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LOGARITHMIC DOUBLE PHASE PROBLEMS WITH CONVECTION: EXISTENCE AND UNIQUENESS RESULTS

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Abstract. In this paper, we deal with a problem driven by a logarithmic double phase operator with variable exponent and a nonlinearity on the right-hand side which also depends on the gradient of the solution. Under very general structure conditions, following a topological approach based on the use of pseudomonotone operators, we establish the existence of a nontrivial weak solutions for the problem under consideration. Moreover, imposing more restrictive conditions on the nonlinearity, we are able to provide the uniqueness of the solution. Finally, we prove the boundedness, closedness and compactness of the related solution set to our problem.

1. Introduction.

Let $N \geq 2$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper, we focus on the following problem:

$$\begin{aligned} -\operatorname{div} A(u) &= f(x,u,\nabla u) \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\operatorname{div} A$ denotes the logarithmic double phase operator with variable exponents defined for $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ (the appropriate Musielak-Orlicz Sobolev space, see Section 2) by

$$\begin{aligned} \operatorname{div} A(u) := \operatorname{div} \Bigg(& |\nabla u|^{p(x)-2}\nabla u \\ & + \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u\Bigg). \end{aligned} \tag{1.2}$$

Additionally, the exponents as well as the weight function satisfy the following conditions:

(H1) $p, q \in C(\overline{\Omega})$ are such that

$$1 < p(x) < N, \quad p(x) < q(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$$

for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

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The nonlinearity $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies general growth and coercivity structure conditions, see the precise assumptions in (H2) and (H4).

We point out that a first interesting phenomenon in our work is the presence of the logarithmic double phase operator $\operatorname{div} A$ which was recently introduced by Arora-Crespo-Blanco-Winkert in [2]. Such operator is closely connected to the variable exponent double phase operator

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \tag{1.3}$$

which exhibits only a power-law type of growth in each of the addends. We emphasize that the first idea that comes up if one wants to consider other types of growth is to modify power-laws with a logarithm. Operators of type (1.3) have their origin in the study of functionals of the form

$$u \mapsto \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right) \mathrm{d}x. \tag{1.4}$$

Such functionals are characterized by a strongly non-uniform ellipticity at the points where μ vanishes. Therefore, they are useful in order to describe the behavior of inhomogeneous materials whose strengthening properties significantly change with the point. We recall that functionals of type (1.4) with constant exponents p, q have been considered by Marcellini [17] and Zhikov [24] in the study of homogenization and elasticity. It is also relevant in the context of duality theory and of the Lavrentiev gap phenomenon.

Note that the functional

$$\omega \to \int_\Omega \left(|D\omega|^p + \sigma(x)|D\omega|^p \log(e + |D\omega|)\right) \mathrm{d}x, \tag{1.5}$$

with $1 < p < \infty$ and $0 \leq \sigma(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ was considered by Baroni-Colombo-Mingione in [3], in which they prove the local Hölder continuity of the gradient of local minimizers of (1.5). The local Hölder continuity of the gradients of local minimizers of the functional

$$\omega \mapsto \int_\Omega \left(|D\omega| \log(1 + |D\omega|) + \sigma(x)|D\omega|^q\right) \mathrm{d}x \tag{1.6}$$

has been proved by De Filippis-Mingione [4] in case $0 \leq \sigma(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ and $1 < q < 1 + \frac{\alpha}{n}$ with $\Omega \subset \mathbb{R}^n$. Note that (1.6) originates from the following functional with nearly linear growth

$$w \mapsto \int_\Omega |Dw| \log(1 + |Dw|)\, \mathrm{d}x \tag{1.7}$$

studied in the works of Fuchs-Mingione [9] and Marcellini-Papi [18]. Functionals as in (1.7) occur in the theory of plasticity with logarithmic hardening, see the paper of Seregin-Frehse [21] and the monograph of Fuchs-Seregin [10].

Another interesting feature of problem (1.1) is the nonlinearity on the right-hand side which depends on the gradient of the solutions. Such functions are called convection. The presence of the gradient-dependence is crucial in the choice of a working strategy, as it inhibits the use of variational methods. For way of this, we use a topological approach in order to get our results.

To the best of our knowledge this is the first work which combines the logarithmic double phase operator given in (1.2) with a convection term. For existence results on equations with convection term and different involved operators we mention the recent papers of Albalawi-Alharthi-Vetro [1] (parametric $(p(x), q(x))$-Laplace type problems), Faria-Miyagaki-Motreanu [7] (positive solutions for problems with the (p, q)-Laplacian), Figueiredo-Madeira [8] (positive maximal and minimal solutions for nonhomogeneous equations), Gasiński-Papageorgiou [12] (positive smooth solutions for nonhomogeneous equations), Gasiński-Winkert [13] (double phase problems with constant exponents), Marano-Winkert [16] (Neumann (p, q)-problems), Papageorgiou-Vetro-Vetro [19] (singular problems), Vetro [22] ($p(x)$-Kirchhoff type problem) and Vetro-Winkert [23] (parametric anisotropic (p, q)-equations). If $p(x) \equiv$ constant and $\mu(x) \equiv 0$, the logarithmic double phase operator reduces to the classical p-Laplacian. In this direction we mention the recent survey artical about different existence results of Guarnotta-Livrea-Marano [14].

The main objective of this paper is to provide existence results for problem (1.1) under very general structure conditions, see Theorems 3.1 and 3.2 in Section 3. We point out that in order to do this, a key role is played by the surjectivity result for pseudomonotone operators given in Theorem 2.5 in Section 2. In addition, we present uniqueness results for problem (1.1) in the case $p = 2$, see Theorems 4.1 and 4.2 in Section 4. Finally, we study the solution set to the problem (1.1) which turns out to be bounded, closed and compact in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$, see Lemmas 5.1, 5.2, 5.3 and 5.4 in Section 5.

2. **Mathematical background.** The function space framework for the analysis of problem (1.1) is provided by the so-called Musielak-Orlicz Sobolev space. Consequently, in this section we collect some basic facts from the theory of such spaces. One can find these topics in the books of Diening-Harjulehto-Hästö-Růžička [5] and Harjulehto-Hästö [15]. Also, we refer to the recent paper of Arora-Crespo-Blanco-Winkert [2].

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $r \in C(\overline{\Omega})$ be such that $r(x) > 1$ for all $x \in \overline{\Omega}$. Here, we put

$$r^- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ := \max_{x\in\overline{\Omega}} r(x)$$

and we write $r'(\cdot)$ for the conjugate variable exponent to $r(\cdot)$, that is, we have

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Let $M(\Omega)$ be the set of all measurable functions $u\colon \Omega \to \mathbb{R}$. Then, we denote by $L^{r(\cdot)}(\Omega)$ the usual variable exponent Lebesgue space, that is,

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega)\colon \varrho_{r(\cdot)}(u) := \int_\Omega |u|^{r(x)} \,\mathrm{d}x < +\infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf \left\{ \alpha > 0\colon \varrho_{r(\cdot)}\left(\frac{u}{\alpha}\right) \leq 1 \right\}.$$

Such norm makes $L^{r(\cdot)}(\Omega)$ a separable, uniformly convex and hence reflexive Banach space with dual space given by $L^{r'(\cdot)}(\Omega)$. In addition, we emphasize that the norm $\|\cdot\|_{r(\cdot)}$ and its modular $\varrho_{r(\cdot)}$ are related as follows.

Proposition 2.1. *Let $r \in C(\overline{\Omega})$ be such that $r(x) > 1$ for all $x \in \overline{\Omega}$. Also, let $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^{r(\cdot)}(\Omega)$. Then, we have:*

(i) $\|u\|_{r(\cdot)} = a \Leftrightarrow \varrho_{r(\cdot)}\left(\frac{u}{a}\right) = 1$ $(a > 0)$;

(ii) $\|u\|_{r(\cdot)} < 1\,(\textit{resp.} = 1, > 1) \Leftrightarrow \varrho_{r(\cdot)}(u) < 1$ (*resp.* $= 1, > 1$);

(iii) $\|u\|_{r(\cdot)} < 1 \Rightarrow \|u\|_{r(\cdot)}^{r^+} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r^-}$;

(iv) $\|u\|_{r(\cdot)} > 1 \Rightarrow \|u\|_{r(\cdot)}^{r^-} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r^+}$;

(v) $\|u_n\|_{r(\cdot)} \to 0\,(\textit{resp.} \to +\infty) \Leftrightarrow \varrho_{r(\cdot)}(u_n) \to 0\,(\textit{resp.} \to +\infty)$.

Also, we know that the Hölder-type inequality

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r^-} + \frac{1}{(r')^-}\right] \|u\|_{r(\cdot)}\ \|v\|_{r'(\cdot)}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. Further, given $r_1, r_2 \in C(\overline{\Omega})$, with $1 < r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, we have that the embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$$

is continuous. Using the Lebesgue space $L^{r(\cdot)}(\Omega)$, we can define the variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$ by

$$W^{1,r(\cdot)}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) \colon |\nabla u| \in L^{r(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

where $\|\nabla u\|_{r(\cdot)} = \|\,|\nabla u|\,\|_{r(\cdot)}$. Moreover, we write $W_0^{1,r(\cdot)}(\Omega)$ by the completion of $C_0^\infty(\Omega)$ in $W^{1,r(\cdot)}(\Omega)$. Recall that $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ are uniformly convex, separable and reflexive Banach spaces and further we know that the Poincaré inequality is valid for $W_0^{1,r(\cdot)}(\Omega)$. So, we can equip $W_0^{1,r(\cdot)}(\Omega)$ with the equivalent norm given by

$$\|u\|_{1,r(\cdot),0} := \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

Let now hypothesis (H1) be satisfied. Then, we consider the nonlinear function $\mathcal{H}_{\log} \colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \log(e+t)$$

for all $x \in \Omega$ and for all $t \geq 0$, where e stands for Euler's number. We point out that $\mathcal{H}_{\log}(\cdot,t)$ is measurable for all $t \geq 0$, $\mathcal{H}_{\log}(x,0) = 0$ and further $\mathcal{H}_{\log}(x,t) > 0$ for all $t > 0$. Moreover, we can easily check that $\mathcal{H}_{\log}$ satisfies the Δ_2-condition. Consequently, we have that the corresponding Musielak-Orlicz space $L^{\mathcal{H}_{\log}}(\Omega)$ is given by

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{u \in M(\Omega) \colon \varrho_{\mathcal{H}_{\log}}(u) < +\infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf\left\{\beta > 0 \colon \varrho_{\mathcal{H}_{\log}}\left(\frac{u}{\beta}\right) \leq 1\right\},$$

where $\varrho_{\mathcal{H}_{\log}}(\cdot)$ denotes the associated modular defined by

$$\varrho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x = \int_\Omega \Big(|u|^{p(x)} + \mu(x)|u|^{q(x)} \log(e+|u|)\Big)\,\mathrm{d}x.$$

We remark that $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable, reflexive Banach space and in addition the modular $\varrho_{\mathcal{H}_{\log}}$ is closely related to the norm $\|\cdot\|_{\mathcal{H}_{\log}}$, see [2, Proposition 3.4].

Proposition 2.2. *Let hypothesis (H1) be satisfied. Then the following hold:*

(i) $\|u\|_{\mathcal{H}_{\log}} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\varrho_{\mathcal{H}_{\log}}(u) < 1$ *(resp.* $> 1, = 1$*);*

(ii) $\min\{\|u\|_{\mathcal{H}_{\log}}^{p^-}, \|u\|_{\mathcal{H}_{\log}}^{q^+ + k}\} \leq \varrho_{\mathcal{H}_{\log}}(u) \leq \max\{\|u\|_{\mathcal{H}_{\log}}^{p^-}, \|u\|_{\mathcal{H}_{\log}}^{q^+ + k}\}$*, where* $k = \frac{e}{e+t_0}$ *with* t_0 *being the only positive solution of* $t_0 = e\log(e + t_0)$*;*

(iii) $\|u\|_{\mathcal{H}_{\log}} \to 0$ *if and only if* $\varrho_{\mathcal{H}_{\log}}(u) \to 0$*;*

(iv) $\|u\|_{\mathcal{H}_{\log}} \to +\infty$ *if and only if* $\varrho_{\mathcal{H}_{\log}}(u) \to +\infty$*.*

Next, using the space $L^{\mathcal{H}_{\log}}(\Omega)$, we introduce the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}_{\log}}(\Omega) \colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}},$$

where as usual $\|\nabla u\|_{\mathcal{H}_{\log}} := \|\,|\nabla u|\,\|_{\mathcal{H}_{\log}}$. Also, we write $W^{1,\mathcal{H}_{\log}}_0(\Omega)$ by the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$. From Arora-Crespo-Blanco-Winkert [2, Propositions 3.6, 3.7 and 3.9] we see that $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $W^{1,\mathcal{H}_{\log}}_0(\Omega)$ are separable, reflexive Banach spaces satisfying the following embeddings.

Proposition 2.3. *Let hypothesis (H1) be satisfied. Then the following hold:*

(i) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ *and* $W^{1,\mathcal{H}_{\log}}_0(\Omega) \hookrightarrow W^{1,p(\cdot)}_0(\Omega)$ *are continuous;*

(ii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *and* $W^{1,\mathcal{H}_{\log}}_0(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *are compact for* $m \in C(\overline{\Omega})$ *with* $1 \leq m(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*;*

(iii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\mathcal{H}_{\log}}(\Omega)$ *is compact and there exists a constant* $\bar{c} > 0$ *such that*

$$\|u\|_{\mathcal{H}_{\log}} \leq \bar{c}\, \|\nabla u\|_{\mathcal{H}_{\log}} \quad \text{for all } u \in W^{1,\mathcal{H}_{\log}}_0(\Omega).$$

Thus, according to Proposition 2.3 (iii), we can take in $W^{1,\mathcal{H}_{\log}}_0(\Omega)$ the equivalent norm given by

$$\|u\| := \|\nabla u\|_{\mathcal{H}_{\log}} \quad \text{for all } u \in W^{1,\mathcal{H}_{\log}}_0(\Omega).$$

In the sequel, given a Banach space X, we denote by X^* its dual space and we use $\langle\,\cdot\,,\cdot\,\rangle$ in order to indicate the dual pairing between X and X^*. Now, let $\mathcal{A}\colon W^{1,\mathcal{H}_{\log}}_0(\Omega) \to [W^{1,\mathcal{H}_{\log}}_0(\Omega)]^*$ be the nonlinear operator defined by

$$\begin{aligned}\langle \mathcal{A}(u), v\rangle := \int_\Omega & |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\ &+ \mu(x)\left[\log(e + |\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \end{aligned} \tag{2.1}$$

for all $u, v \in W^{1,\mathcal{H}_{\log}}(\Omega)$. Such operator has remarkable properties that we summarize in the next proposition, see Arora-Crespo-Blanco-Winkert [2, Theorem 4.4].

Proposition 2.4. *Let hypothesis (H1) be satisfied. Then, the operator* $\mathcal{A}$ *is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone, coercive, that is,*

$$\lim_{\|u\| \to +\infty} \frac{\langle \mathcal{A}(u), u\rangle}{\|u\|} = +\infty,$$

and also of (S_+)*-type, that is,*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle \mathcal{A}(u_n), u_n - u\rangle \le 0$$

imply $u_n \to u$ *in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

We conclude this section by recalling some properties of pseudomonotone operators which will be needed later. An operator $\mathcal{B}\colon X \to X^*$ with X being a reflexive Banach space, is called pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle \mathcal{B}u_n, u_n - u\rangle \le 0$ imply

$$\liminf_{n\to\infty} \langle \mathcal{B}u_n, u_n - v\rangle \ge \langle \mathcal{B}u, u - v\rangle \quad \text{for all } v \in X.$$

If the operator $\mathcal{B}\colon X \to X^*$ is bounded, then the definition of pseudomonotonicity is equivalent to $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle \mathcal{B}u_n, u_n - u\rangle \le 0$ imply $\mathcal{B}u_n \rightharpoonup \mathcal{B}u$ and $\langle \mathcal{B}u_n, u_n\rangle \to \langle \mathcal{B}u, u\rangle$. We are going to use this equivalent condition for bounded operators.

We also mention the following notable surjectivity result for pseudomonotone operators, see, for example, Papageorgiou-Winkert [20, Theorem 6.1.57]. In the next section, it will be a fundamental tool in order to prove the existence of solutions for problem (1.1).

Theorem 2.5. *Let X be a real, reflexive Banach space, let $\mathcal{B}\colon X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then, a solution of the equation $\mathcal{B}u = b$ exists.*

Finally, we recall the following lemma by Gasiński–Papageorgiou [11, Lemma 2.2.27].

Lemma 2.6. *Let X and Y be Banach spaces such that $X \subset Y$, the embedding $X \hookrightarrow Y$ is continuous and X is dense in Y. Then, we have that:*

(i) *the embedding $Y^* \hookrightarrow X^*$ is continuous;*
(ii) *if X is reflexive, then Y^* is dense in X^*.*

3. **Existence results.** In this section, we present our existence results. Precisely, we prove the existence of at least one weak solution for problem (1.1) under very general structure conditions. We point out that our strategy is based on the use of the properties of pseudomonotone operators. In particular, a key role in order to obtain these results is played by Theorem 2.5.

First, we recall that $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x\\
&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&= \int_\Omega f(x,u,\nabla u)\, v\,\mathrm{d}x
\end{aligned} \tag{3.1}$$

is satisfied for all $v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

We suppose the following assumptions on the reaction term f.

(H2) $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(\cdot,0,0)\neq 0$ and the following hold:

(i) there exists $\gamma_1 \in L^{\ell'(\cdot)}(\Omega)$ with $1 < \ell(x) < p^*(x)$ for all $x \in \overline{\Omega}$ such that

$$|f(x,t,\xi)| \leq c_1 \left(\gamma_1(x) + |t|^{\ell(x)-1} + |\xi|^{\frac{p(x)}{\ell'(x)}}\right)$$

for some $c_1 > 0$, for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$;

(ii) there exist $\gamma_2 \in L^1(\Omega)$ and $\tau \in C(\overline{\Omega})$ with $1 < \tau(x) \leq \tau^+ < p^-$ such that

$$f(x,t,\xi)\, t \leq \gamma_2(x) + c_2|t|^{\tau(x)} + c_3|\xi|^{p(x)}$$

for some $c_2 > 0$ and $0 < c_3 < 1$, for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

Now, we are in the position to present our first existence result.

Theorem 3.1. *Let hypotheses (H1) and (H2) be satisfied. Then, problem (1.1) has at least one nontrivial weak solution $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. We are going to apply Theorem 2.5 and introduce first a suitable operator $\mathcal{B}\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to [W_0^{1,\mathcal{H}_{\log}}(\Omega)]^*$ as in the theorem. For this purpose, let $\ell \in C(\overline{\Omega})$ be as given in hypothesis (H2)(i). According to Proposition 2.3 (ii), we have the compact embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\ell(\cdot)}(\Omega)$. Next, let $\mathcal{N}_f^*\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \subset L^{\ell(\cdot)}(\Omega) \to L^{\ell'(\cdot)}(\Omega)$ be the Nemytskij operator associated to f, that is, $\mathcal{N}_f^*$ is such that

$$\mathcal{N}_f^*(u) := f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

Also, let $i^*\colon L^{\ell'(\cdot)}(\Omega) \to [W_0^{1,\mathcal{H}_{\log}}(\Omega)]^*$ be the adjoint operator corresponding to the embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\ell(\cdot)}(\Omega)$. We emphasize that, due to hypothesis (H2)(i), the operator $\mathcal{N}_f^*$ is well-defined, bounded and continuous. In addition, Lemma 2.6 guarantees that i^* is continuous. Keeping this in mind, we conclude that the operator $\mathcal{N}_f := i^* \circ \mathcal{N}_f^*$ is bounded and continuous. Now, we focus on the operator $\mathcal{B}\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to [W_0^{1,\mathcal{H}_{\log}}(\Omega)]^*$ defined by

$$\mathcal{B}(u) := \mathcal{A}(u) - \mathcal{N}_f(u), \tag{3.2}$$

where $\mathcal{A}$ is as given in (2.1). As $\mathcal{N}_f$ is bounded and continuous, according to Proposition 2.4, we deduce that $\mathcal{B}$ is bounded and continuous as well. Therefore, in order to apply Theorem 2.5, we only need to prove that $\mathcal{B}$ is pseudomonotone and coercive.

We start showing the pseudomonotonicity of $\mathcal{B}$. To this end, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle \mathcal{B}(u_n), u_n - u\rangle \leq 0. \tag{3.3}$$

As $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$, we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in its norm. Hence, we deduce that the sequence $\{\mathcal{N}_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Also, as $\ell(x) < p^*(x)$ for all $x \in \overline{\Omega}$ (see hypothesis (H1)), with a view to Proposition 2.3 (ii), we see that

$$u_n \to u \quad \text{in } L^{\ell(\cdot)}(\Omega).$$

Now, using this fact along with the Hölder's inequality, we derive that

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n - u)\,\mathrm{d}x\right| \leq 2\|\mathcal{N}_f^*(u_n)\|_{\ell'(\cdot)}\|u_n - u\|_{\ell(\cdot)}$$

$$\begin{aligned}&\leq 2 \sup_{n\in\mathbb{N}} \|\mathcal{N}_f^*(u_n)\|_{\ell'(\cdot)}\|u_n-u\|_{\ell(\cdot)}\\&\to 0 \quad \text{as} \quad n\to+\infty.\end{aligned} \tag{3.4}$$

This, according to (3.3), gives

$$\limsup_{n\to+\infty} \langle \mathcal{B}(u_n),(u_n-u)\rangle = \limsup_{n\to+\infty} \langle \mathcal{A}(u_n),(u_n-u)\rangle \leq 0.$$

Taking into account that the operator $\mathcal{A}$ is of (S_+)-type (see Proposition 2.4), we conclude that $u_n \to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. This along with the fact that the operator $\mathcal{B}$ is continuous yields

$$\mathcal{B}(u_n)\to\mathcal{B}(u) \quad \text{in } [W_0^{1,\mathcal{H}_{\log}}(\Omega)]^*.$$

Therefore, we have shown that $\mathcal{B}$ is a pseudomonotone operator.

Next, we prove that $\mathcal{B}$ is coercive, that is, we are going to show that

$$\lim_{\|u\|\to+\infty} \frac{\langle\mathcal{B}(u),u\rangle}{\|u\|} = +\infty.$$

Using hypothesis (H2)(ii) we get that

$$\begin{aligned}\langle\mathcal{B}(u),u\rangle &= \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x\\&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)}\,\mathrm{d}x\\&\quad - \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x\\&\geq \varrho_{\mathcal{H}_{\log}}(\nabla u) - \int_\Omega \gamma_2(x)\,\mathrm{d}x - c_2\int_\Omega |u|^{\tau(x)}\,\mathrm{d}x - c_3\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x\\&\geq (1-c_3)\,\varrho_{\mathcal{H}_{\log}}(\nabla u) - \int_\Omega \gamma_2(x)\,\mathrm{d}x - c_2\int_\Omega |u|^{\tau(x)}\,\mathrm{d}x.\end{aligned}$$

Now, we recall that Proposition 2.2 (ii) gives us

$$\varrho_{\mathcal{H}_{\log}}(\nabla u) \geq \min\{\|\nabla u\|_{\mathcal{H}_{\log}}^{p^-}, \|\nabla u\|_{\mathcal{H}_{\log}}^{q^++1}\} = \min\{\|u\|^{p^-}, \|u\|^{q^++1}\}.$$

Also, according to Proposition 2.1 (iii), (iv), we have that

$$\varrho_{\tau(\cdot)} \leq \max\{\|u\|_{\tau(\cdot)}^{\tau^-}, \|u\|_{\tau(\cdot)}^{\tau^+}\}.$$

Further, since $1<\tau(x)<p^*(x)$ (see hypothesis (H1)), from Proposition (2.3) (ii) we see that the embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\tau(\cdot)}(\Omega)$ is compact. Keeping this in mind, for $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ with $\|u\|>1$, we have

$$\begin{aligned}\langle\mathcal{B}(u),u\rangle &\geq (1-c_3)\|u\|^{p^-} - \|\gamma_2\|_1 - c_2\max\left\{\|u\|_{\tau(\cdot)}^{\tau^-}, \|u\|_{\tau(\cdot)}^{\tau^+}\right\}\\&\geq (1-c_3)\|u\|^{p^-} - \|\gamma_2\|_1 - C\,\|u\|^{\tau^+}\end{aligned} \tag{3.5}$$

for some $C>0$. Hence, as $\tau^+<p^-$ and $0<c_3<1$ due to hypothesis (H2)(ii), we conclude that $\mathcal{B}$ is coercive. Then, we can use Theorem 2.5 and deduce that $\mathcal{B}$ is surjective. Consequently, there exists $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ so that $\mathcal{B}(u)=0$. According to (3.1) and (3.2), the function u turns out to be a nontrivial weak solution of problem (1.1). Therefore, the claim is proved. $\square$

Next, we consider the following additional monotonicity condition on p.

(H3) There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ so that for all $x \in \Omega$ the function $p_x \colon \Omega_x \to \mathbb{R}$ defined by

$$p_x(z) := p(x + z\xi_0)$$

is monotone, whereby $\Omega_x = \{z \in \mathbb{R} \colon x + z\,\xi_0 \in \Omega\}$.

A function $p \in C(\overline{\Omega})$ satisfying (H3) can be given, for example, by

$$p((x_1, x_2)) := 2 + x_2 \quad \text{for all } (x_1, x_2) \in \overline{\Omega},$$

where Ω is a bounded domain of $\mathbb{R}^2$ contained in the first quadrant.

Note that hypothesis (H3), according to Fan–Zhang–Zhao [6, Theorem 3.3], guarantees that

$$\tilde{\lambda} := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0. \tag{3.6}$$

Then, using $\tilde{\lambda} > 0$, we can make the following assumption on the Carathéodory function $f \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.

(H4) There exists $\bar{\gamma}_2 \in L^1(\Omega)$ such that

$$f(x,t,\xi)\,t \le \bar{\gamma}_2(x) + \bar{c}_2\,\tilde{\lambda}\,|t|^{p(x)} + \bar{c}_3|\xi|^{p(x)}$$

for some $\bar{c}_2, \bar{c}_3 > 0$, for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$ with

$$\bar{c}_2 + \bar{c}_3 < 1.$$

Now, we are ready to establish our second existence result.

Theorem 3.2. *Let hypotheses (H1), (H2)(i), (H3) and (H4) be satisfied. Then, problem (1.1) has at least one nontrivial weak solution in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. As done in proof of Theorem 3.1, we consider the operator $\mathcal{B} \colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to [W_0^{1,\mathcal{H}_{\log}}(\Omega)]^*$ defined by

$$\mathcal{B}(u) := \mathcal{A}(u) - \mathcal{N}_f(u),$$

with $\mathcal{A}$ as given in (2.1). Then, following the proof of Theorem 3.1, we are able to derive that such operator is bounded and pseudomonotone.

Now, in order to show that $\mathcal{B}$ is also coercive, we recall that, due to (3.6), we know that

$$\tilde{\lambda} \int_\Omega |u|^{p(x)}\,\mathrm{d}x \le \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Using this fact along with hypothesis (H4), we get that

$$\begin{aligned}
\langle \mathcal{B}(u), u\rangle &= \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e + |\nabla u|) + \frac{|\nabla u|}{q(x)(e + |\nabla u|)}\right]|\nabla u|^{q(x)}\,\mathrm{d}x \\
&\quad - \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x \\
&\ge \varrho_{\mathcal{H}_{\log}}(\nabla u) - \int_\Omega \bar{\gamma}_2(x)\,\mathrm{d}x - \bar{c}_2\,\tilde{\lambda}\int_\Omega |u|^{p(x)}\,\mathrm{d}x - \bar{c}_3\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\
&\ge (1 - \bar{c}_3)\,\varrho_{\mathcal{H}_{\log}}(\nabla u) - \|\bar{\gamma}_2(x)\|_1 - \bar{c}_2\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x
\end{aligned}$$

$$= (1 - \bar{c}_2 - \bar{c}_3)\, \varrho_{\mathcal{H}_{\log}}(\nabla u) - \|\bar{\gamma}_2(x)\|_1. \tag{3.7}$$

Hence, taking into account that Proposition 2.2 (v) guarantees that

$$\varrho_{\mathcal{H}_{\log}}(\nabla u) \to +\infty \quad \text{as} \quad \|u\| \to +\infty$$

and $(1 - \bar{c}_2 - \bar{c}_3) > 0$ due to hypothesis (H4), we conclude that $\mathcal{B}$ is coercive. So, we can now apply Theorem 2.5 which shows the assertion of the theorem. $\square$

4. **Uniqueness results.** In the sequel, we denote by $\lambda_{1,2}$ the first eigenvalue of the Dirichlet eigenvalue problem for the Laplace differential operator. We recall that such eigenvalue is positive, isolated, simple and in addition it can be variationally characterized through

$$\lambda_{1,2} = \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_2^2} \colon u \in W_0^{1,2}(\Omega),\ u \neq 0 \right\}. \tag{4.1}$$

Our aim is to provide uniqueness results for problem (1.1). For this purpose, we need stronger assumptions on the reaction term f. Precisely, we suppose the following additional conditions on f:

(H5) (i) there exists a constant $c_4 > 0$ such that

$$(f(x,t,\xi) - f(x,s,\xi))(t-s) \le \lambda_{1,2}\, c_4\, |t-s|^2$$

for a.a. $x \in \Omega$, for all $t, s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$;

(ii) there exists $\gamma_3 \in L^1(\Omega)$ such that the function

$$\xi \to f(x,t,\xi) - \gamma_3(x)$$

is linear for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$. In addition

$$|f(x,t,\xi) - \gamma_3(x)| \le (\lambda_{1,2})^{\frac{1}{2}}\, c_5\, |\xi|$$

for some $c_5 > 0$, for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

Now, our first uniqueness result reads as follows.

Theorem 4.1. *Let hypotheses (H1), (H2) and (H5) be satisfied and let $p(x) = 2$ for all $x \in \overline{\Omega}$. Also, we suppose that*

$$c_4 + c_5 < 1 \tag{4.2}$$

where $c_4, c_5 > 0$ are as given in hypothesis (H5). Then, problem (1.1) admits a unique weak solution.

Proof. We recall that Theorem 3.1 guarantees the existence of at least one nontrivial weak solution for problem (1.1). Thus, we suppose that $u_1, u_2 \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ are two weak solutions for problem (1.1). According to (3.1), taking as test function $v = u_1 - u_2$, we see that

$$\begin{aligned}
&\int_\Omega \nabla u_1 \cdot \nabla(u_1 - u_2)\,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e + |\nabla u_1|) + \frac{|\nabla u_1|}{q(x)(e + |\nabla u_1|)}\right] |\nabla u_1|^{q(x)-2} \nabla u_1 \cdot \nabla(u_1 - u_2)\,\mathrm{d}x \\
&= \int_\Omega f(x, u_1, \nabla u_1)(u_1 - u_2)\,\mathrm{d}x
\end{aligned}$$

and analogously

$$
\begin{aligned}
&\int_\Omega \nabla u_2 \cdot \nabla(u_1-u_2)\,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u_2|)+\frac{|\nabla u_2|}{q(x)(e+|\nabla u_2|)}\right]|\nabla u_2|^{q(x)-2}\nabla u_2\cdot\nabla(u_1-u_2)\,\mathrm{d}x \\
&= \int_\Omega f(x,u_2,\nabla u_2)(u_1-u_2)\,\mathrm{d}x.
\end{aligned}
$$

Then, subtracting these equations we get

$$
\begin{aligned}
&\int_\Omega |\nabla(u_1-u_2)|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\Bigg(\left[\log(e+|\nabla u_1|)+\frac{|\nabla u_1|}{q(x)(e+|\nabla u_1|)}\right]|\nabla u_1|^{q(x)-2}\nabla u_1 \\
&\quad - \left[\log(e+|\nabla u_2|)+\frac{|\nabla u_2|}{q(x)(e+|\nabla u_2|)}\right]|\nabla u_2|^{q(x)-2}\nabla u_2\Bigg)\cdot\nabla(u_1-u_2)\,\mathrm{d}x \\
&= \int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_1))(u_1-u_2)\,\mathrm{d}x \\
&\quad + \int_\Omega (f(x,u_2,\nabla u_1)-f(x,u_2,\nabla u_2))(u_1-u_2)\,\mathrm{d}x.
\end{aligned} \tag{4.3}
$$

Now, using hypotheses (H5) along with (4.1) and Hölder's inequality, according to Proposition 2.3 (i), we derive the following estimate for the right-hand side of (4.3)

$$
\begin{aligned}
&\int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_1))(u_1-u_2)\,\mathrm{d}x \\
&\quad + \int_\Omega (f(x,u_2,\nabla u_1)-f(x,u_2,\nabla u_2))(u_1-u_2)\,\mathrm{d}x \\
&\le \lambda_{1,2}\, c_4\|u_1-u_2\|_2^2 + \int_\Omega \left(f\left(x,u_2,\nabla\left(\frac{1}{2}(u_1-u_2)^2\right)\right)-\gamma_3(x)\right)\mathrm{d}x \\
&\le \lambda_{1,2}\, c_4\|u_1-u_2\|_2^2 + (\lambda_{1,2})^{\frac{1}{2}}\, c_5 \int_\Omega |u_1-u_2|\,|\nabla(u_1-u_2)|\,\mathrm{d}x \\
&\le \lambda_{1,2}\, c_4\|u_1-u_2\|_2^2 + (\lambda_{1,2})^{\frac{1}{2}}\, c_5\,\|u_1-u_2\|_2\,\|\nabla(u_1-u_2)\|_2 \\
&\le (c_4+c_5)\,\|\nabla(u_1-u_2)\|_2^2.
\end{aligned}
$$

Consequently, taking into account that the second term on the left-hand side of (4.3) is nonnegative (see Lemma 4.2 in Arora-Crespo-Blanco-Winkert [2]), we obtain

$$
\|\nabla(u_1-u_2)\|_2^2 = \int_\Omega |\nabla(u_1-u_2)|^2\,\mathrm{d}x \le (c_4+c_5)\,\|\nabla(u_1-u_2)\|_2^2
$$

which implies

$$
[1-(c_4+c_5)]\,\|\nabla(u_1-u_2)\|_2^2 \le 0.
$$

As $1-(c_4+c_5)>0$ due to (4.2), we conclude that $u_1=u_2$. □

We point out that with a proof completely analogous to the one of Theorem 4.1, except that now we use Theorem 3.2 instead of Theorem 3.1, we get the following additional uniqueness result.

Theorem 4.2. *Let hypotheses (H1), (H2)(i) and (H3)–(H5) be satisfied. Also, let $p(x) = 2$ for all $x \in \overline{\Omega}$ and suppose that (4.2) holds. Then, problem (1.1) admits a unique weak solution.*

5. **Properties of the solution set.** In this section, we focus on the set of the solutions of problem (1.1) which we denote by $\mathcal{S}$. With a view to Theorems 3.1 and 3.2 we know that such a solution set is nonempty whenever hypotheses (H1), (H2) or hypotheses (H1), (H2)(i), (H3), (H4) are verified. We will therefore now examine some of the properties of $\mathcal{S}$.

We start with the following result.

Lemma 5.1. *Let hypotheses (H1) and (H2) be satisfied. Then, $\mathcal{S}$ is a bounded subset of $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. First, we point out that from Theorem 3.1 we know that the set $\mathcal{S}$ is nonempty. Thus, in order to prove the claim we argue by contradiction. Therefore, we suppose there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{5.1}$$

As $u_n \in \mathcal{S}$ for all $n \in \mathbb{N}$, from (3.1) taking as test function $v = u_n$, we see that

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(x)(e+|\nabla u_n|)}\right]|\nabla u_n|^{q(x)}\,\mathrm{d}x\\ &- \int_\Omega f(x,u_n,\nabla u_n)u_n\,\mathrm{d}x = 0.\end{aligned}$$

This along with hypothesis (H2)(ii), Proposition 2.2 (ii), Proposition 2.1 (iii), (iv) and Proposition 2.3 (ii) lead to

$$(1-c_3)\|u_n\|^{p^-} - \|\gamma_2\|_1 - C\,\|u_n\|^{\tau^+} \le 0, \tag{5.2}$$

where c_3 and γ_2 are as given in hypothesis (H2)(ii) and C is a positive constant (see (3.5)). Taking into account that $0 < c_3 < 1$ and $\tau^+ < p^-$ (see hypothesis (H2)(ii)), according to (5.1), we have

$$(1-c_3)\|u_n\|^{p^-} - \|\gamma_2\|_1 - C\,\|u_n\|^{\tau^+} \to +\infty \quad \text{as } n \to +\infty$$

and this is clearly in contradiction with (5.2). Hence, the claim holds. □

Next, we show that $\mathcal{S}$ is closed in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

Lemma 5.2. *Let hypotheses (H1) and (H2) be satisfied. Then, $\mathcal{S}$ is a closed subset of $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $u_n \to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. As $u_n \in \mathcal{S}$ for all $n \in \mathbb{N}$, we know that

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla v\,\mathrm{d}x\\ &\quad + \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(x)(e+|\nabla u_n|)}\right]|\nabla u_n|^{q(x)-2}\nabla u_n \cdot \nabla v\,\mathrm{d}x\\ &= \int_\Omega f(x,u_n,\nabla u_n)\,v\,\mathrm{d}x\end{aligned} \tag{5.3}$$

is satisfied for all $v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Then, taking into account that the operators $\mathcal{A}$ and $\mathcal{N}_f$ are continuous (see Proposition 2.4 and proof of Theorem 3.1, respectively), passing to limit as $n \to +\infty$ from (5.3) we get that

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&= \int_\Omega f(x,u,\nabla u)\, v \,\mathrm{d}x
\end{aligned}$$

holds for all $v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. This guarantees that $u \in \mathcal{S}$ and consequently we conclude that $\mathcal{S}$ is closed in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. □

Finally, we prove that $\mathcal{S}$ is also compact.

Proposition 5.3. *Let hypotheses (H1) and (H2) be satisfied. Then, $\mathcal{S}$ is a compact subset of $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. We need to show that any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ admits a subsequence converging to some $u \in \mathcal{S}$. According to Lemma 5.1, we know that $\mathcal{S}$ is a bounded subset of $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and hence we have that any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ is bounded in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Therefore, there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \text{ for some } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

Further, if $\ell \in C(\overline{\Omega})$ is as given in hypothesis (H2)(i), according to Proposition 2.3 (ii), we know that

$$u_n \to u \quad \text{in } L^{\ell(\cdot)}(\Omega).$$

Using this fact along with Hölder's inequality we derive that (3.4) holds.

Next, we remark that $u_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Thus, choosing as test function $v = u_n - u$ in (5.3), we get

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla (u_n-u) \,\mathrm{d}x \\
&\quad + \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(x)(e+|\nabla u_n|)}\right]|\nabla u_n|^{q(x)-2}\nabla u_n \cdot \nabla (u_n-u) \,\mathrm{d}x \\
&= \int_\Omega f(x,u_n,\nabla u_n)\,(u_n-u) \,\mathrm{d}x.
\end{aligned}$$

Now, passing to limit as $n \to +\infty$ in the previous equality, according to (3.4), we infer that

$$\limsup_{n\to+\infty} \langle \mathcal{A}(u_n), (u_n-u)\rangle \le 0.$$

Then, the (S_+)-property of the operator $\mathcal{A}$ (see Proposition 2.4) implies $u_n \to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. As $\mathcal{S}$ is closed due to Lemma 5.2, we conclude that $\mathcal{S}$ is compact. □

By similar arguments we can derive in addition the following result.

Lemma 5.4. *Let hypotheses (H1), (H2)(i), (H3) and (H4) be satisfied. Then, $\mathcal{S}$ is a bounded, closed and compact subset of $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof. In order to show that $\mathcal{S}$ is a bounded set, we can reason as in the proof of Lemma 5.1 except that now we use Theorem 3.2 instead of Theorem 3.1 and hypotheses (H3)-(H4) instead of hypothesis (H2)(ii), Propositions 2.2 (ii), 2.1 (iii), (iv) and 2.3 (ii). In this way, we are able to get that

$$(1-\bar{c}_2-\bar{c}_3)\,\varrho_{\mathcal{H}_{\log}}(\nabla u_n)-\|\bar{\gamma}_2(x)\|_1 \leq 0, \tag{5.4}$$

where $\bar{c}_2, \bar{c}_3$ and $\bar{\gamma}_2$ are as given in hypothesis (H4) (see (3.7)). Now, as $(1-\bar{c}_2-\bar{c}_3)>0$ due to hypothesis (H4), from (5.1) with view to Proposition 2.2 (v), we deduce that

$$(1-\bar{c}_2-\bar{c}_3)\,\varrho_{\mathcal{H}_{\log}}(\nabla u_n)-\|\bar{\gamma}_2(x)\|_1 \to +\infty \quad \text{as} \quad n\to+\infty.$$

This contradict (5.4) and hence we conclude that $\mathcal{S}$ is bounded.

Next, the closedness of $\mathcal{S}$ follows with a proof completely analogous to the one of Lemma 5.2. Finally, as $\mathcal{S}$ is bounded and closed, proceeding in a similar way as in the proof of Lemma 5.3, we can see that $\mathcal{S}$ is compact as well. □

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