|^{2_s^*} dx}.$$

From the observations above, we infer that $w_1(x) \xrightarrow{D_{w_1}} w_2(x) \xrightarrow{T_{w_2}} w_1(x)$.

(ii) Now, we introduce the following equation

$$\begin{cases} (-\Delta)^s u = \phi |u|^{2_s^*-3} u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \quad (3.5)$$

Assume that u, ϕ are positive solutions for (3.5), then

$$(-\Delta)^s (u - \phi) = (\phi - u) |u|^{2_s^*-2}, \quad x \in \mathbb{R}^3.$$

Multiplying both sides of this equation by $(u - \phi)$ and integrating by part gives

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} (u - \phi)|^2 dx + \int_{\mathbb{R}^3} (u - \phi)^2 |u|^{2_s^*-2} dx = 0.$$

Hence we can conclude that $u = \phi = U_{\delta, x_0}$ and so, (3.3) is equivalent to (3.5). Thus, the function U_{δ, x_0} satisfies

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} U_{\delta, x_0}|^2 dx = \int_{\mathbb{R}^3} \phi U_{\delta, x_0} |U_{\delta, x_0}|^{2_s^*-1} dx = \int_{\mathbb{R}^3} |U_{\delta, x_0}|^{2_s^*} dx = S^{\frac{3}{2s}}, \quad (3.6)$$

taking the form

$$U_{\delta, x_0}(x) := \frac{d_s \delta^{\frac{3-2s}{2}}}{(\delta^2 + |x - x_0|^2)^{\frac{3-2s}{2}}}.$$

Now, testing (3.4) by $w_1(x) = U_{\delta, x_0}(x)$, by a direct calculation, we obtain

$$D_{w_1} = \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{4s}} := \kappa. \quad (3.7)$$

Therefore, the ground state solution of (3.2) has the form

$$w_2(x) := \tilde{U}_{\delta, x_0}(x) := \kappa U_{\delta, x_0}(x) \quad \text{for some } x_0 \in \mathbb{R}^3 \text{ and } \delta > 0.$$

The assertions follows. □

Remark 3.2. From (3.6) and (3.7), by a simple computation, we have that

$$\|\tilde{U}_{\delta, x_0}\|^2 = \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}$$

and $U_{\delta, x_0}^*(x) := \varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta, x_0}(x)$ is the unique ground state solution of Eq. (3.1).

Lemma 3.3. The infimum m_ε^∞ is achieved by $U_{\delta, x_0}^*(x) = \varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta, x_0}(x)$ with

$$m_\varepsilon^\infty = J_{\varepsilon, \infty}(U_{\delta, x_0}^*) = \varepsilon^3 \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s[12 + (1 - \sqrt{5})(3 - 2s)]}{6(3 + 2s)} S^{\frac{3}{2s}}.$$

Proof. By Lemma 3.1 and Remark 3.2, we see that

$$U_{\delta, x_0}^*(x) = \varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta, x_0}(x) \quad (3.8)$$

is the positive ground solution of Eq. (3.1). Moreover, by (3.6)–(3.8), a direct computation shows that

$$\begin{aligned} J_{\varepsilon,\infty}(U_{\delta,x_0}^*) &= \frac{\varepsilon^3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,x_0}|^2 dx - \frac{\varepsilon^3}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{\tilde{U}_{\varepsilon,x_0}} |\tilde{U}_{\varepsilon,x_0}|^{2_s^*-1} dx - \frac{\varepsilon^3}{2_s^*} \int_{\mathbb{R}^3} |\tilde{U}_{\varepsilon,x_0}|^{2_s^*} dx \\ &= \varepsilon^3 \left(\frac{\kappa^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} U_{\delta,x_0}|^2 dx - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{U_{\varepsilon,x_0}} |U_{\varepsilon,x_0}|^{2_s^*-1} dx - \frac{\kappa^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |U_{\varepsilon,x_0}|^{2_s^*} dx \right) \\ &= \varepsilon^3 \left(\frac{\kappa^2}{2} S_{2s}^{\frac{3}{2}} - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^* - 1)} S_{2s}^{\frac{3}{2}} - \frac{\kappa^{2_s^*}}{2_s^*} S_{2s}^{\frac{3}{2}} \right) \\ &= \varepsilon^3 \left(\frac{\kappa^2}{2} - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^* - 1)} - \frac{\kappa^{2_s^*}}{2_s^*} \right) S_{2s}^{\frac{3}{2}} \\ &= \varepsilon^3 \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s[12 + (1 - \sqrt{5})(3 - 2s)]}{6(3 + 2s)} S_{2s}^{\frac{3}{2}}. \end{aligned}$$

□

Lemma 3.4. Suppose that the conditions (V₁) and (V₂) are satisfied. Then $m_\varepsilon = m_\varepsilon^\infty$ and m_ε is not attained.

Proof. Let $u \in D^{s,2}(\mathbb{R}^3)$, we define the function

$$\begin{aligned} \Psi(t) &:= \langle J'_\varepsilon(tu), tu \rangle \\ &= t^2 \varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + t^2 \int_{\mathbb{R}^3} V(x) u^2 dx - \varepsilon^{-2s} t^{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \\ &\quad - t^{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= at^2 - bt^{2_s^*} - ct^{2_s^*}, \end{aligned}$$

where

$$a = \int_{\mathbb{R}^3} \left(\varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2 \right) dx, \quad b = \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \quad \text{and} \quad c = \int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

It is easy to check that there exist unique $t(u) > 0, s(u) > 0$ such that $t(u)u \in \mathcal{N}_\varepsilon, s(u)u \in \mathcal{N}_\varepsilon^\infty$ and

$$J_\varepsilon(t(u)u) = \max_{t>0} J_\varepsilon(tu), \quad J_{\varepsilon,\infty}(s(u)u) = \max_{t>0} J_{\varepsilon,\infty}(tu).$$

Moreover, for each $u \in \mathcal{N}_\varepsilon$, we derive that

$$\begin{aligned} m_\varepsilon^\infty &\leq J_{\varepsilon,\infty}(s(u)u) \\ &\leq \frac{\varepsilon^{2s}}{2} \|s(u)u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |s(u)u|^2 dx \\ &\quad - \varepsilon^{-2s} \frac{|s(u)|^{2(2_s^*-1)}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{|s(u)|^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= J_\varepsilon(s(u)u) \\ &\leq J_\varepsilon(u), \end{aligned}$$

consequently, one has

$$m_\varepsilon^\infty \leq m_\varepsilon. \tag{3.9}$$

Let w be a positive solution of (1.10) centered at the origin and take a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ such that $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Set $w_n(x) = \varepsilon^{\frac{3-2s}{2}} w(x - x_n)$ and $t_n = t(w_n)$. Clearly, $w_n \rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$ and by Lemma 2.2, we have

$$\int_{\mathbb{R}^3} V(x) w_n^2 dx \rightarrow 0.$$

At this point, we have

$$J_\varepsilon(t_n w_n) = \frac{t_n^2 \varepsilon^{2s}}{2} \|w_n\|^2 + \frac{t_n^2}{2} o_n(1) - \frac{\varepsilon^{-2s} |t_n|^{2(2_s^*-1)}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2_s^*-1} dx - \frac{|t_n|^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx. \tag{3.10}$$

Since $w_n \in \mathcal{N}_\varepsilon^\infty$ and $t_n w_n \in \mathcal{N}_\varepsilon$, we have

$$\varepsilon^{2s} \|w_n\|^2 = \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2_s^*-1} dx + \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx, \tag{3.11}$$

and

$$\varepsilon^{2s} t_n^2 \|w_n\|^2 + t_n^2 \int_{\mathbb{R}^3} V(x) w_n^2 dx = \varepsilon^{-2s} t_n^{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2_s^*-1} dx + t_n^{2_s^*} \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx. \tag{3.12}$$

From (3.11) and (3.12) it follows

$$\varepsilon^{2s} [t_n^{2(2_s^*-2)} - 1] \|w_n\|^2 + [t_n^{2_s^*-2} - t_n^{2(2_s^*-2)}] \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx = \int_{\mathbb{R}^3} V(x) w_n^2 dx = o_n(1),$$

which implies that $t_n \rightarrow 1$ as $n \rightarrow \infty$. By (3.10), we arrive that $\lim_{n \rightarrow \infty} J_\varepsilon(w_n) = m_\varepsilon^\infty$. Hence, we deduce to $m_\varepsilon \leq m_\varepsilon^\infty$, and then $m_\varepsilon = m_\varepsilon^\infty$ by (3.9).

Next, we show that m_ε cannot be achieved. Assuming the conclusion is not true, then there exists $u_0 \in \mathcal{N}_\varepsilon$ such that $J_\varepsilon(u_0) = m_\varepsilon = m_\varepsilon^\infty$. Therefore, by $s(u_0)u_0 \in \mathcal{N}_\varepsilon^\infty$, we infer to

$$\begin{aligned} m_\varepsilon^\infty &\leq J_{\varepsilon,\infty}(s(u_0)u_0) \\ &\leq \frac{\varepsilon^{2s}}{2} \|s(u_0)u_0\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |s(u_0)u_0|^2 dx \\ &\quad - \frac{1}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{s(u_0)u_0} |s(u_0)u_0|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |s(u_0)u_0|^{2_s^*} dx \\ &= J_\varepsilon(s(u_0)u_0) \leq J_\varepsilon(u_0) = m_\varepsilon^\infty. \end{aligned}$$

Inevitably, we conclude that

$$\int_{\mathbb{R}^3} V(x) |s(u_0)u_0|^2 dx = 0, \quad s(u_0) = 1 \quad \text{imply} \quad u_0 \equiv 0 \quad \text{on} \quad \mathbb{R}^3 \setminus M.$$

Therefore, $u_0 \in \mathcal{N}_\varepsilon^\infty$ with $J_\varepsilon(u_0) = m_\varepsilon^\infty$. Then, by Lemma 3.3, $u_0(x) = \varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta, x_0}(x) > 0$ for all $x \in \mathbb{R}^3$ and for some $\delta > 0, x_0 \in \mathbb{R}^3$, which contradicts $u_0 \equiv 0$ on $\mathbb{R}^3 \setminus M$. \square

Corollary 3.5. Let $c > 0$ and $\{u_n\}_{n \in \mathbb{N}}$ be a $(PS)_c$ -sequence of J_ε restricted on \mathcal{N}_ε . Then $\{u_n\}_{n \in \mathbb{N}}$ is a $(PS)_c$ -sequence of J_ε in $D^{s,2}(\mathbb{R}^3)$. If u is a critical point of J_ε restricted on \mathcal{N}_ε , then u must be a critical point of J_ε in $D^{s,2}(\mathbb{R}^3)$.

Proof. If $\{u_n\}_{n \in \mathbb{N}}$ is a (PS) -sequence of J_ε restricted on \mathcal{N}_ε , then we have

$$\begin{aligned} c + o_n(1) &= \frac{1}{2} \int_{\mathbb{R}^3} (\varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2) dx - \frac{\varepsilon^{-2s}}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx \\ &\quad - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx, \end{aligned}$$

and

$$0 = \int_{\mathbb{R}^3} (\varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2) dx - \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx.$$

Consequently, we obtain

$$\begin{aligned} c + o_n(1) &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^3} [\varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2] dx \\ &\quad + \left(\frac{1}{2_s^*} - \frac{1}{2(2_s^* - 1)}\right) \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx. \end{aligned}$$

Thus, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. By virtue of $u_n \in \mathcal{N}_\varepsilon$ and by Lemma 2.1, we derive that

$$\begin{aligned} & \varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \\ & \leq \varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \int_{\mathbb{R}^3} V(x) |u_n|^2 dx \\ & = \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx + \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \leq \varepsilon^{-2s} S^{-1} |u_n|_{2_s^*}^{2(2_s^*-1)} + |u_n|_{2_s^*}^{2_s^*} \\ & \leq \varepsilon^{-2s} S^{-2_s^*} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^{2_s^*-1} + S^{-\frac{2_s^*}{2}} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^{\frac{2_s^*}{2}}, \end{aligned}$$

which implies that

$$\varepsilon^{-2s} S^{-2_s^*} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^{2_s^*-2} + S^{-\frac{2_s^*}{2}} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^{\frac{2_s^*-2}{2}} \geq \varepsilon^{2s}.$$

Hence

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \geq \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s}. \tag{3.13}$$

On the other hand, by the Lagrangian multiplier rule, we have for $\lambda_n \in \mathbb{R}$ such that

$$o_n(1) = J'_\varepsilon(u_n) - \lambda_n F'_\varepsilon(u_n),$$

with $F_\varepsilon(u) = J'_\varepsilon(u)u$. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$, we get

$$o_n(1) = J'_\varepsilon(u_n)u_n - \lambda_n F'_\varepsilon(u_n)u_n = -\lambda_n F'_\varepsilon(u_n)u_n.$$

In view of $2_s^* > 2$, and by (3.13), we infer that

$$\begin{aligned} F'_\varepsilon(u_n)u_n & = 2\varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + 2 \int_{\mathbb{R}^3} V(x) u_n^2 dx \\ & \quad - 2(2_s^* - 1)\varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - 2_s^* \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ & = (2 - 2_s^*)\varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + (2 - 2_s^*) \int_{\mathbb{R}^3} V(x) |u_n|^2 dx \\ & \quad + (2 - 2_s^*)\varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx \\ & \leq (2 - 2_s^*)\varepsilon^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \\ & \leq [2 - 2_s^*] \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s} < 0. \end{aligned} \tag{3.14}$$

As a result, $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, using the boundedness of $\{u_n\}_{n \in \mathbb{N}}$, we deduce that $\{F'_\varepsilon(u_n)\}_{n \in \mathbb{N}}$ is bounded. So, $J'_\varepsilon(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and then $\{u_n\}_{n \in \mathbb{N}}$ is a $(PS)_c$ -sequence of J_ε in $D^{s,2}(\mathbb{R}^3)$.

Now, if u is a critical point of J_ε restricted on \mathcal{N}_ε , then there exists $\lambda \in \mathbb{R}$ so as to $J'_\varepsilon(u) = \lambda F'_\varepsilon(u)$, and

$$0 = F_\varepsilon(u) = J'_\varepsilon(u)u = \lambda F'_\varepsilon(u)u.$$

By the same calculation as in (3.14), we can obtain that

$$F'_\varepsilon(u)u \leq [2 - 2_s^*] \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s} < 0.$$

Hence, $\lambda = 0$ and so, $J'_\varepsilon(u) = 0$. □

IV. A GLOBAL COMPACTNESS LEMMA

In this section, we intend to establish a global compactness lemma, which is useful in the analyzing the decomposition of the (PS)-sequences, and proving its compactness.

Lemma 4.1. Let the conditions (V_1) , (V_2) be satisfied and let, for each $\varepsilon > 0$, $\{u_n\}_{n \in \mathbb{N}} \subset D^{s,2}(\mathbb{R}^3)$ be a (PS)-sequence of J_ε at the level $c > 0$. Then, replacing u_n with a subsequence if necessary, there exist a number $k \in \mathbb{N}$, sequences of points $x_n^1, \dots, x_n^k \in \mathbb{R}^3$ and radii r_n^1, \dots, r_n^k such that the following hold:

- (i) $u_n^0 \equiv u_n - u^0$ in $D^{s,2}(\mathbb{R}^3)$;
- (ii) $u_n^j \equiv (u_n^{j-1} - u^{j-1})_{r_n^j, x_n^j} u^j$ in $D^{s,2}(\mathbb{R}^3)$, $j = 1, 2, \dots, k$;
- (iii) $\|u_n\|^2 \rightarrow \sum_{j=0}^k \|u^j\|^2$;
- (iv) $J_\varepsilon(u_n) \rightarrow J_\varepsilon(u^0) + \sum_{j=1}^k J_{\varepsilon, \infty}(u^j)$,

As $n \rightarrow \infty$, where u^0 is a solution of Eq. (1.1) and u^j , $1 \leq j \leq k$, are the nontrivial solutions of Eq. (1.10). Moreover, in case $k = 0$ the above holds without u^j .

Proof. Let $\{u_n\}_{n \in \mathbb{N}}$ be a (PS) $_c$ -sequence for J_ε . Then, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. Thus, we may extract a subsequence of $\{u_n\}_{n \in \mathbb{N}}$, still denote by itself, such that $u_n \rightharpoonup u^0$ in $D^{s,2}(\mathbb{R}^3)$ as $n \rightarrow \infty$ and $u_n \rightarrow u^0$ a.e. in \mathbb{R}^3 . Moreover, $J'_\varepsilon(u^0) = 0$. Indeed, for each $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$J'_\varepsilon(u_n)\varphi = \varepsilon^{2s} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi \, dx + \int_{\mathbb{R}^3} V(x) u_n \varphi \, dx - \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s-3} u_n \varphi \, dx - \int_{\mathbb{R}^3} |u_n|^{2^*_s-2} u_n \varphi \, dx.$$

By Lemma 2.1, it follows that $\phi_{u_n} - \phi_{u^0}$ in $D^{s,2}(\mathbb{R}^3)$ and then $\phi_{u_n} - \phi_{u^0}$ in $L^{2^*_s}(\mathbb{R}^3)$. Thus,

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u^0}) |u^0|^{2^*_s-3} u^0 \varphi \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.1}$$

In view of $u_n \rightarrow u$ a.e. in \mathbb{R}^3 and using Hölder's inequality, we infer to

$$\begin{aligned} & \int_{\mathbb{R}^3} |\phi_{u_n} (|u_n|^{2^*_s-3} u_n - |u^0|^{2^*_s-3} u^0)|^{2^*_s} \, dx \\ & \leq C \left(|\phi_{u_n}|_{2^*_s}^{2^*_s} |u_n|_{2^*_s}^{2^*_s(2^*_s-2)} + |\phi_{u_n}|_{2^*_s}^{2^*_s} |u^0|_{2^*_s}^{2^*_s(2^*_s-2)} \right) \leq C. \end{aligned}$$

By Proposition 5.4.7 in Willem,⁵² one has $\phi_{u_n} (|u_n|^{2^*_s-3} u_n - |u^0|^{2^*_s-3} u^0) \rightarrow 0$ in $L^{\frac{2^*_s}{2^*_s-1}}(\mathbb{R}^3)$ and then

$$\int_{\mathbb{R}^3} \phi_{u_n} (|u_n|^{2^*_s-3} u_n - |u^0|^{2^*_s-3} u^0) \varphi \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This together with (4.1) implies

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s-3} u_n \varphi \, dx \rightarrow \int_{\mathbb{R}^3} \phi_{u^0} |u^0|^{2^*_s-3} u^0 \varphi \, dx \quad \text{as } n \rightarrow \infty. \tag{4.2}$$

Since $|u_n|^{2^*_s-2} u_n - |u^0|^{2^*_s-2} u^0$ in $L^{\frac{2^*_s}{2^*_s-1}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |u_n|^{2^*_s-2} u_n \varphi \, dx \rightarrow \int_{\mathbb{R}^3} |u^0|^{2^*_s-2} u^0 \varphi \, dx.$$

Therefore, combining (4.2) with the weak convergence of $u_n \rightharpoonup u^0$ in $D^{s,2}(\mathbb{R}^3)$, we derive to

$$J'_\varepsilon(u^0)\varphi = \lim_{n \rightarrow \infty} J'_\varepsilon(u_n)\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3).$$

This implies that $J'_\varepsilon(u^0) = 0$ and u is a critical point of J_ε by the density of $C_0^\infty(\mathbb{R}^3)$ in $D^{s,2}(\mathbb{R}^3)$. Set

$$v_n^1(x) := u_n(x) - u^0(x).$$

Applying the Brezis–Lieb Lemma⁵³ and Lemma 2.1 (iv), we can easily deduce that

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u^0\|^2 + o_n(1), \quad \|v_n^1\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o_n(1), \tag{4.3}$$

$$J_\varepsilon(v_n^1) = J_\varepsilon(u_n) - J_\varepsilon(u^0) + o_n(1),$$

and

$$J'_\varepsilon(v_n^1) = J'_\varepsilon(u_n) - J'_\varepsilon(u^0) + o_n(1).$$

By $v_n^1 \rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$ and Lemma 2.2, one has

$$\int_{\mathbb{R}^3} V(x)|v_n^1|^2 \, dx = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^3} V(x)v_n^1 \varphi \, dx = o_n(1)\|\varphi\|$$

for each $\varphi \in D^{s,2}(\mathbb{R}^3)$. Consequently, we arrive that

$$J_{\varepsilon,\infty}(v_n^1) = J_\varepsilon(v_n^1) + o_n(1) = J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1) \tag{4.4}$$

$$J'_{\varepsilon,\infty}(v_n^1) = J'_\varepsilon(v_n^1) + o_n(1) = o_n(1).$$

If $v_n^1 \rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$, then we are done. If $v_n^1 \not\rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$, then there exists $\eta > 0$ satisfying

$$J_\varepsilon(v_n^1) > \eta > 0. \tag{4.5}$$

We claim that there exist two sequences $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ and $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ verifying

$$h_n := (v_n^1)_{r_n y_n} - \omega \neq 0 \quad \text{in } D^{s,2}(\mathbb{R}^3),$$

where $(v_n^1)_{r_n y_n} = r_n^{\frac{3-2s}{2}} v_n^1(r_n x + y_n)$. Indeed, by (4.4) we get

$$\varepsilon^{2s} \|v_n^1\|^2 = \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{v_n^1} |v_n^1|^{2_s^*-1} \, dx + \int_{\mathbb{R}^3} |v_n^1|^{2_s^*} \, dx + o_n(1)$$

and

$$J_{\varepsilon,\infty}(v_n^1) = \varepsilon^{-2s} \left(\frac{1}{2} - \frac{1}{2(2_s^* - 1)} \right) \int_{\mathbb{R}^3} \phi_{v_n^1} |v_n^1|^{2_s^*-1} \, dx + \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \int_{\mathbb{R}^3} |v_n^1|^{2_s^*} \, dx + o_n(1).$$

Combining (4.5), Lemma 2.1(ii) and the boundedness of $\{u_n\}_{n \in \mathbb{N}}$, we obtain that

$$0 < d_1 < |v_n^1|_{2_s^*}^{2_s^*-1} < D_1,$$

for some $d_1, D_1 > 0$. Now, we introduce the Lévy concentration function

$$\mathcal{Q}_n(r) := \sup_{x \in \mathbb{R}^3} \int_{B_r(x)} |v_n^1|^{2_s^*} \, dx.$$

In view of $\mathcal{Q}_n(0) = 0$ and $\mathcal{Q}_n(\infty) > d_1^{\frac{6}{3+2s}}$, then there exist sequences $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ and $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ so as to

$$\sup_{z \in \mathbb{R}^3} \int_{B_r(z)} |v_n^1|^{2_s^*} \, dx = \int_{B_{r_n}(y_n)} |v_n^1|^{2_s^*} \, dx = b,$$

where

$$0 < b < \min \left\{ d_1^{\frac{6}{3+2s}}, \left(\frac{S}{2C_{3,s} D_1} \right)^{\frac{6}{6s-3}} \right\}.$$

Recalling that $h_n = (v_n^1)_{r_n y_n}$, without any loss of generality, we may assume that $h_n \rightarrow h$ in $D^{s,2}(\mathbb{R}^3)$ and $h_n \rightarrow h$ a.e. in \mathbb{R}^3 . A simple calculation yields

$$\sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |h_n(x)|^{2_s^*} \, dx = \int_{B_1(0)} |h_n(x)|^{2_s^*} \, dx = \int_{B_{r_n}(y_n)} |v_n^1|^{2_s^*} \, dx = b. \tag{4.6}$$

By virtue of the invariance of the $D^{s,2}(\mathbb{R}^3)$ -norms under translation and dilation, we have

$$\|v_n^1\|^2 = \|h_n\|^2, \quad \|v_n^1\|_{2_s^*}^{2_s^*} = \|h_n\|_{2_s^*}^{2_s^*}, \quad \int_{\mathbb{R}^3} \phi_{h_n} |h_n|^{2_s^*-1} dx = \int_{\mathbb{R}^3} \phi_{v_n^1} |v_n^1|^{2_s^*-1} dx,$$

which lead to

$$J_{\varepsilon,\infty}(h_n) = J_{\varepsilon,\infty}(v_n^1) = J_{\varepsilon,\infty}(u_n) - J_{\varepsilon,\infty}(u^0) + o_n(1), \tag{4.7}$$

and by (4.4), we get

$$J'_{\varepsilon,\infty}(h_n) = J'_{\varepsilon,\infty}(v_n^1) = o_n(1).$$

If $h = 0$, then $h_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^3)$. Suppose that $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } \psi \subset B_1(y^0)$ for some $y^0 \in \mathbb{R}^3$ and $|\nabla \psi(x)| \leq C$ for all $x \in \mathbb{R}^3$, and we obtain

$$\begin{aligned} \|\psi h_n\|^2 &= \iint_{\mathbb{R}^6} \frac{|\psi(x)h_n(x) - \psi(y)h_n(y)|^2}{|x-y|^{3+2s}} dx dy \\ &= \iint_{\mathbb{R}^6} \frac{(h_n(x) - h_n(y))(\psi^2(x)h_n(x) - \psi^2(y)h_n(y))}{|x-y|^{3+2s}} dx dy \\ &\quad + \iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n(x)h_n(y)}{|x-y|^{3+2s}} dx dy. \end{aligned} \tag{4.8}$$

By Hölder's inequality, we have

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n(x)h_n(y)}{|x-y|^{3+2s}} dx dy \\ &\leq \left(\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy \right)^{\frac{1}{2}} \times \left(\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy \right)^{\frac{1}{2}}. \end{aligned} \tag{4.9}$$

We claim that

$$\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy = \iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(y)}{|x-y|^{3+2s}} dx dy = o_n(1). \tag{4.10}$$

To this aim, we derive out

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy \\ &= \int_{B_1(y^0)} \int_{B_1(y^0)} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy \\ &\quad + 2 \int_{B_1(y^0)} \int_{B_1^c(y^0)} \frac{|\psi(x) - \psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}} dx dy \\ &\leq \int_{B_1(y^0)} h_n^2(x) dx \int_{B_1(y^0)} \frac{|\nabla \psi(y + \tau(x-y))|^2 |x-y|^2}{|x-y|^{3+2s}} dy \\ &\quad + C \int_{B_1(y^0)} h_n^2(x) dx \int_{B_1^c(y^0)} \frac{1}{|x-y|^{3+2s}} dy \\ &\leq C_1 \int_{B_1(y^0)} h_n^2(x) dx \int_0^2 \frac{r^2}{r^{1+2s}} dr + C \int_{B_1(y^0)} h_n^2(x) dx \int_1^\infty \frac{r^2}{r^{3+2s}} dr \\ &\leq C_2 \int_{B_1(y^0)} h_n^2(x) dx \rightarrow 0 \end{aligned} \tag{4.11}$$

as $n \rightarrow \infty$, by virtue of $h_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^3)$, where $\tau = \tau(y) \in (0, 1)$. By a similar argument, we can prove

$$\iint_{\mathbb{R}^6} \frac{|\psi(x) - \psi(y)|^2 h_n^2(y)}{|x-y|^{3+2s}} dx dy \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.12}$$

By (4.8)–(4.12), we have

$$\|\psi h_n\|^2 = \iint_{\mathbb{R}^6} \frac{(h_n(x) - h_n(y))(\psi^2(x)h_n(x) - \psi^2(y)h_n(y))}{|x-y|^{3+2s}} dx dy + o_n(1).$$

Therefore, combining the results above and Proposition 2.3, we have

$$\begin{aligned} \varepsilon^{2s} S \|\psi h_n\|_{2_s^*}^2 &\leq \varepsilon^{2s} \|\psi h_n\|^2 \\ &= \varepsilon^{2s} \iint_{\mathbb{R}^6} \frac{(h_n(x) - h_n(y))(\psi^2(x)h_n(x) - \psi^2(y)h_n(y))}{|x - y|^{3+2s}} dx dy + o_n(1) \\ &= \varepsilon^{2s} \int_{\mathbb{R}^3} \phi_{h_n} |h_n|^{2_s^*-1} \psi^2 dx + o_n(1) \\ &\leq \varepsilon^{2s} C_{3,s} \|h_n\|_{2_s^*}^{2_s^*-1} \left(\int_{\mathbb{R}^3} (|h_n|^{2_s^*-1} \psi^2)^{\frac{6}{3+2s}} dx \right)^{\frac{3+2s}{6}} + o_n(1) \\ &= \varepsilon^{2s} C_{3,s} \|h_n\|_{2_s^*}^{2_s^*-1} \left(\int_{\mathbb{R}^3} |h_n|^{\frac{6s-3}{3-2s} \frac{6}{3+2s}} |h_n \psi|^{\frac{12}{3+2s}} dx \right)^{\frac{3+2s}{6}} + o_n(1) \\ &\leq \varepsilon^{2s} C_{3,s} |h_n|_{2_s^*}^{2_s^*-1} \left(\int_{\mathbb{R}^3} |\psi h_n|^{2_s^*} dx \right)^{\frac{3-2s}{3}} \left(\int_{B_1(y^0)} |h_n|^{2_s^*} dx \right)^{\frac{6s-3}{6}} + o_n(1) \\ &\leq \varepsilon^{2s} C_{3,s} D_1 b^{\frac{6s-3}{6}} \left(\int_{\mathbb{R}^3} |\psi h_n|^{2_s^*} dx \right)^{\frac{3-2s}{3}} + o_n(1) < \frac{1}{2} \varepsilon^{2s} S \|\psi h_n\|_{2_s^*}^2 + o_n(1), \end{aligned}$$

which shows that $h_n \rightarrow 0$ in $L_{loc}^{2_s^*}(\mathbb{R}^3)$, a contradiction to (4.6). Thus, $h \neq 0$. By (4.4) and the weakly sequentially continuity of $J'_{\varepsilon,\infty}$, we deduce $J'_{\varepsilon,\infty}(h) = 0$ and so $\{h_n\}_{n \in \mathbb{N}}$, $\{r_n^1\}_{n \in \mathbb{N}}$ and $\{y_n^1\}_{n \in \mathbb{N}}$ are the required sequences.

By iterating this procedure, we have the sequences $v_n^j = u_n^{j-1} - u^{j-1}$, $j \geq 2$, and the re-scaled functions $u_n^j = (v_n^j)_{r_n^j, y_n^j} - u^j$ in $D^{s,2}(\mathbb{R}^3)$, where u^j is a nontrivial solution to (1.10). Furthermore, by (4.3), (4.4), and (4.7), we arrive at

$$\|u_n^j\|^2 = \|v_n^j\|^2 = \|u_n^{j-1}\|^2 - \|u^{j-1}\|^2 + o_n(1) = \dots = \|u_n\|^2 - \sum_{i=0}^{j-1} \|u^i\|^2 + o_n(1),$$

and

$$\begin{aligned} J_{\varepsilon,\infty}(u_n^j) &= J_{\varepsilon,\infty}(v_n^j) = J_{\varepsilon,\infty}(u_n^{j-1}) - J_{\varepsilon,\infty}(u^{j-1}) + o_n(1) \\ &= \dots = J_{\varepsilon}(u_n) - J_{\varepsilon}(u^0) - \sum_{i=1}^{j-1} J_{\varepsilon,\infty}(u^i). \end{aligned}$$

Since

$$\begin{aligned} 0 = J'_{\varepsilon,\infty}(u^j)u^j &= \varepsilon^{2s} \|u^j\|^2 - \varepsilon^{-2s} \int_{\mathbb{R}^3} \phi_{u^j} |u^j|^{2_s^*-1} dx - \int_{\mathbb{R}^3} |u^j|^{2_s^*} dx \\ &\geq \varepsilon^{2s} \|u^j\|^2 - \varepsilon^{-2s} S^{-2s} \|u^j\|^{2(2_s^*-1)} - S^{-\frac{2_s^*}{2}} \|u^j\|^{2_s^*}, \end{aligned}$$

we conclude

$$\|u^j\|^2 \geq \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s},$$

and the iteration must stop at some index $k \geq 0$. □

Corollary 4.2. *If $\{u_n\}_{n \in \mathbb{N}}$ is a nonnegative (PS) $_c$ -sequence for J_{ε} with $c \in (m_{\varepsilon}, 2m_{\varepsilon})$, then, for each $\varepsilon > 0$, $\{u_n\}_{n \in \mathbb{N}}$ is relatively compact in $D^{s,2}(\mathbb{R}^3)$.*

Proof. It follows by Lemma 4.1, that there exist a number $k \in \mathbb{N}$, a solution u^0 of (1.1) and solutions u^1, \dots, u^k of (1.10), fulfilling

$$\|u_n\|^2 \rightarrow \sum_{j=0}^k \|u^j\|^2 \quad \text{and} \quad J_{\varepsilon}(u_n) \rightarrow J_{\varepsilon}(u^0) + \sum_{j=1}^k J_{\varepsilon,\infty}(u^j) = c + o_n(1),$$

as $n \rightarrow \infty$. By Lemma 3.4, if u^0 is a nontrivial solution of (1.1), then $J_{\varepsilon}(u^0) > m_{\varepsilon}$. Noticing that, for each nontrivial solution u^j of (1.10), whether u^j is positive or negative, then we have

$$J_{\varepsilon,\infty}(u^j) = \varepsilon^3 \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s[12 + (1 - \sqrt{5})(3 - 2s)]}{6(3 + 2s)} S^{\frac{3}{2s}} = m_{\varepsilon}.$$

If u^j is a sign-changing solution of (1.10), we can easily deduce that

$$J_{\varepsilon, \infty}(u^j) \geq 2m_\varepsilon,$$

see Zhang–Zhang.⁴⁴ Since

$$m_\varepsilon < c < 2m_\varepsilon,$$

we must have $k = 0$, which implies that $u_n \rightarrow u^0$ in $D^{s,2}(\mathbb{R}^3)$. □

V. PROOF OF THEOREM 1.1

In this section, we focus our attention in showing the multiplicity of high energy semiclassical states by means of the the Lusternik–Schnirelmann category theory. We begin with some preparations. For small $\mu > 0$, we may take $\rho = \rho(\mu) > 0$ such that $M_\mu \subset B_\rho(0)$. Set

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define $\beta : \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^3$ and $\gamma : \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^+$ by

$$\beta(u) = \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} u|^2 dx,$$

and

$$\gamma(u) = \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(u)| |(-\Delta)^{\frac{s}{2}} u|^2 dx.$$

It is not difficulty to see that, for any fixed $\tilde{U}_{\delta,z} \in D^{s,2}(\mathbb{R}^3)$, there exists a unique $t_{\delta,z}$ in $(0, +\infty)$ such that

$$\Phi_{\delta,z}(x) := t_{\delta,z} \varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta,z}(x) \in \mathcal{N}_\varepsilon.$$

We define the set

$$\Gamma = \Gamma(\rho, \delta_1, \delta_1) := \{(x, \delta) \in \mathbb{R}^3 \times \mathbb{R} : |x| < \rho/2, \delta_1 < \delta < \delta_2\}. \tag{5.1}$$

A direct calculation shows that, for any fixed $z \in \mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} V(x) \tilde{U}_{\delta,z}^2(x) dx \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Thus, for each $\varepsilon > 0$ and any fixed $z \in \mathbb{R}^3$, we derive that $\lim_{\delta \rightarrow 0} t_{\delta,z} = 1$. So, for each $\varepsilon > 0$, there exist $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ with $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \rightarrow 0$ as $\varepsilon \rightarrow 0$, satisfying the inequality

$$\begin{aligned} & \sup \{J_\varepsilon(\Phi_{\delta,z}) : (z, \delta) \in \Lambda\} \\ & < \varepsilon^3 \left(\left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s[12 + (1-\sqrt{5})(3-2s)]}{6(3+2s)} S^{\frac{3}{2s}} + h(\varepsilon) \right), \end{aligned} \tag{5.2}$$

where $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Lemma 5.1. *There holds $\lim_{\delta \rightarrow 0} \gamma(\Phi_{\delta,z}) = 0$ uniformly for $|z| \leq \frac{\rho}{2}$.*

Proof. Let $\xi > 0$ be such that $0 < 2\xi < \rho$. By $t_{\delta,z} = 1 + o_\delta(1)$, we have

$$\begin{aligned} \gamma(\Phi_{\delta,z}) &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx + o_\delta(1) \\ &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3 \setminus B_\xi(z)} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx \\ &\quad + \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{B_\xi(z)} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx + o_\delta(1) \\ &:= I_1 + I_2 + o_\delta(1). \end{aligned}$$

By Lemma 4.1 of He-Zhao-Zou,⁵⁴ we derive

$$\begin{aligned} I_1 &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3 \setminus B_\xi(0)} |\chi(x+z) - \beta(\Phi_{\delta,0})| (-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,0}|^2 dx \\ &\leq C\rho \int_{\mathbb{R}^3 \setminus B_\xi(0)} |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,0}|^2 dx \rightarrow 0 \end{aligned}$$

as $\delta \rightarrow 0$.

For I_2 , by Remark 3.1 we infer to

$$\begin{aligned} \beta(\Phi_{\delta,z}) &= \beta\left(\varepsilon^{\frac{3-2s}{2}} \tilde{U}_{\delta,z}\right) + o_\delta(1) \\ &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}(x)|^2 dx + o_\delta(1) \\ &= z + \frac{\delta^{3+2s}}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} [\chi(\delta x+z) - z] |(-\Delta)^{\frac{s}{2}} \tilde{U}_{1,0}|^2 dx + o_\delta(1) = z + o_\delta(1). \end{aligned} \tag{5.3}$$

As a result, we get

$$\begin{aligned} &\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} I_2 \\ &\leq \int_{B_\xi(z)} |\chi(x) - \chi(z)| |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx + \int_{B_\xi(z)} |\chi(z) - \beta(\Phi_{\delta,z})| |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx \\ &\leq 2 \int_{B_\xi(z)} |x-z| |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx + 2\xi \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \\ &\quad + \int_{B_\xi(z)} |\chi(z) - z| |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\delta,z}|^2 dx + o_\delta(1) \\ &\leq 4\xi \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} + o_\delta(1), \end{aligned}$$

where we have used Lemma 2 of Chabrowski-Yang,⁵⁵ which states that

$$|\chi(x) - \chi(z)| \leq 2|x-z| + 2\xi, \quad x \in B_\xi(z).$$

As $\xi > 0$ can be arbitrarily small, we infer that $\lim_{\delta \rightarrow 0} \gamma(\Phi_{\delta,z}) = 0$, uniformly for $|z| \leq \rho/2$. □

Now, we define a set $\tilde{\mathcal{N}}_\varepsilon \subset \mathcal{N}_\varepsilon$ by

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \varepsilon^3 c^* < J_\varepsilon(u) < \varepsilon^3(c^* + h(\varepsilon)), (\beta(u), \gamma(u)) \in \Gamma\},$$

where Γ is given as (5.1) satisfying (5.2) and the number c^* is defined as

$$c^* := \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s[12 + (1-\sqrt{5})(3-2s)]}{6(3+2s)} S^{\frac{3}{2s}}.$$

According to Lemma 5.1, we can adjust $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$ small enough.

Lemma 5.2. *There holds $\lim_{\varepsilon \rightarrow 0} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} \text{dist}(\beta(u), M_\mu) = 0$ for any $\mu > 0$.*

Proof. Let $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be so as to

$$\text{dist}(\beta(u_n), M_\mu) = \sup_{u \in \tilde{\mathcal{N}}_{\varepsilon_n}} \text{dist}(\beta(u), M_\mu) + o_n(1),$$

and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. It is sufficient to obtain a sequence $z_n \in M_\mu$ such that

$$\beta(u_n) = z_n + o_n(1). \tag{5.4}$$

In view of $u_n \in \mathcal{N}_{\varepsilon_n}$, we have

$$\int_{\mathbb{R}^3} \left(\varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2 \right) dx = \varepsilon_n^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s - 1} dx + \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx. \tag{5.5}$$

For further discussion, we set

$$\begin{aligned} \ell_n &:= \int_{\mathbb{R}^3} \left(\varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2 \right) dx, \\ a_n &:= \varepsilon_n^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s - 1} dx, \quad b_n := \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx. \end{aligned}$$

Multiplying both sides of the second equation of (1.1) (with ε replaced by ε_n) by $|u_n|$ and integrating by parts, using Young's inequality, we infer, for $\varepsilon_n > 0$ small enough,

$$\begin{aligned} \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx &= \varepsilon_n^{2s} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \phi_{u_n} (-\Delta)^{\frac{s}{2}} |u_n| dx \\ &\leq \frac{\varepsilon_n^{2s} \tau^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} |u_n||^2 dx + \frac{\varepsilon_n^{2s}}{2\tau^2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \phi_{u_n}|^2 dx \\ &\leq \frac{\tau^2}{2} \int_{\mathbb{R}^3} \varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{1}{2\tau^2} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s - 1} dx \\ &\leq \frac{\tau^2}{2} \int_{\mathbb{R}^3} \left(\varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2 \right) dx + \frac{\varepsilon_n^{-2s}}{2\tau^2} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2^*_s - 1} dx. \end{aligned}$$

From the definition of ℓ_n, a_n and b_n , it follows that

$$b_n \leq \frac{1}{2\tau^2} a_n + \frac{\tau^2}{2} \ell_n. \tag{5.6}$$

Choosing $\tau^2 = \frac{\sqrt{5}-1}{2}$ and using (5.5) as well as (5.6), we obtain that $a_n \geq \frac{3-\sqrt{5}}{2} \ell_n$. It follows from (5.5) and (5.6) that

$$\begin{aligned} J_{\varepsilon_n}(u_n) &= J_{\varepsilon_n}(u_n) - \frac{1}{2} J'_{\varepsilon_n}(u_n) u_n \\ &= \frac{2s}{3+2s} a_n + \frac{s}{3} b_n \\ &= \frac{s(3-2s)}{3(3+2s)} a_n + \frac{s}{3} \ell_n \\ &\geq \frac{s(15-2s-\sqrt{5}(3-2s))}{6(3+2s)} \ell_n. \end{aligned} \tag{5.7}$$

On the other hand, by the estimate (3.13), we have

$$\begin{aligned} \ell_n &= \int_{\mathbb{R}^3} \left(\varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2 \right) dx \\ &> \varepsilon_n^{2s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \\ &\geq \varepsilon_n^3 \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}. \end{aligned} \tag{5.8}$$

Then from (5.7) and (5.8) we obtain

$$\begin{aligned} J_{\varepsilon_n}(u_n) &\geq \frac{s(15 - 2s - \sqrt{5}(3 - 2s))}{6(3 + 2s)} \ell_n \\ &> \varepsilon_n^3 \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s(15 - 2s - \sqrt{5}(3 - 2s))}{6(3 + 2s)} S^{\frac{3}{2s}} := \varepsilon_n^3 c^*. \end{aligned}$$

Consequently, we have

$$\varepsilon_n^3 c^* < J_{\varepsilon_n}(u_n) - \frac{1}{2} J'_{\varepsilon_n}(u_n) u_n < \varepsilon_n^3 (c^* + h(\varepsilon_n)). \tag{5.9}$$

Letting $w_n := \varepsilon_n^{\frac{2s-3}{2}} u_n$, we have from (5.7) and (5.9) that

$$\begin{aligned} \varepsilon_n^3 c^* &< J_{\varepsilon_n}(u_n) - \frac{1}{2} J'_{\varepsilon_n}(u_n) u_n \\ &= \frac{s(3 - 2s)}{3(3 + 2s)} \varepsilon_n^{-2s} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2s^*-1} dx + \frac{s}{3} \int_{\mathbb{R}^3} \left(\varepsilon_n^{2s} |(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x) |u_n|^2 \right) dx \\ &= \varepsilon_n^3 \left[\frac{s(3 - 2s)}{3(3 + 2s)} \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2s^*-1} dx + \frac{s}{3} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx + \frac{s}{3} \varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) |w_n|^2 dx \right] \\ &< \varepsilon_n^3 (c^* + h(\varepsilon_n)), \end{aligned}$$

from which we conclude that

$$\varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n^2 dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Again by $u_n \in \mathcal{N}_{\varepsilon_n}$, we obtain

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx + \varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n^2 dx = \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2s^*-1} dx + \int_{\mathbb{R}^3} |w_n|^{2s} dx,$$

that is, $\{w_n\}_{n \in \mathbb{N}}$ is a (PS)-sequence for J_1 . It derives from Lemmas 3.4 and 4.1 with $\varepsilon = 1$ that there exist a number $k \in \mathbb{N}$, sequences of points $x_n^1, \dots, x_n^k \in \mathbb{R}^3$ and radii r_n^1, \dots, r_n^k such that:

- (1) $w_n^0 \equiv w_n \rightarrow w^0$ in $D^{s,2}(\mathbb{R}^3)$;
- (2) $w_n^j \equiv (w_n^{j-1} - w^{j-1})_{r_n^j, x_n^j} \rightarrow w^j$ in $D^{s,2}(\mathbb{R}^3)$, $j = 1, 2, \dots, k$;
- (3) $\|w_n\|^2 \rightarrow \sum_{j=0}^k \|w^j\|^2$;
- (4) $J_1(w_n) \rightarrow J_1(w^0) + \sum_{j=1}^k J_{1,\infty}(w^j)$,

as $n \rightarrow \infty$, where w^0 is a solution of Eq. (1.1) with $\varepsilon = 1$ and $w^j, 1 \leq j \leq k$, are the nontrivial solutions of Eq. (3.2). Assume that $w^0 \neq 0$, then by Lemma 3.4, we have

$$J_1(w^0) > c^* := \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s(15 - 2s - \sqrt{5}(3 - 2s))}{6(3 + 2s)} S^{\frac{3}{2s}},$$

which is a contradiction to conclusion (4) above, as $J_{1,\infty}(w^j) \geq c^*$ and $J_1(w_n) \rightarrow c^*$. Consequently, $w^0 = 0$ and we must have $k = 1$ and w^1 is a ground state solution of (3.2) with $J_{1,\infty}(w^1) = c^*$. Hence, there exist $\delta_1 > 0$ and $z_1 \in \mathbb{R}^3$ such that $w^1 = \tilde{U}_{\delta_1, z_1}$ and there exist $(r_n^1, x_n^1) \in \mathbb{R} \times \mathbb{R}^3$ such that

$$\|(w_n)_{r_n^1, x_n^1} - w^1\| \rightarrow 0.$$

Thus, there exist a sequence of points $\{z_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ and a sequence of $\{\sigma_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that $\|q_n\| := \|w_n - \tilde{U}_{\sigma_n, z_n}\| \rightarrow 0$ with $z_n = x_n^1 + r_n^1 z_1$ and $\sigma_n = r_n^1 \delta_1$. We assert that

$$\sigma_n \rightarrow 0 \quad \text{and} \quad \{z_n\}_{n \in \mathbb{N}} \text{ is bounded.} \tag{5.10}$$

In fact, setting $\Psi_{\sigma_n, z_n} = \varepsilon_n^{\frac{3-2s}{2}} \tilde{U}_{\sigma_n, z_n}$, as $q_n \rightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} w_n|^2 dx - \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 dx \right| \\ &= \left| \int_{\mathbb{R}^3} \chi(x) (|(-\Delta)^{\frac{s}{2}} w_n|^2 - |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2) dx \right| \\ &\leq \int_{\mathbb{R}^3} |\chi(x)| \times \left\| |(-\Delta)^{\frac{s}{2}} w_n|^2 - |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 \right\| dx \\ &\leq \rho \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} w_n| + |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|) \left\| |(-\Delta)^{\frac{s}{2}} w_n| - |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}| \right\| dx \\ &\leq \rho \left\{ \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} w_n| + |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|)^2 dx \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} w_n| - |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|)^2 dx \right\}^{\frac{1}{2}} \\ &\leq \sqrt{2\rho} \sqrt{\|w_n\|^2 + \|\tilde{U}_{\sigma_n, z_n}\|^2} \times \left\{ \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n - (-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 dx \right\}^{\frac{1}{2}} \\ &\leq C \|w_n - \tilde{U}_{\sigma_n, z_n}\| = C \|q_n\| \rightarrow 0 \end{aligned} \tag{5.11}$$

as $n \rightarrow \infty$. Then, by (5.11), we have

$$\begin{aligned} \beta(u_n) &= \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \\ &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} w_n|^2 dx \\ &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 dx + o_n(1) \\ &= \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx + o_n(1) \\ &= \beta(\Psi_{\sigma_n, z_n}) + o_n(1). \end{aligned} \tag{5.12}$$

By $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$, it can be assumed that

$$\beta(\Psi_{\sigma_n, z_n}) \subset B_{\rho/2}(0). \tag{5.13}$$

If $\sigma_n \rightarrow \infty$, then for each $R > 0$, by Proposition 2.2 of Di Nezza–Palatucci–Valdinoci³ and Lemma 4.1 of He–Zhao–Zou,⁵⁴ we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n^{3-2s}} \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx &= \lim_{n \rightarrow \infty} \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \kappa \int_{B_R(0)} |\nabla U_{\sigma_n, z_n}|^2 dx = 0. \end{aligned}$$

Combining this fact, the definition of the map γ and (5.13), we have

$$\begin{aligned} \gamma(\Psi_{\sigma_n, z_n}) &= \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(\Psi_{\sigma_n, z_n})| |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx \\ &\geq \frac{1}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x)| |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx - |\beta(\Psi_{\sigma_n, z_n})| \\ &\geq \frac{\rho}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3 \setminus B_R(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx - \frac{\rho}{2} + o_n(1) \\ &= \frac{\rho}{\varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx - \frac{\rho}{2} + o_n(1) \\ &= \frac{\rho}{2} + o_n(1). \end{aligned} \tag{5.14}$$

By $\|h_n\| := \|u_n - \Psi_{\sigma_n, z_n}\| \rightarrow 0$, (5.12) and Hölder's inequality, we have, as $n \rightarrow \infty$,

$$\begin{aligned} &|\gamma(u_n) - \gamma(\Psi_{\sigma_n, z_n})| \\ &= \frac{1}{c_\varepsilon^*} \left| \int_{\mathbb{R}^3} (|\chi(x) - \beta(u_n)| |(-\Delta)^{\frac{s}{2}} u_n|^2 - |\chi(x) - \beta(\Psi_{\sigma_n, z_n})| |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2) dx \right| \\ &\leq \frac{1}{c_\varepsilon^*} \int_{\mathbb{R}^3} |\chi(x) - \beta(u_n)| \times \left| |(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 \right| dx \\ &\quad + \frac{1}{c_\varepsilon^*} \int_{\mathbb{R}^3} \left| |\chi(x) - \beta(u_n)| - |\chi(x) - \beta(\Psi_{\sigma_n, z_n})| \right| \times |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx \\ &\leq \frac{1}{c_\varepsilon^*} \int_{\mathbb{R}^3} |\chi(x) - \beta(u_n)| \times (|(-\Delta)^{\frac{s}{2}} u_n| + |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|) \left| |(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 \right| dx \\ &\quad + \frac{1}{c_\varepsilon^*} \int_{\mathbb{R}^3} |\beta(u_n) - \beta(\Psi_{\sigma_n, z_n})| \times |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx \\ &\leq \frac{\sqrt{2}\rho}{c_\varepsilon^*} \left\{ \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 \right) \right\}^{\frac{1}{2}} \times \left\{ \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} (u_n - \Psi_{\sigma_n, z_n})|^2 dx \right\}^{\frac{1}{2}} \\ &\quad + o_n(1) \times \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx \\ &\leq C \|h_n\| + o_n(1) \times \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \Psi_{\sigma_n, z_n}|^2 dx \rightarrow 0, \end{aligned}$$

where $c_\varepsilon^* := \varepsilon^{3-2s} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}$. Thus,

$$\gamma(u_n) = \gamma(\Psi_{\sigma_n, z_n}) + o_n(1),$$

and so,

$$\gamma(u_n) > \frac{\rho}{2} + o_n(1). \tag{5.15}$$

But from $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$, we have

$$\delta_1(\varepsilon_n) < \gamma(u_n) < \delta_2(\varepsilon_n), \tag{5.16}$$

where $\delta_i(\varepsilon_n) \rightarrow 0, i = 1, 2$ as $n \rightarrow \infty$, a contradiction to (5.15). Thus, $\{\sigma_n\}_{n \in \mathbb{N}}$ is bounded, and there exists some $\bar{\sigma} \geq 0$ such that $\sigma_n \rightarrow \bar{\sigma}$, as $n \rightarrow \infty$. Now, if $\bar{\sigma} > 0$, then we must have that $|z_n| \rightarrow \infty$. Otherwise, $\tilde{U}_{\sigma_n, z_n}$ would converge strongly in $D^{s,2}(\mathbb{R}^3)$ and so it would w_n . Therefore, J_1 has a nontrivial minimizer on \mathcal{N}_1 , which contradicts to Lemma 3.4. Thus, for each $R > 0$, by the fact that $\lim_{n \rightarrow \infty} |z_n| = \infty$ and due to Proposition 2.2 of Di Nezza–Palatucci–Valdinoci³ and Lemma 4.1 of He–Zhao–Zou,⁵⁴ we have that

$$\lim_{n \rightarrow \infty} \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n, z_n}|^2 dx = 0.$$

Therefore, we can obtain the estimation similar to (5.14), a contradiction to (5.16). By a similar argument, we can show the boundedness of the sequence $\{z_n\}_{n \in \mathbb{N}}$. So, (5.10) holds true.

Based on the above analysis, we can assume that $z_n \rightarrow z^*$ and $\sigma_n \rightarrow 0$. By extracting subsequences of $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\varepsilon_n\}_{n \in \mathbb{N}}$, still denoted as $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\varepsilon_n\}_{n \in \mathbb{N}}$, such that $\frac{\sigma_{n_i}}{\varepsilon_{n_i}} = o_{n_i}(1)$ as $n_i \rightarrow \infty$, we may change $\{\sigma_{n_i}\}_{n \in \mathbb{N}}$ by $\{\varepsilon_{n_i}\}_{n \in \mathbb{N}}$ and relabel $\{\varepsilon_{n_i}\}_{n \in \mathbb{N}}$ by $\{\varepsilon_n\}_{n \in \mathbb{N}}$. Define

$$v_n(x) := \varepsilon_n^{\frac{3-2s}{2}} w_n(\varepsilon_n x + z_n).$$

Then $v_n \rightarrow \tilde{U}_{1,0}$ in $D^{s,2}(\mathbb{R}^3)$ and we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + z_n) v_n(x)^2 dx = \lim_{n \rightarrow \infty} \varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n(x)^2 dx = 0,$$

which shows that $\int_{\mathbb{R}^3} V(z^*) \tilde{U}_{1,0}^2(x) dx = 0$. Thus, $V(z^*) = 0$ and $z^* \in M$, and so, $z_n \in M_\mu$ for large n . Returning to (5.12), we get

$$\begin{aligned} \beta(u_n) &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} \tilde{U}_{\sigma_n z_n}|^2 dx + o_n(1) \\ &= \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} [\chi(\varepsilon_n x + z_n) - \chi(z_n)] |(-\Delta)^{\frac{s}{2}} \tilde{U}_{1,0}|^2 dx + z_n + o_n(1). \end{aligned}$$

Since $\varepsilon_n x + z_n \rightarrow z^* \in M$, we have that $\beta(u_n) = z_n + o_n(1)$ and the sequence $\{z_n\}_{n \in \mathbb{N}}$ is what we desire. Consequently, (5.4) follows and the proof is complete. □

Proof of Theorem 1.1. For any fixed $\mu > 0$, let $\varepsilon = \varepsilon_\mu > 0$ be small. Therefore, $\Phi : [\delta_1, \delta_2] \times M \rightarrow \tilde{\mathcal{N}}_\varepsilon$ given by $\Phi(\delta, z) = \Phi_{\delta,z}$ is well defined and by Lemma 5.2, we see that $\beta(\tilde{\mathcal{N}}_\varepsilon) \subset M_\mu$. By (5.3), we infer to

$$\beta(\Phi_{\delta,z}) = z + o_\delta(1) \quad \text{uniformly in } z \in M.$$

For $\delta \in [\delta_1, \delta_2]$, we denote $\beta(\Phi_{\delta,z}) = z + \vartheta(z)$ for $z \in M$, with $|\vartheta(z)| < \mu/2$ uniformly for $z \in M$. We introduce the map

$$\mathcal{H}(t, (\delta, z)) := (\delta, z + (1-t)\vartheta(z)).$$

It is easy to check that $\mathcal{H} : [0, 1] \times [\delta_1, \delta_2] \times M \rightarrow [\delta_1, \delta_2] \times M_\mu$ is continuous. Clearly,

$$\mathcal{H}(0, (\delta, z)) = (\delta, \beta(\Phi_{\delta,z})), \quad \mathcal{H}(1, (\delta, z)) = (\delta, z),$$

which implies the map

$$H(\delta, z) := (\delta, \beta(\Phi_{\delta,z})) : [\delta_1, \delta_2] \times M \rightarrow [\delta_1, \delta_2] \times M_\mu$$

is homotopic to the inclusion mapping $\text{Id} : [\delta_1, \delta_2] \times M \rightarrow [\delta_1, \delta_2] \times M_\mu$. Thus, applying the Lusternik–Schnirelmann category theory,⁵⁰ we have

$$\text{cat}(\tilde{\mathcal{N}}_\varepsilon) \geq \text{cat}_{[\delta_1, \delta_2] \times M_\mu}([\delta_1, \delta_2] \times M) = \text{cat}_{M_\mu}(M).$$

By Corollaries 3.5 and 4.2, the functional J_ε satisfies the $(\text{PS})_c$ -condition on $\tilde{\mathcal{N}}_\varepsilon$. Hence, the Lusternik–Schnirelmann category theory of critical points shows that J_ε has at least $\text{cat}_{M_\mu}(M)$ solutions. □

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for the careful reading of this paper and valuable comments in order to improve it. X. He was supported by the NSFC (Grant Nos. 12171497, 11771468, and 11971027).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Xiaoming He: Formal analysis (equal); Writing – review & editing (equal). **Yuxi Meng:** Formal analysis (equal); Writing – review & editing (equal). **Patrick Winkert:** Formal analysis (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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