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Asymmetric $(p, 2)$-equations, superlinear at $+\infty$, resonant at $-\infty$

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We consider a Dirichlet problem driven by the sum of a p-Laplacian and a Laplacian (known as a $(p, 2)$-equation) and with a nonlinearity which exhibits asymmetric behavior as $s \to \pm\infty$. More precisely, it is $(p-1)$-superlinear near $+\infty$ (but without satisfying the Ambrosetti–Rabinowitz condition) and it is $(p-1)$-sublinear near $-\infty$ and possibly resonant with respect to the principal eigenvalue of the p-Laplacian. Using variational tools along with Morse theory we prove a multiplicity theorem generating five nontrivial solutions (one is negative, two are positive, one is nodal and for the fifth we do not have any information about its sign).

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous Dirichlet problem (a $(p, 2)$-equation)

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$$
\begin{aligned}
-\Delta_p u - \Delta u &= f(x,u) \qquad \text{in } \Omega,\\
u &= 0 \qquad\qquad\quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $2 < p < \infty$ and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function which is C^1 in the second variable, that is $f(x,\cdot) \in C^1(\mathbb{R})$ for a.a. $x \in \Omega$. Moreover, we assume that $f(x,\cdot)$ exhibits an asymmetric behavior as $s \to \pm\infty$. To be more precise, we suppose that $f(x,\cdot)$ is $(p-1)$-sublinear as $s \to -\infty$ and resonance is possible with respect to the principal eigenvalue of the Dirichlet p-Laplacian. On the other hand, $f(x,\cdot)$ is $(p-1)$-superlinear as $s \to +\infty$ but without satisfying the usual Ambrosetti–Rabinowitz condition (AR-condition for short). Instead we use a weaker condition which incorporates in our framework also superlinear nonlinearities with slower growth near $+\infty$ which fail to satisfy the AR-condition.

Our work here was motivated by the recent paper of Recova–Rumbos [38] who study a semilinear Dirichlet problem driven by the Laplacian that has a right-hand side nonlinearity $f \in C^1\left(\overline{\Omega} \times \mathbb{R}\right)$ which has an analogous asymmetric behavior as $s \to \pm\infty$. However, the superlinearity in the positive direction is expressed using the AR-condition and the overall hypotheses on f are more restrictive. Related results can also be found in the works of Arcoya–Villegas [6], Cuesta–de Figueiredo–Srikanth [12], de-Figueiredo–Ruf [16], de Paiva–Presoto [27], Motreanu–Motreanu–Papageorgiou [23,24], Perera [35] and Recova–Rumbos [37].

We point out that $(p,2)$-equations arise in many physical applications. We mention the works of Benci–D'Avenia–Fortunato–Pisani [7] (quantum physics) and Cherfils–Il'yasov [9] (plasma physics). Recently there have been some existence and multiplicity results for such equations but all for problems with a symmetric nonlinearity. We mention, for example, the works of Aizicovici–Papageorgiou–Staicu [2], Cingolani–Degiovanni [10], Papageorgiou–Rădulescu [30], Papageorgiou–Smyrlis [31], Papageorgiou–Winkert [32, 33], Sun [39], Sun–Zhang–Su [40].

Our approach uses and combines tools from critical point theory along with Morse theory in terms of critical groups. The main goal of our paper is to present a multiplicity theorem which states the existence of at least five nontrivial solutions of (1.1) including sign information about the solutions obtained. Indeed, we can show that one is negative, two are positive and one is nodal (i.e. has changing sign). For the fifth solution we do not have any information about its sign. To the best of our knowledge, our work is the first establishing the existence of nodal solutions for such asymmetric problems.

2. Preliminaries

Let X be a Banach space and X^* its topological dual while $\langle\cdot,\cdot\rangle$ denotes the duality brackets to the pair (X^*, X). We have the following definition.

Definition 2.1. The functional $\varphi \in C^1(X,\mathbb{R})$ fulfills the Cerami condition (the C-condition for short) if the following holds: every sequence $(u_n)_{n\geq 1} \subseteq X$ such that $(\varphi(u_n))_{n\geq 1}$ is bounded in $\mathbb{R}$ and $(1+\|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ which is more general than the usual Palais–Smale condition. It leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. Central in that theory is the well-known mountain pass theorem due to Ambrosetti–Rabinowitz [4] which we recall here in a slightly more general form (see, for example, Gasiński–Papageorgiou [18]).

Theorem 2.2. *Let $\varphi \in C^1(X)$ be a functional satisfying the C-condition and let $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \geq m_\rho$ with c being a critical value of φ.

By $L^p(\Omega)$ $\left(\text{or } L^p\left(\Omega;\mathbb{R}^N\right)\right)$ and $W^{1,p}_0(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_p$ and $\|\cdot\|$. Thanks to the Poincaré inequality we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W^{1,p}_0(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W^{1,p}_0(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is denoted by $|\cdot|_N$ and for a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define the Nemytskij operator corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

Evidently, $x \mapsto N_h(u)(x)$ is measurable.

In addition to the Sobolev space $W^{1,p}_0(\Omega)$ we will also use the ordered Banach space

$$C^1_0(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

and its positive cone

$$C^1_0(\overline{\Omega})_+ = \left\{u \in C^1_0(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)=\left\{u\in C_0^1(\overline{\Omega})_+ : u(x)>0 \text{ for all } x\in\Omega,\ \frac{\partial u}{\partial n}(x)<0 \text{ for all } x\in\partial\Omega\right\},$$

where $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$.

Now, let $f_0:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying the subcritical polynomial growth condition

$$|f_0(x,s)|\leq a_0(x)\left(1+|s|^{r-1}\right)\qquad \text{for a.a. } x\in\Omega \text{ and for all } s\in\mathbb{R},$$

with $a_0\in L^\infty(\Omega)_+$ and $1<r<p^*$, where p^* is the critical exponent of p given by

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if } p<N,\\ +\infty & \text{if } p\geq N.\end{cases}$$

Setting $F_0(x,s)=\int_0^s f_0(x,t)dt$ we define the C^1-functional $\varphi_0: W_0^{1,p}(\Omega)\to\mathbb{R}$ by

$$\varphi_0(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega F_0(x,u)dx.$$

The next result is a particular case of a more general result developed by Gasiński–Papageorgiou [19] (see also Winkert [41] for nonsmooth functionals). The result is actually an outgrowth of the nonlinear regularity theory of Lieberman [22].

Proposition 2.3. *If $u_0\in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0>0$ such that*

$$\varphi_0(u_0)\leq\varphi_0(u_0+h)\quad \text{for all } h\in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})}\leq\rho_0,$$

then $u_0\in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1>0$ such that

$$\varphi_0(u_0)\leq\varphi_0(u_0+h)\quad \text{for all } h\in W_0^{1,p}(\Omega) \text{ with } \|h\|\leq\rho_1.$$

Given $1<r<\infty$, we denote by $-\Delta_r: W_0^{1,r}(\Omega)\to W^{-1,r'}(\Omega)$ with $\frac{1}{r}+\frac{1}{r'}=1$ the negative r-Laplacian defined by

$$\langle-\Delta_r u,v\rangle=\int_\Omega\|\nabla u\|_{\mathbb{R}^N}^{r-2}(\nabla u,\nabla v)_{\mathbb{R}^N}dx\quad \text{for all } u,v\in W_0^{1,r}(\Omega). \tag{2.1}$$

If $r=2$, then $\Delta_r=\Delta$ becomes the well-known Laplace operator and we have $\Delta\in\mathcal{L}\left(H_0^1(\Omega),H^{-1}(\Omega)\right)$, where $\mathcal{L}\left(H_0^1(\Omega),H^{-1}(\Omega)\right)$ denotes the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. For the general case, we have the following result (see, Motreanu–Motreanu–Papageorgiou [25, p. 40]).

Proposition 2.4. *The map* $-\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$*, defined in (2.1), is continuous, strictly monotone (hence maximal monotone) and of type* $(S)_+$*, that is, if* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle -\Delta_r u_n, u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$*.*

Next we present some basic facts about the spectra of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ for $1 < r < \infty$ and of $\left(-\Delta, H_0^1(\Omega)\right)$. So, we consider the following nonlinear eigenvalue problem (linear if $r = 2$)

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2} u \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

We say that a number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ if problem (2.2) possesses a nontrivial solution $\hat{u} \in W_0^{1,r}(\Omega)$ which is said to be an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. The set of all eigenvalues of (2.2) is denoted by $\hat{\sigma}(r)$ and it is known that $\hat{\sigma}(r)$ has a smallest element $\hat{\lambda}_1(r)$ which has the following properties:

- $\hat{\lambda}_1(r)$ is positive;
- $\hat{\lambda}_1(r)$ is isolated, that is, there exists $\varepsilon > 0$ such that $\left(\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \varepsilon\right) \cap \hat{\sigma}(r) = \emptyset$;
- $\hat{\lambda}_1(r)$ is simple, that is, if u, v are two eigenfunctions corresponding to $\hat{\lambda}_1(r)$, then $u = kv$ for some $k \in \mathbb{R} \setminus \{0\}$;
-
$$\hat{\lambda}_1(r) = \inf \left[\frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right]. \tag{2.3}$$

The infimum in (2.3) is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(r) > 0$. From (2.3) it is clear that the elements of this eigenspace do not change sign. In what follows we denote by $\hat{u}_1(r)$ the positive L^r-normalized eigenfunction (i.e. $\|\hat{u}_1(r)\|_r = 1$) associated to $\hat{\lambda}_1(r)$. The nonlinear regularity theory (see Lieberman [22]) and the nonlinear maximum principle (see, for example, Gasiński–Papageorgiou [18]) imply $\hat{u}_1(r) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. The isolation of $\hat{\lambda}_1(r) > 0$ and since the set $\hat{\sigma}(r) \subseteq (0, +\infty)$ is closed, lead to a straightforward definition of the second eigenvalue $\hat{\lambda}_2(r)$ given by

$$\hat{\lambda}_2(r) = \inf \left[\hat{\lambda} \in \hat{\sigma}(r) : \hat{\lambda} > \hat{\lambda}_1(r) \right].$$

In addition, the Lusternik–Schnirelmann minimax scheme gives a whole strictly increasing sequence $\left(\hat{\lambda}_k(r)\right)_{k\ge 1} \subseteq \hat{\sigma}(r)$ such that $\hat{\lambda}_k(r) \to +\infty$ as $k \to +\infty$. We do not know if this sequence exhausts the whole spectrum of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ but in case $N = 1$ (ordinary differential equations) or $r = 2$ (linear eigenvalue problem) the answer is positive. We mention that $\hat{\lambda}_1(r) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have eigenfunctions being nodal.

In the linear case $r = 2$ we have $\hat{\sigma}(2) = \left(\hat{\lambda}_k(2)\right)_{k \geq 1}$ and the corresponding eigenspaces $E\left(\hat{\lambda}_k(2)\right)$ are linear spaces satisfying

$$H_0^1(\Omega) = \overline{\bigoplus_{k \geq 1} E\left(\hat{\lambda}_k(2)\right)}.$$

These eigenspaces have the so-called unique continuation property (ucp for short) meaning that if $u \in E\left(\hat{\lambda}_k(2)\right), k \geq 1$ vanishes on a set of positive Lebesgue measure, then $u(x) \equiv 0$ (see de Figueiredo–Gossez [15]). Standard regularity theory implies that $E\left(\hat{\lambda}_k(2)\right) \subseteq C_0^1(\overline{\Omega})$ for all $k \geq 1$. For $m \in \mathbb{N}$ let

$$\overline{H}_m = \bigoplus_{k=1}^{m} E\left(\hat{\lambda}_k(2)\right) \quad \text{and} \quad \hat{H}_m = \overline{\bigoplus_{k \geq m} E\left(\hat{\lambda}_k(2)\right)}.$$

Using these spaces we can have precise variational characterizations for all eigenvalues. Therefore, we have

$$\hat{\lambda}_1(2) = \inf \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right] \tag{2.4}$$

and for $m \geq 2$

$$\begin{aligned} \hat{\lambda}_m(2) &= \max \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_m, u \neq 0\right] \\ &= \min \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \hat{H}_m, u \neq 0\right]. \end{aligned} \tag{2.5}$$

Evidently, (2.4) is a particular case of (2.3) (when $r = 2$) and the infimum is realized on $E\left(\hat{\lambda}_1(2)\right)$. In (2.5) both the maximum and the minimum are realized on $E\left(\hat{\lambda}_m(2)\right)$.

All the above facts lead to the following useful inequalities (see Papageorgiou–Kyritsi [29, pp. 356, 365]).

Proposition 2.5. *Let $\vartheta \in L^\infty(\Omega)_+$ be such that $\vartheta(x) \leq \hat{\lambda}_1(r)$ for a.a. $x \in \Omega$ and the inequality is strict on a set of positive measure. Then there exists a number $\hat{c}_0 > 0$ such that*

$$\|\nabla u\|_r^r - \int_\Omega \vartheta(x)|u|^r dx \geq \hat{c}_0 \|u\|^r \quad \textit{for all } u \in W_0^{1,r}(\Omega).$$

Proposition 2.6.

(1) Let $\vartheta \in L^\infty(\Omega)_+$ and $\vartheta(x) \geq \hat{\lambda}_m(2)$ for a.a. $x \in \Omega$ with strict inequality on a set of positive measure, then there exists a number $\hat{c}_1 > 0$ such that

$$\|\nabla u\|_2^2 - \int_\Omega \vartheta(x)|u|^2 dx \leq -\hat{c}_1 \|u\|_{H_0^1(\Omega)}^2 \quad \textit{for all } u \in \overline{H}_m.$$

(2) *Let $\vartheta \in L^\infty(\Omega)_+$ and $\vartheta(x) \leq \hat{\lambda}_m(2)$ for a.a. $x \in \Omega$ with strict inequality on a set of positive measure, then there exists a number $\hat{c}_2 > 0$ such that*

$$\|\nabla u\|_2^2 - \int_\Omega \vartheta(x)|u|^2 dx \geq \hat{c}_2 \|u\|^2_{H_0^1(\Omega)} \quad \text{for all } u \in \hat{H}_m.$$

We will also use the weighted version of the linear (i.e. $r = 2$) eigenvalue problem stated in (2.2):

$$\begin{aligned} -\Delta u &= \hat{\lambda} m(x) u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.6}$$

where $m \in L^\infty(\Omega), m(x) \geq 0$ for a.a. $x \in \Omega$ and the inequality is strict on a set of positive measure. Then the spectrum of (2.6) consists of a sequence $\left(\hat{\lambda}_k(2,m)\right)_{k\geq 1}$ of distinct eigenvalues which satisfy

$$0 < \hat{\lambda}_1(2,m) < \hat{\lambda}_2(2,m) < \ldots < \hat{\lambda}_k(2,m) \to +\infty \quad \text{as } k \to +\infty.$$

These eigenvalues and the corresponding eigenfunctions as well as the eigenspaces have the same properties as before. In this case the Rayleigh quotient involved in the variational characterizations of the eigenvalues is given by $\dfrac{\|\nabla u\|_2^2}{\int_\Omega m(x)u^2dx}$. Then, exploiting the ucp of the eigenspaces, we have the following strict monotonicity property for the map $m \to \hat{\lambda}_k(2,m)$ for all $k \in \mathbb{N}$ (see Motreanu–Motreanu–Papageorgiou [25, p. 252]).

Proposition 2.7. *If $m, m' \in L^\infty(\Omega) \setminus \{0\}, m \neq m'$ and $0 \leq m(x) \leq m'(x)$ for a.a. $x \in \Omega$, then $0 < \hat{\lambda}_k(2,m') < \hat{\lambda}_k(2,m)$ for all $k \in \mathbb{N}$.*

Next, let us recall some basic definitions and facts about Morse theory which will need in the sequel. Let X be a Banach space and let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ the term $H_k(Y_1, Y_2)$ stands for the $k \stackrel{th}{=}$-relative singular homology group with integer coefficients. Recall that

$$H_k(Y_1,Y_2) = Z_k(Y_1,Y_2) \big/ B_k(Y_1,Y_2) \quad \text{for all } k \in \mathbb{N}_0,$$

where $Z_k(Y_1, Y_2)$ is the group of relative singular k-cycles of Y_1 mod Y_2 (that is, $Z_k(Y_1, Y_2) = \ker \partial_k$ with ∂_k being the boundary homomorphism) and $B_k(Y_1, Y_2)$ is the group of relative singular k-boundaries of Y_1 mod Y_2 (that is, $B_k(Y_1, Y_2) = \operatorname{im} \partial_{k+1}$). We know that $\partial_{k-1} \circ \partial_k = 0$ for all $k \in \mathbb{N}$, hence $B_k(Y_1, Y_2) \subseteq Z_k(Y_1, Y_2)$ and so the quotient

$$Z_k(Y_1,Y_2) \big/ B_k(Y_1,Y_2)$$

makes sense.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \le c\} \qquad \text{(the sublevel set of } \varphi \text{ at } c\text{)},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \qquad \text{(the critical set of } \varphi\text{)},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\} \qquad \text{(the critical set of } \varphi \text{ at the level } c\text{)}.$$

For every isolated critical point $u \in K_\varphi^c$ the critical groups of φ at $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \ge 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the definition of critical groups above is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \ge 0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. This is a consequence of the second deformation theorem (see, for example, Gasiński–Papageorgiou [18, p. 628]).

We now assume that K_φ is finite and introduce the following series in $t \in \mathbb{R}$:

$$M(t, u) = \sum_{k \ge 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \ge 0} \operatorname{rank} C_k(\varphi, \infty) t^k.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.7}$$

with $Q(t) = \sum_{k \ge 0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Suppose next that $X = H$ is a Hilbert space and let U be a neighborhood of a given point $u \in H$. We further assume that $\varphi \in C^2(U)$, K_φ is finite and $u \in K_\varphi$. The Morse index of u, denoted by $\mu = \mu(u)$, is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u) \in \mathcal{L}(H)$ is negative definite. The nullity of u, denoted by $\nu = \nu(u)$, is defined to be the dimension of $\ker \varphi''(u)$. We say that $u \in K_\varphi$ is nondegenerate if $\varphi''(u)$ is invertible, that is, $\nu = \nu(u) = 0$. The inverse function theorem implies that a nondegenerate critical point is always isolated. If the nullity of u is finite, then $\varphi''(u) \in \mathcal{L}(H)$ is a Fredholm operator of index zero. More details on critical groups and related topics can be found in the books of Ambrosetti–Malchiodi [3] and Motreanu–Motreanu–Papageorgiou [25].

3. Solutions of constant sign

In this section we prove the existence of constant sign solutions for problem (1.1). We impose the following conditions on the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

(H) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(x, 0) = 0, f(x, \cdot) \in C^1(\mathbb{R})$ for a.a. $x \in \Omega$ and

- (i) there holds
  $|f'_s(x, s)| \leq a(x) \left(1 + |s|^{r-2}\right)$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$;
  with $a \in L^\infty(\Omega)_+$ and $p < r < p^*$;
- (ii) if $F(x, s) = \int_0^s f(x, t)dt$, then
  $$\lim_{s\to+\infty} \frac{F(x, s)}{s^p} = +\infty \text{ uniformly for a.a. } x \in \Omega;$$
  there exist $\tau \in \left((r - p) \max \left\{\frac{N}{p}, 1\right\}, p^*\right)$ and $\xi_0 > 0$ such that
  $$0 < \xi_0 \leq \liminf_{s\to+\infty} \frac{f(x, s)s - pF(x, s)}{s^\tau} \text{ uniformly for a.a. } x \in \Omega;$$
- (iii) there exist a function $w_+ \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and a constant $c_+ > 0$ such that
  - $0 < c_+ \leq w_+(x)$ for all $x \in \overline{\Omega}$;
  - $f(x, w_+(x)) \leq 0$ for a.a. $x \in \Omega$;
  - $0 \leq -\Delta_p(w_+) - \Delta(w_+)$ in $W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*$;
- (iv) there exist $\xi_1, c_0 > 0$ such that
  - $-\xi_1 \leq \liminf_{s\to-\infty} \frac{f(x, s)}{|s|^{p-2}s} \leq \limsup_{s\to-\infty} \frac{f(x, s)}{|s|^{p-2}s} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$;
  - $-c_0 \leq f(x, s)s - pF(x, s)$ for a.a. $x \in \Omega$ and for all $s \leq 0$;
- (v) there exists $m \in \mathbb{N}, m \geq 2$ such that
  - $f'_s(x, 0) = \lim_{s\to0} \frac{f(x, s)}{s}$ uniformly for a.a. $x \in \Omega$;
  - $\hat{\lambda}_m(2) \leq f'_s(x, 0) \leq \hat{\lambda}_{m+1}(2)$ for a.a. $x \in \Omega$ and the two inequalities are strict on sets (not necessarily the same) of positive measure.

Remark 3.1. Hypothesis (H)(ii) implies that $f(x, \cdot)$ is $(p - 1)$-superlinear near $+\infty$ for a.a. $x \in \Omega$. Note that we do not use the AR-condition which says in its unilateral version that we can find numbers $q > p$ and $M > 0$ such that

- $$0 < qF(x, s) \leq f(x, s)s \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M; \tag{3.1}$$
- $$\operatorname*{ess\,inf}_{\Omega} F(\cdot, M) > 0, \tag{3.2}$$

see Ambrosetti–Rabinowitz [4] and Mugnai [26]. Integrating (3.1) and using (3.2) gives the following weaker condition

$$c_1 s^q \leq F(x,s) \quad \text{for a.a.}\, x \in \Omega, \ \text{for all } s \geq M \text{ with } c_1 > 0. \tag{3.3}$$

From (3.3) it follows the much weaker condition on $F(x,\cdot)$ which says that

$$\lim_{s\to+\infty} \frac{F(x,s)}{s^p} = +\infty \quad \text{uniformly for a.a.}\, x \in \Omega. \tag{3.4}$$

Next we employ condition (3.4), which expresses the p-superlinearity of the primitive, along with the second statement in (H)(ii). These two conditions lead to the $(p-1)$-superlinearity of $f(x,\cdot)\big|_{\mathbb{R}_+}$, that is

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a.}\, x \in \Omega.$$

Hypothesis (H)(ii) is weaker than the AR-condition (see (3.1), (3.2)). In fact we may assume in (3.1) that $q > (r-p)\max\left\{\frac{N}{p},1\right\}$, then, assuming (3.1) and (3.2), gives

$$\begin{aligned}
\frac{f(x,s)s - pF(x,s)}{s^q} &= \frac{f(x,s)s - qF(x,s)}{s^q} + (q-p)\frac{F(x,s)}{s^q}\\
&\geq (q-p)\frac{F(x,s)}{s^q}\\
&\geq (q-p)c_1 \quad \text{for a.a.}\, x\in\Omega, \ \text{for all } s \geq M.
\end{aligned}$$

Thus,

$$\liminf_{s\to+\infty} \frac{f(x,s)s - pF(x,s)}{s^q} = \xi_0 = (q-p)c_1 > 0 \quad \text{uniformly for a.a.}\, x \in \Omega.$$

Therefore, hypothesis (H)(ii) is satisfied.

Consider now a function defined in the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$ (for the sake of simplicity we drop the x-dependence) that has the form

$$f(s) = \begin{cases} \eta s & \text{if } 0 \leq s \leq 1,\\ \eta\left(s^{p-1}\ln(s) + 1\right) & \text{if } s > 1, \end{cases}$$

where $\eta \in \left(\hat\lambda_m, \hat\lambda_{m+1}\right)$ for some $m \geq 2$. Then $f \in C^1(0,\infty)$ satisfies hypothesis (H)(ii) but fails to satisfy the AR-condition.

The third inequality in Hypothesis (H)(iii) means that

$$0 \leq \int_\Omega \|\nabla w_+\|_{\mathbb{R}^N}^{p-2} (\nabla w_+, \nabla h)_{\mathbb{R}^N}\, dx + \int_\Omega (\nabla w_+, \nabla h)_{\mathbb{R}^N}\, dx$$

for all $h \in W^{1,p}(\Omega), h \geq 0$. Hypothesis (H)(iii) is satisfied if, for example, there exists $c_+ > 0$ such that

$$f(x, c_+) \leq 0 \quad \text{for a.a.}\, x \in \Omega.$$

Combined with hypothesis (H)(v) it dictates a kind of oscillatory behavior near zero. Hypothesis (H)(iv) implies that $f(x,\cdot)$ is $(p-1)$-sublinear near $-\infty$ for a.a. $x \in \Omega$ and allows the occurrence of resonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$. Finally, hypothesis (H)(v) implies that we stay strictly above the principal eigenvalue $\hat{\lambda}_1(2) > 0$ at zero and only nonuniform nonresonance is possible.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional of problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx.$$

Proposition 3.2. *If hypotheses (H) are satisfied, then the functional φ fulfills the C-condition.*

Proof. Let $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for all } n \geq 1 \tag{3.5}$$

with some $M_1 > 0$ and

$$(1 + \|u_n\|)\, \varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*. \tag{3.6}$$

By means of (3.6) we obtain

$$\left|\langle -\Delta_p u_n, h\rangle + \langle -\Delta u_n, h\rangle - \int_\Omega f(x,u_n)hdx\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{3.7}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. We claim that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Arguing indirectly, suppose that $\|u_n^-\| \to \infty$. We set $y_n = \frac{u_n^-}{\|u_n^-\|}$ for all $n \geq 1$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \geq 1$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega), \quad y \geq 0. \tag{3.8}$$

Choosing $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.7) results in

$$\left\|\nabla u_n^-\right\|_p^p + \left\|\nabla u_n^-\right\|_2^2 - \int_\Omega f\left(x, -u_n^-\right)\left(-u_n^-\right)dx \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

which gives

$$\|\nabla y_n\|_p^p + \frac{1}{\|u_n^-\|^{p-2}} \|\nabla y_n\|_2^2 - \int_\Omega \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} (-y_n)\, dx \le \frac{\varepsilon_n}{\|u_n^-\|^p} \tag{3.9}$$

for all $n \in \mathbb{N}$. Hypotheses (H)(i),(iv) imply that

$$|f(x,s)| \le c_2 \left(1 + |s|^{p-1}\right) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \le 0 \text{ and some } c_2 > 0.$$

Therefore,

$$\left(\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}}\right)_{n\ge 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{3.10}$$

Note that $u_n^-(x) \to +\infty$ for a.a. $x \in \{y > 0\}$. Using this fact along with (3.10) as well as hypothesis (H)(iv) and by passing to a suitable subsequence if necessary, we obtain

$$\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \rightharpoonup -\xi y^{p-1} \quad \text{in } L^{p'}(\Omega) \tag{3.11}$$

with $\xi \in L^\infty(\Omega)$, $\xi(x) \le \hat{\lambda}_1(p)$ for a.a. $x \in \Omega$ (see Aizicovici–Papageorgiou–Staicu [1, Proposition 16]). Thus, if we pass in (3.9) to the limit as $n \to \infty$ by applying (3.8), (3.11) and recalling $p > 2$, we have

$$\|\nabla y\|_p^p \le \int_\Omega \xi(x) y^p dx. \tag{3.12}$$

If $\xi \ne \hat{\lambda}_1(p)$, then from (3.12) and Proposition 2.5 it follows that $\hat{c}_0 \|y\|^p \le 0$, which means $y = 0$. Hence, due to (3.9), $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

If $\xi = \hat{\lambda}_1(p)$ for a.a. $x \in \Omega$, then (3.12) and (2.3) imply $\|\nabla y\|_p^p = \hat{\lambda}_1(p) \|y\|_p^p$ and so, by means of (3.8), $y = c_3 \hat{u}_1(p)$ for some $c_3 \ge 0$. If $c_3 = 0$, then $y = 0$ and we reach a contradiction as above. Hence, $c_3 > 0$ meaning $y \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Then

$$u_n^-(x) \to +\infty \quad \text{for a.a. } x \in \Omega. \tag{3.13}$$

Because of (3.5), we have

$$\begin{aligned}
&\frac{1}{p}\left\|\nabla u_n^+\right\|_p^p+\frac{1}{2}\left\|\nabla u_n^+\right\|_2^2\\
&=\frac{1}{p}\left\|\nabla u_n\right\|_p^p+\frac{1}{2}\left\|\nabla u_n\right\|_2^2-\frac{1}{p}\left\|\nabla u_n^-\right\|_p^p-\frac{1}{2}\left\|\nabla u_n^-\right\|_2^2\\
&\quad+\int_\Omega F\left(x,u_n\right)dx-\int_\Omega F\left(x,u_n\right)dx\\
&=\varphi\left(u_n\right)-\frac{1}{p}\left\|\nabla u_n^-\right\|_p^p-\frac{1}{2}\left\|\nabla u_n^-\right\|_2^2+\int_\Omega F\left(x,u_n\right)dx\\
&\leq M_1+\frac{1}{p}\left[\int_\Omega pF\left(x,u_n\right)dx-\left\|\nabla u_n^-\right\|_p^p-\frac{p}{2}\left\|\nabla u_n^-\right\|_2^2\right]
\end{aligned}\tag{3.14}$$

for all $n \in \mathbb{N}$. Now, we set $h=-u_n^- \in W_0^{1,p}(\Omega)$ in (3.7) to get

$$\left|\left\|\nabla u_n^-\right\|_p^p+\left\|\nabla u_n^-\right\|_2^2-\int_\Omega f\left(x,-u_n^-\right)\left(-u_n^-\right)dx\right|\leq\varepsilon_n,$$

hence

$$-\left\|\nabla u_n^-\right\|_p^p-\left\|\nabla u_n^-\right\|_2^2\leq\varepsilon_n-\int_\Omega f\left(x,-u_n^-\right)\left(-u_n^-\right)dx\quad\text{for all } n\in\mathbb{N}.\tag{3.15}$$

Using (3.15) in (3.14) and recalling $p>2$ yields

$$\begin{aligned}
&\frac{1}{p}\left\|\nabla u_n^+\right\|_p^p+\frac{1}{2}\left\|\nabla u_n^+\right\|_2^2\\
&\leq M_2+\frac{1}{p}\left[\int_\Omega pF\left(x,u_n\right)dx-\int_\Omega f\left(x,-u_n^-\right)\left(-u_n^-\right)dx\right]
\end{aligned}\tag{3.16}$$

for some $M_2>0$ and for all $n \in \mathbb{N}$. Note that

$$pF\left(x,u_n\right)=pF\left(x,u_n^+\right)+pF\left(x,u_n^-\right)\quad\text{for all } n\in\mathbb{N}.\tag{3.17}$$

In addition, hypothesis (H)(iv) implies

$$pF\left(x,-u_n^-\right)-f\left(x,-u_n^-\right)\left(-u_n^-\right)\leq c_0\quad\text{for a.a. } x\in\Omega \text{ and for all } n\in\mathbb{N}.\tag{3.18}$$

Returning to (3.16) and using (3.17), (3.18) gives

$$\frac{1}{p}\left\|\nabla u_n^+\right\|_p^p+\frac{1}{2}\left\|\nabla u_n^+\right\|_2^2\leq M_3+\int_\Omega F\left(x,u_n^+\right)dx$$

for some $M_3 > 0$ and for all $n \in \mathbb{N}$. Therefore,

$$\varphi\left(u_n^+\right) \leq M_3 \quad \text{for all } n \in \mathbb{N}.$$

From this bound and (3.5) it follows that

$$\varphi\left(-u_n^-\right) \leq M_4 \quad \text{for some } M_4 > 0 \text{ and for all } n \in \mathbb{N}.$$

Then, taking (2.3) into account, we derive

$$\frac{\hat{\lambda}_1(p)}{p}\left\|u_n^-\right\|_p^p - \int_\Omega F\left(x, -u_n^-\right) dx + \frac{1}{2}\left\|\nabla u_n^-\right\|_2^2 \leq M_4 \quad \text{for all } n \in \mathbb{N}. \tag{3.19}$$

Moreover, thanks to hypothesis (H)(iv), we have, for a.a. $x \in \Omega$ and for all $s \leq 0$,

$$\begin{aligned}
\frac{d}{ds}\left(\frac{F(x,s)}{|s|^p}\right) &= \frac{f(x,s)|s|^p - p|s|^{p-2}sF(x,s)}{|s|^{2p}} \\
&= \frac{|s|^{p-2}s\left[f(x,s)s - pF(x,s)\right]}{|s|^{2p}} \\
&= \frac{f(x,s)s - pF(x,s)}{|s|^p s} \\
&\geq \frac{-c_0}{|s|^p s},
\end{aligned}$$

which shows that

$$\frac{F(x,t)}{|t|^p} - \frac{F(x,s)}{|s|^p} \geq \frac{c_0}{p}\left[\frac{1}{|t|^p} - \frac{1}{|s|^p}\right] \tag{3.20}$$

for a.a. $x \in \Omega$ and for all $t < s < 0$. Furthermore, hypothesis (H)(iv) implies that

$$\limsup_{s \to -\infty} \frac{pF(x,s)}{|s|^p} \leq \hat{\lambda}_1(p) \quad \text{uniformly for a.a.}\, x \in \Omega.$$

So, if we let in (3.20) $t \to -\infty$, then

$$\frac{\hat{\lambda}_1(p)}{p} - \frac{F(x,s)}{|s|^p} \geq -\frac{c_0}{p}\frac{1}{|s|^p} \quad \text{for a.a.}\, x \in \Omega \text{ and for all } s < 0.$$

Hence,

$$\hat{\lambda}_1(p)|s|^p - pF(x,s) \geq -c_0 \quad \text{for a.a.}\, x \in \Omega \text{ and for all } s \leq 0. \tag{3.21}$$

Returning to (3.19) and using (3.21) results in

$$\frac{1}{2}\left\|\nabla u_n^-\right\|_2^2 \leq M_5 \quad \text{for some } M_5>0 \text{ and for all } n\in\mathbb{N},$$

which implies, due to the representation in (2.4), that

$$\frac{\hat{\lambda}_1(2)}{2}\int_\Omega \left(u_n^-\right)^2 dx \leq M_5 \quad \text{for all } n\in\mathbb{N}. \tag{3.22}$$

Taking (3.13) and Fatou's lemma into account we have

$$\int_\Omega \left(u_n^-\right)^2 dx \to +\infty \quad \text{as } n\to+\infty. \tag{3.23}$$

Comparing (3.22) and (3.23) we reach a contradiction. So, we have proved that

$$\left(u_n^-\right)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.24}$$

Next we are going to show that $(u_n^+)_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ is bounded. To this end, we argue again by contradiction and suppose that $\|u_n^+\|\to\infty$ as $n\to\infty$. We set $v_n=\frac{u_n^+}{\|u_n^+\|}$ for all $n\geq 1$. Then, $\|v_n\|=1, v_n\geq 0$ for all $n\geq 1$ and we may assume that

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad v_n\to v \quad \text{in } L^r(\Omega), \quad v\geq 0.$$

Choosing $h=u_n^+\in W_0^{1,p}(\Omega)$ in (3.7) yields

$$-\left\|\nabla u_n^+\right\|_p^p - \left\|\nabla u_n^+\right\|_2^2 + \int_\Omega f\left(x,u_n^+\right)u_n^+ dx \leq \varepsilon_n \quad \text{for all } n\in\mathbb{N}. \tag{3.25}$$

On the other hand, from (3.5) and (3.24), we obtain

$$\left\|\nabla u_n^+\right\|_p^p + \frac{p}{2}\left\|\nabla u_n^+\right\|_2^2 - \int_\Omega pF\left(x,u_n^+\right)dx \leq M_6 \quad \text{for all } n\in\mathbb{N} \tag{3.26}$$

and for some $M_6>0$. Adding (3.25) and (3.26) gives

$$\left(\frac{p}{2}-1\right)\left\|\nabla u_n^+\right\|_2^2 + \int_\Omega \left[f\left(x,u_n^+\right)u_n^+ - pF\left(x,u_n^+\right)\right]dx \leq M_7$$

for some $M_7>0$ and for all $n\in\mathbb{N}$. Since $p>2$ it results in

$$\int_\Omega \left[f\left(x,u_n^+\right)u_n^+ + pF\left(x,u_n^+\right)\right]dx \leq M_7 \quad \text{for all } n\in\mathbb{N}. \tag{3.27}$$

Hypotheses (H)(i),(ii) imply the existence of $\xi_2\in(0,\xi_0)$ and $c_4>0$ such that

$$\xi_2 s^\tau - c_4 \leq f(x,s)s - pF(x,s) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.28}$$

Using (3.28) in (3.27) yields

$$\xi_2 \left\|u_n^+\right\|_\tau^\tau \leq M_8 \quad \text{for some } M_8 > 0 \text{ and for all } n \in \mathbb{N}.$$

Therefore

$$\left(u_n^+\right)_{n\geq 1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{3.29}$$

Let us first suppose that $N \neq p$. Because of hypothesis (H)(ii) it is clear that, without any loss of generality, we may assume that $\tau < r < p^*$. So, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}.$$

Applying the interpolation inequality (see, for example Gasiński–Papageorgiou [18, p. 905]) we have

$$\left\|u_n^+\right\|_r \leq \left\|u_n^+\right\|_\tau^{1-t} \left\|u_n^+\right\|_{p^*}^t,$$

which by (3.29) and the Sobolev embedding theorem results in

$$\left\|u_n^+\right\|_r^r \leq M_9 \left\|u_n^+\right\|^{tr} \quad \text{with some } M_9 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.30}$$

Choosing $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.7) gives

$$\left\|\nabla u_n^+\right\|_p^p + \left\|\nabla u_n^+\right\|_2^2 - \int_\Omega f\left(x, u_n^+\right) u_n^+ dx \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}.$$

Then, from hypothesis (H)(i) along with (3.30) we have

$$\left\|\nabla u_n^+\right\|_p^p \leq c_5 \left(1 + \left\|u_n^+\right\|_r^r\right) \leq c_6 \left(1 + \left\|u_n^+\right\|^{tr}\right) \tag{3.31}$$

for some $c_5, c_6 > 0$ and for all $n \in \mathbb{N}$. Since the hypothesis on τ (see (H)(ii)) implies $tr < p$, we conclude from (3.31) that

$$\left(u_n^+\right)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$

which in combination with (3.24) finally gives

$$\left(u_n\right)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.32}$$

Let us now consider the case $p = N$. Here, $p^* = +\infty$ and the Sobolev embedding theorem yields that

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact for all } 1 \leq q < p^* = +\infty.$$

So, for the argument above to work, we need to replace $p^* = +\infty$ by $q > r > \tau$. Again, we choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{q}$$

to obtain

$$tr = \frac{q(r-\tau)}{q-\tau}. \tag{3.33}$$

We see that $\frac{q(r-\tau)}{q-\tau} \to r-\tau$ as $q \to p^* = +\infty$, but by hypothesis (H)(ii) we have $r-\tau < p$. Therefore for $q > r$ large enough, we will have $tr < p$ (see (3.33)) and then the previous argument works and we reach (3.32).

Because of (3.32) we may assume that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega). \tag{3.34}$$

Taking $h = u_n - u \in W_0^{1,p}(\Omega)$ in (3.7), passing to the limit as $n \to \infty$ and using (3.34) yields

$$\lim_{n\to\infty} \left[\langle -\Delta_p u_n, u_n - u\rangle + \langle -\Delta u_n, u_n - u\rangle\right] = 0,$$

which implies due to the monotonicity of the negative Laplacian $-\Delta$ that

$$\limsup_{n\to\infty} \left[\langle -\Delta_p u_n, u_n - u\rangle + \langle -\Delta u, u_n - u\rangle\right] \leq 0.$$

Taking again (3.34) into account, this leads to

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \leq 0.$$

From this in combination with (3.34) and the fact that $-\Delta_p$ fulfills the $(S)_+$-property (see Proposition 2.4), we derive

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

This proves that φ satisfies the C-condition. □

Let $f_-(x,s)$ denote the negative truncation of the nonlinearity $f(x,\cdot)$, that is $f_-(x,s) = f(x,-s^-)$ which is known to be a Carathéodory function. We set $F_-(x,s) = \int_0^s f_-(x,t)dt$ and consider the C^1-functional $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_-(x,u)dx.$$

Proposition 3.3. *If hypotheses (H) are fulfilled, then the functional φ_- is coercive.*

Proof. We argue by contradiction and may assume that φ_- is not coercive. Then there exist a sequence $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ and a number $M_{10} > 0$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad \varphi_-(u_n) \leq M_{10} \quad \text{for all } n \in \mathbb{N}. \tag{3.35}$$

The second assertion in (3.35) reads as

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 - \int_\Omega F_-(x,u_n)dx \leq M_{10} \quad \text{for all } n \in \mathbb{N}. \tag{3.36}$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(\Omega). \tag{3.37}$$

Using the representation of y_n, (3.36) can be rewritten as

$$\frac{1}{p}\|\nabla y_n\|_p^p + \frac{1}{2\|u_n\|^{p-2}}\|\nabla y_n\|_2^2 - \int_\Omega \frac{F_-(x,-u_n^-)}{\|u_n\|^p}dx \leq \frac{M_{10}}{\|u_n\|^p} \tag{3.38}$$

for all $n \in \mathbb{N}$. Hypotheses (H)(i),(iv) imply that

$$|F(x,s)| \leq c_7(1+|s|^p) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \leq 0 \text{ and some } c_7 > 0. \tag{3.39}$$

Then, from (3.37) and (3.39) it follows that

$$\left(\frac{F(\cdot,-u_n^-(\cdot))}{\|u_n\|^p}\right)_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

So, by the Dunford–Pettis theorem and by passing to a subsequence if necessary, we may assume that

$$\frac{F(\cdot,-u_n^-(\cdot))}{\|u_n\|^p} \rightharpoonup \gamma \quad \text{in } L^1(\Omega). \tag{3.40}$$

Note that hypothesis (H)(iv) implies that

$$-\xi_1 \leq \liminf_{s\to-\infty}\frac{pF(x,s)}{|s|^p} \leq \limsup_{s\to-\infty}\frac{pF(x,s)}{|s|^p} \leq \hat{\lambda}_1(p)$$

uniformly for a.a. $x \in \Omega$. Then, from (3.35) it follows that

$$\gamma = \frac{1}{p}g\left(y^-\right)^p \quad \text{with} \quad -\xi_1 \leq g(x) \leq \hat{\lambda}_1(p) \quad \text{for a.a. } x \in \Omega, \tag{3.41}$$

see Aizicovici–Papageorgiou–Staicu [1, proof of Proposition 14]. So, if we pass to the limit as $n \to \infty$ in (3.38) and apply (3.37), (3.40) as well as (3.41), then

$$\|\nabla y\|_p^p \leq \int_\Omega g(x) \left(y^-\right)^p dx, \tag{3.42}$$

where $p > 2$ was taken into account as well. In particular, it holds

$$\left\|\nabla y^-\right\|_p^p \leq \int_\Omega g(x) \left(y^-\right)^p dx. \tag{3.43}$$

Recall that $g(x) \leq \hat{\lambda}_1(p)$ for a.a. $x \in \Omega$, see (3.41). If this inequality is strict on a set of positive measure, then from (3.43) and Proposition 2.5 we obtain

$$\hat{c}_0 \left\|y^-\right\|^p \leq 0,$$

which implies

$$y^- = 0. \tag{3.44}$$

Combining (3.42) and (3.44) we have $y^+ = 0$ and so $y = 0$. Hence, (3.38) implies that

$$y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Next suppose that $g(x) = \hat{\lambda}_1(p)$ for a.a. $x \in \Omega$. By means of (3.43) and (2.3) it follows that

$$\left\|\nabla y^-\right\|_p^p = \hat{\lambda}_1(p) \left\|y^-\right\|_p^p,$$

which gives

$$y^- = k\hat{u}_1(p) \quad \text{for some } k \geq 0.$$

If $k = 0$, then $y^- = 0$ and as above we obtain a contradiction. If $k > 0$, then $y^- \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $y = -y^- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We have

$$u_n(x) = -u_n^-(x) \to -\infty \quad \text{for a.a. } x \in \Omega. \tag{3.45}$$

Recall that

$$\hat{\lambda}_1(p)|s|^p - pF(x,s) \geq -c_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leq 0, \tag{3.46}$$

see (3.21). Returning to (3.36) we have

$$\frac{1}{2}\left\|\nabla u_n^-\right\|_2^2 \leq M_{10} - \frac{1}{p}\int\limits_\Omega \left[\hat{\lambda}_1(p)\left(u_n^-\right)^p - pF\left(x, -u_n^-\right)\right] dx,$$

which results in, due to (2.4) and (3.46), that

$$\frac{\hat{\lambda}_1(2)}{2}\int\limits_\Omega \left(u_n^-\right)^2 dx \leq M_{10} + c_0 \quad \text{for all } n \in \mathbb{N}. \tag{3.47}$$

On the other side, Fatou's Lemma and (3.45) imply

$$\int\limits_\Omega \left(u_n^-\right)^2 dx \to +\infty \quad \text{as } n \to \infty,$$

which contradicts (3.47). Therefore we conclude that φ must be coercive. □

Now we can prove the existence of a nontrivial negative solution of problem (1.1).

Proposition 3.4. *If hypotheses (H) hold, then problem (1.1) admits a negative solution $u_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which is a local minimizer of the energy functional φ.*

Proof. From Proposition 3.3 we know that φ_- is coercive and taking the Sobolev embedding theorem into account, we can show that φ_- is also sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_-\left(u_0\right) = \inf\left[\varphi_-(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.48}$$

Hypothesis (H)(v) implies that we can find $\delta > 0$ small enough and $\eta > \hat{\lambda}_1(2)$ such that

$$F(x,s) \geq \frac{\eta}{2}|s|^2 \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \delta. \tag{3.49}$$

Since $\hat{u}_1(2) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, let $t \in (0,1)$ be small enough such that

$$t\hat{u}_1(2)(x) \in [0,\delta] \quad \text{for all } x \in \overline{\Omega}. \tag{3.50}$$

Then, applying (3.49) and (3.50) we derive

$$\varphi_-\left(-t\hat{u}_1(2)\right) \leq \frac{t^p}{p}\left\|\nabla \hat{u}_1(2)\right\|_p^p + \frac{t^2}{2}\left[\hat{\lambda}_1(2) - \eta\right].$$

Since $t \in (0,1), 2 < p$ and $\eta > \hat{\lambda}_1(2)$, by choosing $t \in (0,1)$ even smaller if necessary, we have $\varphi_-\left(-t\hat{u}_1(2)\right) < 0$. This gives, due to (3.48), that

$$\varphi_-(u_0) < 0 = \varphi_-(0)$$

and so $u_0 \neq 0$.

Relation (3.48) reads as $\varphi'_-(u_0) = 0$ which means

$$-\Delta_p u_0 - \Delta u_0 = N_{f_-}(u_0). \tag{3.51}$$

Taking $u_0^+ \in W_0^{1,p}(\Omega)$ as test function in (3.51) yields

$$\|\nabla u_0^+\|_p^p + \|\nabla u_0^+\|_2^2 = 0,$$

which implies $u_0^+ = 0$ and so $u_0 \leq 0, u_0 \neq 0$. Hence, equation (3.51) becomes

$$-\Delta_p u_0 - \Delta u_0 = N_f(u_0),$$

which means that u_0 is a solution to our original problem

$$\begin{aligned} -\Delta_p u - \Delta u &= f(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By means of the boundedness results of Ladyzhenskaya–Ural'tseva [21, Theorem 7.1, p. 286] we have $u_0 \in L^\infty(\Omega)$.

Now let $\rho = \|u_0\|_\infty$. Taking hypotheses (H)(i), (v) into account we may find $\hat{\xi}_\rho > 0$ such that

$$f(x,s)s + \hat{\xi}_\rho |s|^p \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \rho. \tag{3.52}$$

Using (3.52) in (3.51) implies

$$\Delta_p(-u_0)(x) + \Delta(-u_0)(x) + \hat{\xi}_\rho |u_0(x)|^{p-2} u_0(x) \leq 0 \quad \text{for a.a. } x \in \Omega$$

respectively,

$$\Delta_p(-u_0)(x) + \Delta(-u_0)(x) \leq \hat{\xi}_\rho (-u_0(x))^{p-1} \quad \text{for a.a. } x \in \Omega. \tag{3.53}$$

Let $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2} \xi + \xi$ for all $\xi \in \mathbb{R}^N$. Then

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

and

$$\nabla a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2} \left(I + (p-2) \frac{\xi \otimes \xi}{\|\xi\|_{\mathbb{R}^N}^2} \right) + I \quad \text{for all } \xi \in \mathbb{R}^N.$$

This implies

$$(\nabla a(\xi) y, y)_{\mathbb{R}^N} \geq \|y\|_{\mathbb{R}^N}^2 \quad \text{for all } \xi, y \in \mathbb{R}^N.$$

Therefore, we may use the tangency principle of Pucci–Serrin [36, p. 35] and infer that $u_0(x) < 0$ for all $x \in \Omega$. Then, from (3.53) and the boundary point theorem of Pucci–Serrin [36, p. 120], we conclude that $u_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Note that $\varphi\big|_{-C_0^1(\overline{\Omega})_+} = \varphi_-\big|_{-C_0^1(\overline{\Omega})_+}$ and so u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Invoking Proposition 2.3 we infer that u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of the energy functional φ. □

Proposition 3.5. *Let hypotheses (H) be satisfied. Then problem (1.1) admits at least two positive solutions*

$$\hat{u}, \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad \tilde{u} - \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\},$$

with $\hat{u}$ being a local minimizer of the energy functional φ.

Proof. Using $w_+ \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ from hypothesis (H)(iii), we introduce the following Carathéodory function

$$g(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ f(x,s) & \text{if } 0 \le s \le w_+(x), \\ f\left(x, w_+(x)\right) & \text{if } w_+(x) < s. \end{cases} \tag{3.54}$$

We set $G(x,s) = \int_0^s g(x,t)dt$ and consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega G(x,u)dx.$$

It is clear that ψ is coercive, due to the truncation defined in (3.54), and it is also sequentially weakly lower semicontinuous. Hence, the Weierstrass theorem implies the existence of $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi\left(\hat{u}\right) = \inf\left[\psi(u) : u \in W_0^{1,p}(\Omega)\right] = \hat{m}. \tag{3.55}$$

As in the proof of Proposition 3.4, using hypothesis (H)(v), we can show that for $t \in (0,1)$ small enough (at least such that $t\hat{u}_1(2) \le w_+$), we have $\psi\left(t\hat{u}_1(2)\right) < 0$ implying $\psi\left(\hat{u}\right) = \hat{m} < 0 = \psi(0)$. Hence, $\hat{u} \neq 0$.

From (3.55) we have $\psi'\left(\hat{u}\right) = 0$ which reads as

$$-\Delta_p \hat{u} - \Delta \hat{u} = N_g\left(\hat{u}\right). \tag{3.56}$$

Taking $-\hat{u}^- \in W_0^{1,p}(\Omega)$ as test function in (3.56) gives

$$\left\|\nabla \hat{u}^-\right\|_p^p + \left\|\nabla \hat{u}^-\right\|_2^2 = 0,$$

which implies $\hat{u} \geq 0, \hat{u} \neq 0$. Next, choosing $(\hat{u} - w_+)^+ \in W_0^{1,p}(\Omega)$ as test function in (3.56) and using the definition of the truncation defined in (3.54) as well as hypothesis (H)(iii), we obtain

$$\begin{aligned} &\left\langle -\Delta_p \hat{u}, (\hat{u} - w_+)^+ \right\rangle + \left\langle -\Delta \hat{u}, (\hat{u} - w_+)^+ \right\rangle \\ &= \int_\Omega f(x, w_+)(\hat{u} - w_+)^+ \, dx \\ &\leq \left\langle -\Delta_p w_+, (\hat{u} - w_+)^+ \right\rangle + \left\langle -\Delta w_+, (\hat{u} - w_+)^+ \right\rangle. \end{aligned}$$

Thus

$$\left\langle -\Delta_p \hat{u} + \Delta_p w_+, (\hat{u} - w_+)^+ \right\rangle + \left\langle -\Delta \hat{u} + \Delta w_+, (\hat{u} - w_+)^+ \right\rangle \leq 0,$$

meaning $\hat{u} \leq w_+$. So, we have proved that

$$\hat{u} \in [0, w_+] = \left\{ u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq w_+(x) \text{ for a.a. } x \in \Omega \right\}. \tag{3.57}$$

Then, by means of (3.54) and (3.57), we see that (3.56) becomes

$$-\Delta_p \hat{u} - \Delta \hat{u} = N_f(\hat{u}),$$

meaning that

$$\begin{aligned} -\Delta_p \hat{u} - \Delta \hat{u} &= f(x, \hat{u}) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.58}$$

It is clear that $\hat{u} \in L^\infty(\Omega)$ (see Ladyzhenskaya–Ural'tseva [21, Theorem 7.1, p. 286]) and the regularity results of Lieberman [22, Theorem 1.1] imply $\hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Now, for $\rho = \|\hat{u}\|_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by (3.52). Using this and (3.58) it follows

$$\Delta_p \hat{u}(x) + \Delta \hat{u}(x) \leq \hat{\xi}_\rho \hat{u}(x)^{p-1} \quad \text{for a.a. } x \in \Omega.$$

Then as in the proof of Proposition 3.4, first using the tangency principle of Pucci–Serrin [36, p. 35] we have that $\hat{u}(x) > 0$ for all $x \in \Omega$ and then the boundary point theorem of Pucci–Serrin [36, p. 120] implies that

$$\hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.59}$$

Moreover, hypothesis (H)(iii) and the tangency principle of Pucci–Serrin [36, p. 35] lead to

$$\hat{u}(x) < w_+(x) \quad \text{for all } x \in \overline{\Omega}. \tag{3.60}$$

Hence, from (3.59) and (3.60), we conclude that

$$\hat{u} \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [0, w_+] . \tag{3.61}$$

Note that $\psi\big|_{[0,w_+]} = \varphi\big|_{[0,w_+]}$ which due to (3.61) implies that $\hat{u}$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Invoking Proposition 2.3 gives

$$\hat{u} \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi.$$

Next we introduce the Carathéodory function $k : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k(x,s) = \begin{cases} f\left(x, \hat{u}(x)\right) & \text{if } s < \hat{u}(x), \\ f(x,s) & \text{if } \hat{u}(x) \le s. \end{cases} \tag{3.62}$$

We set $K(x,s) = \int_0^s k(x,t)dt$ and consider the C^1-functional $\hat{\psi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega K(x,u)dx.$$

As it was done for $\hat{u}$, using the definition of the truncation in (3.62), we can show that

$$K_{\hat{\psi}} \subseteq [\hat{u}, \infty) = \left\{ u \in W_0^{1,p}(\Omega) : \hat{u}(x) \le u(x) \text{ for a.a. } x \in \Omega \right\}. \tag{3.63}$$

Without loss of generality we may assume that

$$K_{\hat{\psi}} \cap [\hat{u}, w_+] = \{\hat{u}\} , \tag{3.64}$$

otherwise, because of (3.63), we already have a second positive solution of (1.1) bounded below by $\hat{u}$. Furthermore, with straightforward minor changes in the proof of Proposition 3.2 we can show that

$$\hat{\psi} \text{ satisfies the } C\text{-condition.} \tag{3.65}$$

Claim. *$\hat{u}$ is a local minimizer of $\hat{\psi}$.*

Consider the Carathéodory function $\tilde{k} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{k}(x,s) = \begin{cases} k(x,s) & \text{if } s < w_+(x), \\ k\left(x, w_+(x)\right) & \text{if } w_+(x) \le s. \end{cases} \tag{3.66}$$

We define the C^1-functional $\tilde{\psi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\tilde{\psi}(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega \tilde{K}(x,u)dx,$$

where $\tilde{K}(x,s) = \int_0^s \tilde{k}(x,t)dt$. As before we can check that

$$K_{\tilde{\psi}} \subseteq [\hat{u}, w_+] . \tag{3.67}$$

Taking (3.66) into account it is clear that $\tilde{\psi}$ is coercive and the Sobolev embedding theorem implies that it is sequentially weakly lower semicontinuous as well. Therefore, we find $\hat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\tilde{\psi}\left(\hat{u}_0\right) = \inf\left[\tilde{\psi}(u) : u \in W_0^{1,p}(\Omega)\right].$$

Thanks to (3.67) we have $\hat{u}_0 \in [\hat{u}, w_+]$. Owing to (3.66) it holds $\hat{\psi}'\big|_{[\hat{u},w_+]} = \tilde{\psi}'\big|_{[\hat{u},w_+]}$. Thus, $\hat{u}_0 \in K_{\hat{\psi}}$ and so, regarding (3.64), $\hat{u}_0 = \hat{u}$. But from (3.66) it is clear that

$$\hat{\psi}\big|_{[0,w_+]} = \tilde{\psi}\big|_{[0,w_+]}.$$

Because of (3.61) this implies that $\hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a local $C_0^1(\overline{\Omega})$-minimizer of $\hat{\psi}$. As before, applying Proposition 2.3 we infer that $\hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a local $W_0^{1,p}(\Omega)$-minimizer of $\hat{\psi}$. This proves the Claim.

Now we assume that $K_{\hat{\psi}}$ is finite, otherwise we would have found a whole sequence of distinct positive solutions of (1.1) bounded below by $\hat{u}$ and so we would be done. Because of the Claim, we can find $\rho \in (0,1)$ small enough such that

$$\hat{\psi}\left(\hat{u}\right) < \inf\left[\hat{\psi}(u) : \|u - \hat{u}\| = \rho\right] = m_\rho, \tag{3.68}$$

see Aizicovici–Papageorgiou–Staicu [1, proof of Proposition 29]. Moreover, hypothesis (H)(ii) implies that for every $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ there holds

$$\hat{\psi}(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.69}$$

Then (3.65), (3.68) and (3.69) permit the usage of the mountain pass theorem stated as Theorem 2.2. Hence, we find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\tilde{u} \in K_{\hat{\psi}} \quad \text{and} \quad m_\rho \le \hat{\psi}\left(\tilde{u}\right). \tag{3.70}$$

From (3.68) and (3.70) we see that $\hat{u} \neq \tilde{u}$. In addition, thanks to (3.62), (3.63) and (3.70) we conclude that $\tilde{u}$ is a positive solution of (1.1), $\tilde{u} - \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ and as before the nonlinear regularity theory of Lieberman [22] gives that $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. $\square$

4. Nodal solutions – multiplicity theorem

In this section we are going to prove the existence of nodal (sign-changing) solutions for problem (1.1) and then we state the complete multiplicity theorem for problem (1.1). To do this, we first show the existence of extremal constant sign solutions, that is, the smallest positive solution $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the greatest negative solution $v_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for problem (1.1). Then concentrating on the order interval

$$[v_*, u_*] = \left\{u \in W_0^{1,p}(\Omega) : v_*(x) \le u(x) \le u_*(x) \text{ for a.a. } x \in \Omega\right\},$$

and applying variational tools combined with Morse theory, we can show the existence of a nontrivial solution $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$. Evidently, y_0 must be nodal. Finally we use Morse theory to generate a fifth nontrivial solution.

Now, let $\mathcal{S}_+$ (resp. $\mathcal{S}_-$) be the set of positive (resp. negative) solutions for problem (1.1). From the results of Section 3 we know that

$$\emptyset \subseteq \mathcal{S}_+ \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad \emptyset \subseteq \mathcal{S}_- \subseteq -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

Moreover, as in Filippakis–Kristály–Papageorgiou [17], exploiting the monotonicity of the map $u \to -\Delta_p u - \Delta u$, we have that $\mathcal{S}_+$ is downward directed, that is, if $u_1, u_2 \in \mathcal{S}_+$, then there exists $u \in \mathcal{S}_+$ such that $u \le u_1$ and $u \le u_2$. Analogously, it is known that $\mathcal{S}_-$ is upward directed, that is, if $v_1, v_2 \in \mathcal{S}_-$, then there exists $v \in \mathcal{S}_-$ such that $v_1 \le v$ and $v_2 \le v$.

First, we will prove the existence of lower (resp. upper) bounds for $\mathcal{S}_+$ (resp. $\mathcal{S}_-$). To this end, note that for a given $\varepsilon > 0$ we can find $c_8 = c_8(\varepsilon) > 0$ such that

$$f(x,s)s \ge \left(f_s'(x,0) - \varepsilon\right) s^2 - c_8|s|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{4.1}$$

This unilateral growth estimate on the nonlinearity leads to the consideration of the following auxiliary Dirichlet problem

$$\begin{aligned} -\Delta_p u - \Delta u &= \left(f_s'(\cdot,0) - \varepsilon\right) u - c_8|u|^{r-2}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

Proposition 4.1. *For every $\varepsilon > 0$ small enough problem (4.2) has a unique positive solution $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and since (4.2) is odd, $\overline{v} = -\overline{u} \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique negative solution.*

Proof. Let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{c_8}{r}\left\|u^+\right\|_r^r - \frac{1}{2}\int_\Omega \left(f_s'(\cdot,0) - \varepsilon\right)\left(u^+\right)^2 dx$$

Obviously σ_+ is coercive since $r > p > 2$ and sequentially weakly lower semicontinuous. Therefore we can find $\overline{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\overline{u}) = \inf\left[\sigma_+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{4.3}$$

Using $\|\hat{u}_1(2)\|_2 = 1$ and hypothesis (H)(v) we have for $t > 0$

$$\sigma_+(t\hat{u}_1(2)) \leq \frac{t^p}{p}\|\nabla\hat{u}_1(2)\|_p^p + \frac{t^2}{2}\left(\hat{\lambda}_1(2) - \hat{\lambda}_m(2) + \varepsilon\right) + \frac{t^r}{r}c_8\|\hat{u}_1(2)\|_r^r. \tag{4.4}$$

Recalling $m \geq 2$, let $\varepsilon \in \left(0, \hat{\lambda}_m(2) - \hat{\lambda}_1(2)\right)$. Then, since $2 < p < r$, choosing $t \in (0,1)$ small enough, (4.4) implies

$$\sigma_+(t\hat{u}_1(2)) < 0,$$

which shows, due to (4.3), that $\sigma_+(\overline{u}) < 0 = \sigma_+(0)$. Hence, $\overline{u} \neq 0$.

Since $\overline{u}$ is a critical point of σ_+ there holds $\sigma_+'(\overline{u}) = 0$ meaning that

$$-\Delta_p\overline{u} - \Delta\overline{u} = (f_s'(\cdot,0) - \varepsilon)\overline{u}^+ - c_8\left(\overline{u}^+\right)^{r-1}. \tag{4.5}$$

Acting on (4.5) with $-\overline{u}^- \in W_0^{1,p}(\Omega)$ yields

$$\left\|\nabla\overline{u}^-\right\|_p^p + \left\|\nabla\overline{u}^-\right\|_2^2 = 0,$$

which gives $\overline{u} \geq 0, \overline{u} \neq 0$. Therefore, (4.5) becomes

$$\begin{aligned} -\Delta_p\overline{u} - \Delta\overline{u} &= (f_s'(\cdot,0) - \varepsilon)\overline{u} - c_8(\overline{u})^{r-1} \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

As before, applying the regularity results of Lieberman [22, Theorem 1.1] implies that $\overline{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ is a positive solution of (4.2).

Moreover, we have

$$\Delta_p\overline{u}(x) + \Delta\overline{u}(x) \leq c_8\|\overline{u}\|_\infty^{r-p}\overline{u}(x)^{p-1} \quad \text{for a.a. } x \in \Omega.$$

Then, due to Pucci–Serrin [36, pp. 35 or 111 and 120], we obtain $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Next, we will show the uniqueness of this positive solution. For this purpose, let $G_0 : \mathbb{R}_+ = [0,\infty) \to \mathbb{R}_+$ be defined by

$$G_0(t) = \frac{1}{p}t^p + \frac{1}{2}t^2.$$

Then, $G_0 \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing and strictly convex. Additionally, $t \to G_0\left(t^{\frac{1}{2}}\right)$ is convex as well. We set $G(\xi) = G_0(\|\xi\|_{\mathbb{R}^N})$ for all $\xi \in \mathbb{R}^N$ and consider the integral functional $I_+ : L^1(\Omega) \to \overline{R} = \mathbb{R} \cup \{+\infty\}$ defined by

$$I_+(u) = \begin{cases} \displaystyle\int_\Omega G\left(\nabla u^{\frac{1}{2}}\right)dx & \text{if } u \geq 0, u^{\frac{1}{2}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} I_+ = \left\{u \in L^1(\Omega) : I_+(u) < +\infty\right\}$ being the effective domain of I_+. We set $w_1 = u_1^{\frac{1}{2}}, w_2 = u_2^{\frac{1}{2}}$ and by definition, we have $w_1, w_2 \in W_0^{1,p}(\Omega)$. We define

$$w = (tu_1 + (1-t)u_2)^{\frac{1}{2}}$$

with $t \in [0,1]$. Using Lemma 1 of Díaz–Saá [13] (see also Benguria–Brézis–Lieb [8]), we obtain

$$\|\nabla w(x)\|_{\mathbb{R}^N} \leq \left(t\,\|\nabla w_1(x)\|_{\mathbb{R}^N}^2 + (1-t)\,\|\nabla w_2(x)\|^2\right)^{\frac{1}{2}} \quad \text{for a.a.}\, x \in \Omega.$$

Hence

$$\begin{aligned} G_0\left(\|\nabla w(x)\|_{\mathbb{R}^N}\right) &\leq G_0\left(\left(t\,\|\nabla w_1(x)\|_{\mathbb{R}^N}^2 + (1-t)\,\|\nabla w_2(x)\|_{\mathbb{R}^N}^2\right)^{\frac{1}{2}}\right) \\ &\leq tG_0\left(\|\nabla w_1(x)\|_{\mathbb{R}^N}\right) + (1-t)G_0\left(\|\nabla w_2(x)\|_{\mathbb{R}^N}\right), \end{aligned}$$

since G_0 is increasing and $t \to G_0\left(t^{\frac{1}{2}}\right)$ is convex. As $G(\xi) = G_0\left(\|\xi\|_{\mathbb{R}^N}\right)$ for all $\xi \in \mathbb{R}^N$, it follows that

$$G(\nabla w(x)) \leq tG\left(\nabla w_1(x)\right) + (1-t)G\left(\nabla w_2(x)\right) \quad \text{for a.a.}\, x \in \Omega.$$

Therefore, I_+ is convex. Furthermore, using Fatou's Lemma, we see that I_+ is lower semicontinuous.

If $u \in W_0^{1,p}(\Omega)$ is a positive solution of problem (4.2), then from the first part of the proof we know that $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, hence $u^2 \in \operatorname{dom} I_+$. Then, for every $h \in C_0^1(\overline{\Omega})$ and for $|t| > 0$ small enough, we have $u^2 + th \in \operatorname{dom} I_+$.

Using this fact and the chain rule we see that the Gateaux derivative of I_+ at u^2 in the direction h exists and is equal to

$$I_+'\left(u^2\right)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p u - \Delta u}{u} h dx.$$

Suppose now that $v \in W_0^{1,p}(\Omega)$ is another positive solution of (4.2). As before, we obtain $v \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), v^2 \in \operatorname{dom} I_+$ and

$$I_+'\left(v^2\right)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p v - \Delta v}{v} h dx \quad \text{for all } h \in C_0^1(\overline{\Omega}).$$

The convexity of I_+ implies that I_+' is monotone. Thus

$$\begin{aligned} 0 &\leq \left\langle I_+'\left(u^2\right) - I_+'\left(v^2\right), u^2 - v^2\right\rangle_{L^1(\Omega)} \\ &= \frac{1}{2}\int_\Omega \left(\frac{-\Delta_p u - \Delta u}{u} - \frac{-\Delta_p v - \Delta v}{v}\right)\left(u^2 - v^2\right) dx \end{aligned}$$

$$= \frac{1}{2} \int_\Omega c_8 \left(v^{r-1} - u^{r-1}\right)\left(u^2 - v^2\right) dx.$$

Since $s \to s^{r-1}$ is strictly increasing on $\mathbb{R}_+$, we have $u = v$. This proves the uniqueness of the positive solution $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Since (4.2) is odd, $\overline{v} = -\overline{u} \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique negative solution of problem (4.2). $\square$

Now we can use the solutions of Proposition 4.1 to create bounds for the elements of $\mathcal{S}_+$ and $\mathcal{S}_-$.

Proposition 4.2. *If hypotheses (H) hold, then $\overline{u} \leq u$ for all $u \in \mathcal{S}_+$ and $v \leq \overline{v}$ for all $v \in \mathcal{S}_-$ with $\overline{u}, \overline{v}$ being the nontrivial unique constant sign solutions of problem (4.2) obtained in Proposition 4.1.*

Proof. We will do the proof only the elements of $\mathcal{S}_+$, the proof for the elements of $\mathcal{S}_-$ works similar.

Let $\varepsilon > 0$ be small enough as postulated by Proposition 4.1 and let $u \in \mathcal{S}_+$. We introduce the Carathéodory function $e : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ \left(f_s'(x,0) - \varepsilon\right) s - c_8 s^{r-1} & \text{if } 0 \leq s \leq u(x), \\ \left(f_s'(x,0) - \varepsilon\right) u(x) - c_8 u(x)^{r-1} & \text{if } u(x) < s. \end{cases} \tag{4.6}$$

We set $E(x,s) = \int_0^s e(x,t)dt$ and consider the C^1-functional $\chi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\chi(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega E(x,u)dx.$$

Due to the truncation defined in (4.6), it is clear that χ is coercive and as before it is also sequentially weakly lower semicontinuous. Hence, we find $\overline{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\chi\left(\overline{u}_0\right) = \inf\left[\chi(u) : u \in W_0^{1,p}(\Omega)\right].$$

Since $u \in \mathcal{S}_+ \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, using Lemma 3.3 of Filippakis–Kristály–Papageorgiou [17], we can find $t \in (0,1)$ small enough such that $t\hat{u}_1(p) \leq u$. Moreover, since $2 < p < r$ by taking $t \in (0,1)$ even smaller if necessary, we have $\chi\left(t\hat{u}_1(2)\right) < 0$ and so $\chi\left(\overline{u}_0\right) < 0 = \chi(0)$. Hence, $\overline{u}_0 \neq 0$.

Since $\overline{u}_0$ is the global minimizer of χ, there holds $\chi'\left(\overline{u}_0\right) = 0$ which reads as

$$-\Delta_p \overline{u}_0 - \Delta \overline{u}_0 = N_e\left(\overline{u}_0\right). \tag{4.7}$$

First, we choose $-\overline{u}_0^- \in W_0^{1,p}(\Omega)$ as test function in (4.7). This gives

$$\left\|\nabla\overline{u}_0^-\right\|_p^p + \left\|\nabla\overline{u}_0^-\right\|_2^2 = 0$$

and so $\overline{u}_0 \geq 0, \overline{u}_0 \neq 0$. Next, we act with $(\overline{u}_0 - u)^+ \in W_0^{1,p}(\Omega)$ on (4.7). This leads to, based on (4.1), (4.6) and $u \in \mathcal{S}_+$,

$$\begin{aligned}
&\left\langle -\Delta_p \overline{u}_0, (\overline{u}_0 - u)^+ \right\rangle + \left\langle -\Delta \overline{u}_0, (\overline{u}_0 - u)^+ \right\rangle \\
&= \int_\Omega e\left(x, \overline{u}_0\right)\left(\overline{u}_0 - u\right)^+ dx \\
&= \int_\Omega \left(\left(f_s'(x,0) - \varepsilon\right) u - c_8 u^{r-1}\right)\left(\overline{u}_0 - u\right)^+ dx \\
&\leq \int_\Omega f(x,u)\left(\overline{u}_0 - u\right)^+ dx \\
&= \left\langle -\Delta_p u, (\overline{u}_0 - u)^+ \right\rangle + \left\langle -\Delta u, (\overline{u}_0 - u)^+ \right\rangle,
\end{aligned}$$

from which we infer that

$$\left\langle -\Delta_p \overline{u}_0 + \Delta_p u, (\overline{u}_0 - u)^+ \right\rangle + \left\|\nabla\left(\overline{u}_0 - u\right)^+\right\|_2^2 \leq 0.$$

Therefore, $\overline{u}_0 \leq u$. So, we have proved that

$$\overline{u}_0 \in [0,u] = \left\{ y \in W_0^{1,p}(\Omega) : 0 \leq y(x) \leq u(x) \text{ for a.a. } x \in \Omega \right\}. \tag{4.8}$$

Taking (4.6) and (4.8) into account we see that equation (4.7) becomes

$$-\Delta_p \overline{u}_0 - \Delta \overline{u}_0 = \left(f_s'(\cdot,0) - \varepsilon\right)\overline{u}_0 - c_8 \overline{u}_0^{r-1},$$

which implies, due to Proposition 4.1, that $\overline{u}_0 = \overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Hence, $\overline{u} \leq u$.

Since $u \in \mathcal{S}_+$ is arbitrary, we have the conclusion of the proposition for the set $\mathcal{S}_+$. Similarly, we show that $v \leq \overline{v}$ for all $v \in \mathcal{S}$. □

These bounds lead to the existence of extremal constant sign solutions for problem (1.1) stated in the next proposition.

Proposition 4.3. *Let hypotheses (H) be satisfied. Then problem (1.1) has a smallest positive solution $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and a greatest negative solution $v_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Proof. By applying Lemma 3.10 on p. 178 of Hu–Papageorgiou [20] there exists a decreasing sequence $(u_n)_{n\geq 1} \subseteq \mathcal{S}_+$ such that

$$\inf \mathcal{S}_+ = \inf_{n\geq 1} u_n.$$

Since $u_n \in \mathcal{S}_+$ for every $n \in \mathbb{N}$, we have

$$-\Delta_p u_n - \Delta u_n = N_f(u_n). \tag{4.9}$$

Thanks to hypothesis (H)(i) we easily check that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^p(\Omega). \tag{4.10}$$

Owing to Proposition 4.2 we know that $\overline{u} \leq u_n$ for all $n \in \mathbb{N}$. Hence,

$$\overline{u} \leq u_* \neq 0.$$

Acting on (4.9) with $u_n - u_* \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (4.10) yields

$$\lim_{n\to\infty} \left[\langle -\Delta_p u_n, u_n - u_*\rangle + \langle -\Delta u_n, u_n - u_*\rangle\right] = 0,$$

which in view of the monotonicity of $-\Delta$ results in

$$\limsup_{n\to\infty} \left[\langle -\Delta_p u_n, u_n - u_*\rangle + \langle -\Delta u_*, u_n - u_*\rangle\right] \leq 0.$$

Thanks to the convergence properties in (4.10) the second term above goes to zero and so we derive

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u_*\rangle \leq 0,$$

which implies, due to Proposition 2.4 along with (4.10) that

$$u_n \to u_* \quad \text{in } W_0^{1,p}(\Omega). \tag{4.11}$$

So, if we pass in (4.9) to the limit as $n \to \infty$ and use (4.11), then

$$-\Delta_p u_* - \Delta u_* = N_f(u_*).$$

Hence, $u_* \in \mathcal{S}_*$ and $u_* = \inf \mathcal{S}_+$.

Similarly we generate $v_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ being the greatest negative solution of (1.1). $\square$

Following the approach outlined in the beginning of this section, we focus now on the order interval $[v_*, u_*]$ looking for a nontrivial solution of (1.1) different from u_* and v_*. Such a solution necessarily must be nodal. In order to show the existence of such a solution, we will combine variational tools with Morse theory in terms of critical groups. We start by computing the critical groups of φ at the origin.

Proposition 4.4. *If hypotheses (H) hold, then* $C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$ *with* $d_m = \dim \overline{H}_m \geq 2$ *and* $\overline{H}_m = \bigoplus_{i=1}^m E\left(\hat{\lambda}_i(2)\right)$.

Proof. Regarding hypothesis (H)(i) we have that $m(x) := f_s'(x,0) \in L^\infty(\Omega)$. Let us consider the C^2-functional $\psi : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega m(x)u^2 dx.$$

We claim that $K_\psi = \{0\}$. Indeed, if $u \in K_\psi$, then

$$\begin{aligned} -\Delta u &= m(x)u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.12}$$

Hypothesis (H)(v) and Proposition 2.7 yield

$$\hat{\lambda}_m(2, m) < \hat{\lambda}_m\left(2, \hat{\lambda}_m(2)\right) = 1, \quad 1 = \hat{\lambda}_{m+1}\left(2, \hat{\lambda}_{m+1}(2)\right) < \hat{\lambda}_{m+1}(2, m). \tag{4.13}$$

Hence, (4.12) and (4.13) imply that $K_\psi = \{0\}$. Moreover, since

$$\langle \psi''(0)v, h\rangle = \int_\Omega (\nabla v, \nabla h)_{\mathbb{R}^N} dx - \int_\Omega m(x)vh dx \quad \text{for all } v, h \in H_0^1(\Omega),$$

from the argument above and using Proposition 2.6, we infer that $u = 0$ is a nondegenerate critical point of ψ with Morse index $d_m = \dim \overline{H}_m, \overline{H}_m = \bigoplus_{i=1}^m E\left(\hat{\lambda}_i(2)\right)$. It follows that

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.14}$$

Let $\psi_0 = \psi\big|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, from Palais [28], we have

$$C_k(\psi_0, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

which gives, due to (4.14),

$$C_k(\psi_0, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.15}$$

We consider the homotopy $h : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h(t,u) = t\varphi(u) + (1-t)\psi_0(u).$$

Suppose that we can find $(t_n)_{n\geq 1} \subseteq [0,1]$ and $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1], \quad u_n \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}_0. \tag{4.16}$$

The last assertion in (4.16) reads as

$$t_n(-\Delta_p u_n) - \Delta u_n = t_n N_f(u_n) + (1-t_n) m u_n \quad \text{for all } m \in \mathbb{N}. \tag{4.17}$$

We set $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and (4.17) yields

$$t_n \|u_n\|^{p-2}(-\Delta_p y_n) - \Delta y_n = t_n \frac{N_f(u_n)}{\|u_n\|} + (1-t_n) m y_n \tag{4.18}$$

for all $n \in \mathbb{N}$. We have that

$$(-\Delta_p y_n)_{n\geq 1} \subseteq W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^* \text{ is bounded.} \tag{4.19}$$

Passing to a subsequence if necessary we may assume that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(\Omega). \tag{4.20}$$

Hypotheses (H)(i), (v) imply that

$$|f(x,s)| \leq c_9\left(|s| + |s|^{r-1}\right) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \in \mathbb{R}$$

and for some $c_9 > 0$. This gives

$$\left|\frac{f(x,u_n(x))}{\|u_n\|}\right| \leq c_9\left(1 + \|u_n\|^{r-2}\right)|y_n(x)| \quad \text{for a.a. } x \in \Omega \tag{4.21}$$

and for all $n \in \mathbb{N}$. Due to (4.20) we conclude that

$$\left(\frac{N_f(u_n)}{\|u_n\|}\right)_{n\geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

This fact, hypothesis (H)(v) and by passing to a subsequence if necessary, imply that

$$\frac{N_f(u_n)}{\|u_n\|} \rightharpoonup my \quad \text{in } L^2(\Omega) \tag{4.22}$$

see, for example Aizicovici–Papageorgiou–Staicu [1]. We return to (4.18), pass to the limit as $n \to \infty$ and use (4.16), (4.19), (4.20) and (4.22) to obtain $-\Delta y = my$ meaning that

$$\begin{aligned} -\Delta y &= m(x) y \qquad \text{in } \Omega, \\ y &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

As before, using hypothesis (H)(v) and Proposition 2.7, we infer that

$$y = 0. \tag{4.23}$$

On the other hand from (4.18), (4.21) and Theorem 7.1 of Ladyzhenskaya–Ural'tseva [21] (with $m = 2$), we know that there exists $M_{12} > 0$ such that

$$y_n \in L^\infty(\Omega) \quad \text{and} \quad \|y_n\|_\infty \le M_{12} \quad \text{for all } n \in \mathbb{N}.$$

But then Theorem 1 of Lieberman [22] (with $m = 2$) implies that there exist $\beta \in (0,1)$ and $M_{13} > 0$ such that

$$y_n \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|y_n\|_{C_0^{1,\beta}(\overline{\Omega})} \le M_{13} \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\beta}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and using (4.20), we have

$$y_n \to y \quad \text{in } C_0^1(\overline{\Omega}),$$

which gives

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega).$$

Therefore $\|y\| = 1$ which contradicts (4.23). This means that (4.16) can not happen and then using the homotopy invariance property of critical groups, we get

$$C_k\left(h(0,\cdot),0\right) = C_k\left(h(1,\cdot),0\right) \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k\left(\psi_0,0\right) = C_k\left(\varphi,0\right) \quad \text{for all } k \in \mathbb{N}_0.$$

Finally, due to (4.15), it follows

$$C_k\left(\varphi,0\right) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \quad \square$$

Remark 4.5. An alternative proof of this result can be based on the following argument. Let $\hat{\psi}_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\hat{\psi}_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega m(x)u^2 dx.$$

Note that

$$\left|\hat{\psi}_0(u) - \psi_0(u)\right| = \frac{1}{p}\|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Moreover, there holds, for all $u, h \in W_0^{1,p}(\Omega)$,

$$\left|\left\langle \hat{\psi}_0'(u) - \psi_0'(u), h \right\rangle\right| = |\langle -\Delta_p u, h\rangle| \leq \|\nabla u\|_p^{p-1} \|h\|.$$

From these estimates and Theorem 5.1 of Corvellec–Hantoute [11] (continuity of the critical groups in the C^1-norm), we derive

$$C_k\left(\hat{\psi}_0, 0\right) = C_k\left(\psi_0, 0\right) \quad \text{for all } k \in \mathbb{N}_0,$$

which in view of (4.15) gives

$$C_k\left(\hat{\psi}_0, 0\right) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Note that hypothesis (H)(v) implies that for a given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x,s) - m(x)s| \leq \varepsilon \quad \text{and} \quad \left|F(x,s) - \frac{1}{2}m(x)s^2\right| \leq \varepsilon$$

for a.a. $x \in \Omega$ and for all $|s| \leq \delta$. Then, we have

$$\left|\hat{\psi}_0(u) - \varphi(u)\right| \leq \int_\Omega \left|F(x,u) - \frac{1}{2}m(x)u^2\right| dx$$

and

$$\left|\left\langle \hat{\psi}_0'(u) - \varphi'(u), h \right\rangle\right| \leq \int_\Omega |f(x,u) - m(x)u|\, dx.$$

Hence,

$$C_k\left(\hat{\psi}_0\big|_{C_0^1(\overline{\Omega})}, 0\right) = C_k\left(\varphi\big|_{C_0^1(\overline{\Omega})}, 0\right)$$

which implies $C_k\left(\hat{\psi}_0, 0\right) = C_k\left(\varphi, 0\right)$ for all $k \in \mathbb{N}_0$, see Palais [28]. Therefore, we conclude that

$$C_k\left(\varphi, 0\right) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

In what follows $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ are the two extremal constant sign solutions of problem (1.1) obtained in Proposition 4.3.

Proposition 4.6. *Let hypotheses (H) be satisfied. Then problem (1.1) admits a nodal solution* $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$.

Proof. First, we introduce the following truncation of the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$

$$f_*(x,s) = \begin{cases} f(x, v_*(x)) & \text{if } s < v_*(x), \\ f(x,s) & \text{if } v_*(x) \le s \le u_*(x), \\ f(x, u_*(x)) & \text{if } u_*(x) < s, \end{cases} \tag{4.24}$$

which is clearly a Carathéodory function. We set $F_*(x,s) = \int_0^s f_*(x,t)dt$ and consider the C^1-functional $\psi_* : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_*(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega F_*(x,u)dx.$$

Furthermore, we define the Carathéodory functions

$$f_*^+(x,s) = f_*\left(x, s^+\right) \quad \text{and} \quad f_*^-(x,s) = f_*\left(x, -s^-\right).$$

We set $F_*^\pm(x,s) = \int_0^s f_*^\pm(x,t)dt$ and consider the C^1-functionals $\psi_*^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_*^\pm(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega F_*^\pm(x,u)dx.$$

Reasoning as in the proof of Proposition 3.5 we can show that

$$K_{\psi_*} \subseteq [v_*, u_*], \quad K_{\psi_*^+} \subseteq [0, u_*], \quad K_{\psi_*^-} \subseteq [v_*, 0],$$

where

$$[0,u_*] = \left\{u \in W_0^{1,p}(\Omega) : 0 \le u(x) \le u_*(x) \text{ for a.a. } x \in \Omega\right\},$$
$$[v_*,0] = \left\{u \in W_0^{1,p}(\Omega) : v_*(x) \le u(x) \le 0 \text{ for a.a. } x \in \Omega\right\}.$$

The extremality of $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ implies that

$$K_{\psi_*} \subseteq [v_*, u_*], \quad K_{\psi_*^+} = \{0, u_*\}, \quad K_{\psi_*^-} = \{0, v_*\}. \tag{4.25}$$

Next we show that u_* and v_* are local minimizers of ψ_*. From (4.24) it is clear that ψ_*^+ is coercive and because of the Sobolev embedding theorem it is sequentially weakly lower semicontinuous as well. Thus, there exists $\hat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\psi_*^+(\hat{u}_*) = \inf\left[\psi_*^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{4.26}$$

Since $m \ge 2$, $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $2 < p$ we can choose $t \in (0,1)$ small enough such that

$$t\hat{u}_1(2) \leq u_* \quad \text{and} \quad \psi_*^+ \left(t\hat{u}_1(2)\right) < 0,$$

see hypotheses (H)(v) and use Lemma 3.3 of Filippakis–Kristály–Papageorgiou [17]. Then, from (4.26) it follows that $\psi_*^+(\hat{u}_*) < 0 = \psi_*^+(0)$ meaning that

$$\hat{u}_* \neq 0.$$

Because of (4.25) and since $\hat{u}_*$ is a nontrivial critical point of ψ_*^+ we know that $\hat{u}_* = u_*$. Note that $\psi_*\big|_{C_0^1(\overline{\Omega})_+} = \psi_*^+\big|_{C_0^1(\overline{\Omega})_+}$ and $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Hence, u_* is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_* and by means of Proposition 2.3 it is also a $W_0^{1,p}(\Omega)$-minimizer of ψ_*. Similarly we can show this for v_* working with the functional ψ_*^-.

Without any loss of generality we may assume that $\psi_*(v_*) \leq \psi_*(u_*)$, the analysis is similar if the opposite inequality holds. Moreover we may assume that K_{ψ_*} is finite, otherwise (4.25) implies the existence of a whole sequence of distinct nodal solutions.

Recall that $v_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a local minimizer of ψ_*. Hence, we can find $\rho \in (0,1)$ small enough such that $\|v_* - u_*\| > \rho$ and

$$\psi_*(v_*) \leq \psi_*(u_*) < \inf\left[\psi_*(u) : \|u - u_*\| = \rho\right] = m_\rho, \tag{4.27}$$

see Aizicovici–Papageorgiou–Staicu [1, proof of Proposition 29]. We know that ψ_* is coercive and so it satisfies the C-condition, see Papageorgiou–Winkert [32, Proposition 3.2]. This fact and (4.27) permit the usage of the mountain pass theorem stated as Theorem 2.2. Therefore, we find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\psi_*} \quad \text{and} \quad m_\rho \leq \psi_*(y_0). \tag{4.28}$$

From (4.25), (4.27) (4.28) and the nonlinear regularity theory of Lieberman [22] it follows

$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad y_0 \notin \{v_*, u_*\}. \tag{4.29}$$

Since y_0 is a critical point of mountain pass type for the functional ψ_* there holds

$$C_1(\psi_*, y_0) \neq 0, \tag{4.30}$$

see Motreanu–Motreanu–Papageorgiou [25, p. 176].

We need to show that y_0 is nontrivial in order to conclude that it is nodal. To this end, we compute the critical groups of ψ_* at $u = 0$ in order to compare them with (4.30).

Claim. *$C_k(\psi_*, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_m = \dim \overline{H}_m$ and $\overline{H}_m = \bigoplus_{i=1}^m E\left(\hat{\lambda}_i(2)\right)$.*

Consider the homotopy $\hat{h} : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{h}(t,u) = t\varphi(u) + (1-t)\psi_*(u).$$

Suppose that we can find $(t_n)_{n\geq 1} \subseteq [0,1]$ and $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1], \quad \|u_n\| \to 0 \quad \text{and} \quad \hat{h}'_u(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.31}$$

The last assertion in (4.31) reads as

$$-\Delta_p u_n - \Delta u_n = t_n N_f(u_n) + (1-t_n) N_{f_*}(u_n) \quad \text{for all } n \in \mathbb{N}$$

meaning that

$$\begin{aligned} -\Delta_p u_n - \Delta u_n &= t_n f(x,u_n) + (1-t_n) f_*(x,u_n) \qquad && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Theorem 7.1 of Ladyzhenskaya–Ural'tseva [21] implies that we can find $M_{14} > 0$ such that

$$u_n \in L^\infty(\Omega) \quad \text{and} \quad \|u_n\|_\infty \leq M_{14} \quad \text{for all } n \in \mathbb{N}. \tag{4.32}$$

Then, (4.32) and Theorem 1 of Lieberman [22] imply the existence of $\alpha \in (0,1)$ and $M_{15} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_{15} \quad \text{for all } n \in \mathbb{N}. \tag{4.33}$$

From (4.31), (4.33) and because $C_0^{1,\alpha}(\overline{\Omega})$ is compactly embedded into $C_0^1(\overline{\Omega})$ we obtain

$$u_n \to 0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

Thus, we find a number n_0 such that $u_n \in [v_*, u_*]$ for all $n \geq n_0$. Then, due to (4.25), $(u_n)_{n\geq n_0} \subseteq K_{\psi_*}$. But this contradicts our hypothesis that K_{ψ_*} is finite. Therefore, (4.31) can not occur and then exploiting the homotopy invariance of critical groups it holds

$$C_k(\varphi, 0) = C_k(\psi_*, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

which, by virtue of Proposition 4.4, gives

$$C_k(\psi_*, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

This proves the claim.

Combining the Claim with (4.30) we see that $y_0 \neq 0$. Then, (4.29) implies that $y_0 \in C_0^1(\overline{\Omega})$ is a nodal solution of (1.1).

Now, let $\rho = \max\{\|v_*\|_\infty, \|u_*\|_\infty\}$. The differentiability of $f(x,\cdot)$ and hypotheses (H)(i),(iv) imply the existence of $\hat{\xi}_\rho > 0$ such that

$$s \to f(x,s) + \hat{\xi}_\rho |s|^{p-2}s \text{ is nondecreasing on } [-\rho, \rho] \text{ for a.a. } x \in \Omega.$$

From this and $y_0 \leq u_*$ it follows

$$-\Delta_p y_0 - \Delta y_0 + \hat{\xi}_\rho |y_0|^{p-2} y_0 = f(x, y_0) + \hat{\xi}_\rho |y_0|^{p-2} y_0 \\ \leq f(x, u_*) + \hat{\xi}_\rho u_*^{p-1} \\ = -\Delta_p u_* - \Delta u_+ + \hat{\xi}_\rho u_*^{p-1} \tag{4.34}$$

Applying the tangency principle of Pucci–Serrin [36, p. 35] yields

$$y_0(x) < u_*(x) \quad \text{for all } x \in \Omega.$$

Then, for every compact set $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon(K) > 0$ such that

$$f(x, y_0) + \hat{\xi}_\rho |y_0|^{p-2} y_0 + \varepsilon \leq f(x, u_*) + \hat{\xi}_\rho u_*^{p-1} \quad \text{for a.a. } x \in K.$$

So, from (4.34) and the strong comparison principle (see Arcoya–Ruiz [5] and Papageorgiou–Winkert [34]), we have

$$u_* - y_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{4.35}$$

In the same way we can show that

$$y_0 - v_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{4.36}$$

From (4.35) and (4.36) we conclude that

$$y_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [v_*, u_*]. \quad \square$$

Proposition 4.7. *If hypotheses (H) hold, then $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof. As in the proof of Proposition 4.6, using the homotopy invariance of critical groups, the nonlinear regularity results of Ladyzhenskaya–Ural'tseva [21] and Lieberman [22] as well as the fact that $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v_*, u_*]$ (see Proposition 4.6), we establish that

$$C_k(\varphi, y_0) = C_k(\psi_*, y_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{4.37}$$

whereby we recall that by hypothesis K_φ is finite. From (4.30) and (4.37) it follows that

$$C_1(\varphi, y_0) \neq 0.$$

But $\varphi \in C^2\left(W_0^{1,p}(\Omega)\right)$. So, from Papageorgiou–Smyrlis [31] (see also Papageorgiou–Rădulescu [30]), we conclude that

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \quad \square$$

Proposition 4.8. *If hypotheses (H) hold and K_φ is finite, then $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $\partial B_1^+ = \left\{u \in W_0^{1,p}(\Omega) : \|u\| = 1, u^+ \neq 0\right\}$ and consider the deformation $h : [0,1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h(t,u) = \frac{(1-t)u + t\hat{u}_1(p)}{\|(1-t)u + t\hat{u}_1(p)\|}.$$

Note that

$$h(0,\cdot)\big|_{\partial B_1^+} = \mathrm{id}\big|_{\partial B_1^+} \quad \text{and} \quad h(1,u) = \frac{\hat{u}_1(p)}{\|\hat{u}_1(p)\|} \in \partial B_1^+ \quad \text{for all } u \in \partial B_1^+.$$

This shows that ∂B_1^+ is contractible.

From hypothesis (H)(ii) we see that for any given $\eta > 0$ we can find $M_{16} = M_{16}(\eta) > 0$ such that

$$F(x,s) \geq \frac{\eta}{p} s^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M_{16}. \tag{4.38}$$

Hypothesis (H)(iv) implies that we can find $c_{10} > 0$ and $\hat{M}_{16} > 0$ such that

$$-\frac{c_{10}}{p}|s|^p \leq F(x,s) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leq -\hat{M}_{16}. \tag{4.39}$$

Finally, hypothesis (H)(i) implies the existence of $c_{11} > 0$ such that

$$|F(x,s)| \leq c_{11} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \left[\hat{M}_{16}, M_{16}\right]. \tag{4.40}$$

Now let $u \in \partial B_1^+$, $t \geq 1$ and define

$$\begin{aligned}
&\Omega_+ := \{x \in \Omega : tu(x) \geq M_{16}\}, \quad \Omega^- := \left\{x \in \Omega : tu(x) \leq -\hat{M}_{16}\right\},\\
&\Omega_+^- := \left\{x \in \Omega : -\hat{M}_{16} < tu(x) < M_{16}\right\}.
\end{aligned}$$

Using (4.38), (4.39), (4.40) and the fact that $\|u\| = 1$ yields

$$\begin{aligned}
\varphi(tu) &= \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,tu)dx\\
&= \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \int_{\Omega_+} F(x,tu)dx - \int_{\Omega^-} F(x,tu)dx\\
&\quad - \int_{\Omega_+^-} F(x,tu)dx
\end{aligned} \tag{4.41}$$

$$\leq \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{t^p}{p}\eta\int_{\Omega_+} u^p dx + \frac{t^p}{p}c_{10}\int_{\Omega^-}|u|^p dx + c_{11}|\Omega|_N$$

$$\leq \frac{t^p}{p}\left[c_{12} - \eta\int_{\Omega_+} u^p dx\right] + \frac{t^2}{2}\|\nabla u\|_2^2 + c_{11}|\Omega|_N$$

for some $c_{12} > 0$. Since $u \in \partial B_1^+$ we know that $u^+ \neq 0$. Hence, we can find numbers $t_0 > 0$ and $\xi_0 > 0$ such that

$$\int_{\Omega_+} u^p dx \geq \xi_0 \quad \text{for all } t \geq t_0. \tag{4.42}$$

Using (4.42) in (4.41) results in

$$\varphi(tu) \leq \frac{t^p}{p}[c_{12} - \eta\xi_0] + \frac{t^2}{2}\|\nabla u\|_2^2 + c_{11}|\Omega|_N \quad \text{for all } t \geq t_0. \tag{4.43}$$

Recall that $\eta > 0$ is arbitrary, so we can choose $\eta > 0$ large enough such that $\eta\xi_0 > c_{12}$. Then, from (4.43) and since $2 < p$ we have

$$\varphi(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.44}$$

Hypotheses (H)(i),(ii), (iv) imply the existence of $c_{13} > 0$ such that

$$-c_{13} + \int_\Omega pF(x,u)dx \leq \int_\Omega f(x,u)udx. \tag{4.45}$$

Then, using the chain rule, (4.45) and recall $p > 2$, we obtain

$$\begin{aligned}
\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u\rangle \\
&= \frac{1}{t}\langle \varphi'(tu), tu\rangle \\
&= \frac{1}{t}\left[\|\nabla u\|_p^p + \|\nabla u\|_2^2 - \int_\Omega f(x,tu)tudx\right] \\
&\leq \frac{1}{t}\left[p\varphi(tu) + c_{13}\right].
\end{aligned}$$

Using (4.44) we see that for $t \geq 1$ large enough we have

$$\frac{d}{dt}\varphi(tu) < 0.$$

Let $\vartheta < \min\left\{\frac{c_{13}}{p}, \inf_{\overline{B}_1^+}\varphi\right\}$. By the implicit function theorem we see that there exists an unique $k \in C\left(\partial B_1^+\right), k \geq 1$ such that

$$\varphi(tu) = \begin{cases} > \vartheta & \text{if } t \in [0, k(u)), \\ = \vartheta & \text{if } t = k(u), \\ < \vartheta & \text{if } t > k(u). \end{cases} \tag{4.46}$$

Due to the choice of ϑ and (4.46) we have

$$\varphi^\vartheta \subseteq \left\{tu : u \in \partial B_1^+, t \geq k(u)\right\}.$$

Let $D_+ = \left\{tu : u \in \partial B_1^+, t \geq 1\right\}$. Then $\varphi^\vartheta \subseteq D_+$. We consider the deformation $h_0 : [0,1] \times D_+ \to D_+$ defined by

$$h_0(s, tu) = \begin{cases} (1-s)tu + sk(u)u & \text{if } t \in [1, k(u)], \\ tu & \text{if } t > k(u). \end{cases}$$

Then

$$h_0\left(1, D_+\right) \subseteq \varphi^\vartheta \quad \text{and} \quad h_+(s, \cdot)\big|_{\varphi^\vartheta} = \text{id}\,\big|_{\varphi^\vartheta} \quad \text{for all } s \in [0,1].$$

Therefore, φ^ϑ is a strong deformation retract of D_+. Using the radial retraction and Theorem 6.5 of Dugundji [14], we see that D_+ and ∂B_1^+ are homotopy equivalent.

So, we have

$$H_k\left(W_0^{1,p}(\Omega), \varphi^\vartheta\right) = H_k\left(W_0^{1,p}(\Omega), D_+\right) = H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) \quad \text{for all } k \in \mathbb{N}_0,$$

see Motreanu–Motreanu–Papageorgiou [25, p. 143]. Recall that ∂B_1^+ is contractible, thus

$$H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

see Motreanu–Motreanu–Papageorgiou [25, p. 147], which gives

$$H_k\left(W_0^{1,p}(\Omega), \varphi^\vartheta\right) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Taking $\vartheta < 0$ even more negative if necessary, we conclude that

$$C_k\left(\varphi, \infty\right) = 0 \quad \text{for all } k \in \mathbb{N}_0. \quad \square$$

Now we are ready to state and prove the complete multiplicity theorem for problem (1.1).

Theorem 4.9. *Let hypotheses (H) be satisfied. Then problem (1.1) has at least five nontrivial solutions, namely*

- $u_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad \hat{u}, \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *with* $\tilde{u} - \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$,
- $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [u_0, \hat{u}]$ *nodal,*
- $\hat{y} \in C_0^1(\overline{\Omega})$.

Proof. From Proposition 3.4 we have a negative solution $u_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ being a local minimizer of the energy functional φ. Hence

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.47}$$

Proposition 3.5 provides two positive solutions $\hat{u}, \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ with $\hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ being a local minimizer of φ. Therefore

$$C_k(\varphi, \hat{u}) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.48}$$

In addition, Proposition 3.5 gives $\hat{u} \le \tilde{u}$. Let $\rho = \|\tilde{u}\|_\infty$. From the proof of Proposition 4.6 we know that we can find $\hat{\xi}_\rho > 0$ such that $s \to f(x,s) + \hat{\xi}_\rho s^{p-1}$ is nondecreasing on $[0, \rho]$. Using this it results in

$$\begin{aligned} -\Delta_p \hat{u} - \Delta \hat{u} + \hat{\xi}_\rho \hat{u}^{p-1} &= f(x, \hat{u}) + \hat{\xi}_\rho \hat{u}^{p-1} \\ &\le f(x, \tilde{u}) + \hat{\xi}_\rho \tilde{u}^{p-1} \\ &= -\Delta_p \tilde{u} - \Delta \tilde{u} + \hat{\xi}_\rho \tilde{u}^{p-1} \quad \text{for a.a. } x \in \Omega. \end{aligned} \tag{4.49}$$

As before, see the proof of Proposition 4.6, the tangency principle of Pucci–Serrin [36, p. 35] implies that

$$\hat{u}(x) < \tilde{u}(x) \quad \text{for all } x \in \Omega,$$

which yields

$$f(x, \hat{u}) + \hat{\xi}_\rho \hat{u}^{p-1} < f(x, \tilde{u}) + \hat{\xi}_\rho \tilde{u}^{p-1} \quad \text{for a.a. } x \in \Omega.$$

Then, from (4.49) and the strong comparison principle (see Arcoya–Ruiz [5] and Papageorgiou–Winkert [34]) we obtain

$$\tilde{u} - \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{4.50}$$

We assume that K_φ is finite, otherwise we already have infinitely many solutions. Let $\hat{\psi} \in C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$ be as in the proof of Proposition 3.5. Consider the homotopy

$$\tilde{h}(t,u) = t\varphi(u) + (1-t)\hat{\psi}$$

and reasoning as in the proof of Proposition 4.6, via the homotopy invariance of critical groups and (4.50), we establish that

$$C_k(\varphi, \tilde{u}) = C_k\left(\hat{\psi}, \tilde{u}\right) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.51}$$

From the proof of Proposition 3.5 we know that $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a critical point of $\hat{\psi}$ of mountain pass type. Thus,

$$C_1\left(\hat{\psi}, \tilde{u}\right) \neq 0,$$

which gives, due to (4.51), that

$$C_1(\varphi, \tilde{u}) \neq 0.$$

Then, from Papageorgiou–Smyrlis [31], we have

$$C_k(\varphi, \tilde{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.52}$$

In Proposition 4.6 we have shown that

$$y_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [u_0, \hat{u}]$$

is a nodal solution of (1.1) and Proposition 4.7 says that

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.53}$$

Finally, from Propositions 4.4 and 4.8, we have

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \tag{4.54}$$

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.55}$$

Let us now suppose that $K_\varphi = \{0, u_0, \hat{u}, \tilde{u}, y_0\}$. Then, by applying (4.47), (4.48), (4.52), (4.53), (4.54), (4.55) and the Morse relation stated in (2.7) with $t = -1$, it follows

$$2(-1)^0 + 2(-1)^1 + (-1)^{d_m} = 0,$$

which implies $(-1)^{d_m} = 0$, a contradiction. Hence, there exists $\hat{y} \in K_\varphi$ with $\hat{y} \notin \{0, u_0, \hat{u}, \tilde{u}, y_0\}$. As before, the nonlinear regularity theory shows that $\hat{y} \in C_0^1(\overline{\Omega})$. □

Remark 4.10. We mention that Recova–Rumbos [38] proved the existence of only three solutions, under similar conditions but with considerably more restrictive hypotheses on the nonlinearity, for semilinear Dirichlet problems driven by the Laplacian, see Theorem 1.2 in [38].

Conflict of interest statement

There is no conflict of interest.

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