Research Article

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Boundedness, existence and uniqueness results for coupled gradient dependent elliptic systems with nonlinear boundary condition

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Abstract: In this paper, we study coupled elliptic systems with gradient dependent right-hand sides and nonlinear boundary conditions, where the left-hand sides are driven by so-called double phase operators. By applying a surjectivity result for pseudomonotone operators along with an equivalent norm in the function space, we prove that the system has at least one nontrivial solution under very general assumptions on the data. Under slightly stronger conditions, we are also able to show that this solution is unique. As a result of independent interest, we further prove the boundedness of solutions to such elliptic systems by employing Moser's iteration scheme.

Keywords: boundedness results, convection term, coupled elliptic systems, equivalent norm, gradient dependence, nonlinear boundary condition, pseudomonotone operators, Robin and Steklov eigenvalues

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1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 1$, with Lipschitz boundary $\partial\Omega$, in this paper, we consider coupled elliptic systems of the form

$$\begin{aligned} -\operatorname{div}(\mathcal{D}_{p_1,q_1,\mu_1}(u_1)) &= f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) && \text{in } \Omega,\\ -\operatorname{div}(\mathcal{D}_{p_2,q_2,\mu_2}(u_2)) &= f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) && \text{in } \Omega,\\ \mathcal{D}_{p_1,q_1,\mu_1}(u_1)\cdot \vartheta &= g_1(x, u_1, u_2) && \text{on } \partial\Omega,\\ \mathcal{D}_{p_2,q_2,\mu_2}(u_2)\cdot \vartheta &= g_2(x, u_1, u_2) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $1 < p_i < N$, $p_i < q_i < p_i^*$, $0 \le \mu_i(\cdot) \in L^\infty(\Omega)$ and Carathéodory functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $g_i : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ that satisfy appropriate growth and coercivity properties, see hypotheses (H1) and (H2) in Sections 3 and 4, respectively. Moreover, we denote by $\vartheta : \partial\Omega \to \mathbb{R}^N$ the outer unit normal vector field of Ω and $\operatorname{div}(\mathcal{D}_{p_i,q_i,\mu_i})$ stands for the double phase operator, whereby

$$\mathcal{D}_{p_i,q_i,\mu_i}(u_i) := |\nabla u_i|^{p_i-2}\nabla u_i + \mu_i(x)|\nabla u_i|^{q_i-2}\nabla u_i$$

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for a suitable measurable function $u_i : \Omega \to \mathbb{R}$, see Section 2 for details on the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$.

The main goal of this paper is to establish sufficient conditions to prove the existence of at least one nontrivial weak solution of problem (1.1). The assumptions on the data are very general and easy to verify for concrete problems. The idea in the proof is the combination of a surjectivity result for pseudomonotone operators along with an equivalent norm in $W^{1,\mathcal{H}}(\Omega)$ and the properties of appropriate eigenvalue problems for the p_i-Laplacian for $i = 1, 2$. More precisely, we use the first eigenvalues and corresponding estimates of the Robin and Steklov eigenvalue problems for the p_i-Laplacian for $i = 1, 2$. Under slightly stronger conditions, we are also going to prove the uniqueness of the solution of system (1.1) by using again suitable eigenvalue problems. As a result of independent interest, we further show the boundedness of any weak solution to more general elliptic systems than in (1.1) by applying Moser's iteration technique. The novelty in our work is the fact that we combine a fully coupled gradient dependent double phase system with a coupled nonlinear boundary condition. It should be noted that the operators in (1.1) reduce to the p_i-Laplacians if $\mu_i \equiv 0$ and to the (q_i, p_i)-Laplacians if $\inf \mu_i > 0$ for $i = 1, 2$, respectively.

Consider the double phase operator $\operatorname{div}(\mathcal{D}_{p,q,\mu})$ for $1 < p < q$ and $0 \le \mu(\cdot) \in L^\infty(\Omega)$, then the corresponding energy functional is given by

$$J : u \mapsto \int_\Omega \left(\frac{1}{p} |\nabla u|^p + \frac{\mu(x)}{q} |\nabla u|^q \right) \mathrm{d}x. \tag{1.2}$$

The functional J in (1.2) changes its ellipticity on the set $\{x \in \Omega : \mu(x) = 0\}$, so the operator shifts between two different phases. Thanks to this change of ellipticity, the double phase operator qualifies for modeling the behavior of inhomogeneous materials. Indeed, suppose our domain Ω consists of two different materials, then we can choose μ according to the geometry of the domain. Several mathematical applications can be found in elasticity theory or in the study of duality theory as well as the Lavrentiev gap phenomenon, see, for example, Zhikov [41,42]. Functionals of the form (1.2) have been considered for the first time in Zhikov [40]. Note that the integrand $R(x, \xi) = \frac{1}{p} |\xi|^p + \frac{\mu(x)}{q} |\xi|^q$ for all $(x, \xi) \in \Omega \times \mathbb{R}^N$ of J has unbalanced growth, that is,

$$c_1 |\xi|^p \le \frac{1}{p} |\xi|^p + \frac{\mu(x)}{q} |\xi|^q \le c_2 \|\mu\|_\infty (1 + |\xi|^q)$$

for a.a. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$ with $c_1, c_2 > 0$. This new class of functionals can be used to provide models for strongly anisotropic materials, so materials whose properties depend on the direction of the strain, like wood for example.

Besides the double phase operator, problem (1.1) has nonlinear boundary conditions and nonlinear, gradient dependent right-hand sides. Functions, which depend on the gradient of the solution, are also called convection term. The difficulty in the study of such terms is their nonvariational character, that is, standard variational tools cannot be applied, even in the scalar case. In the past years, several interesting works for single equations with convection terms have been published, we refer to the papers of Ajagjal and Meskine [1], de Araujo and Faria [10], El Manouni et al. [12], Faraci et al. [13], Faraci and Puglisi [14], Figueiredo and Madeira [16], Gasiński and Papageorgiou [17], Gasiński and Winkert [18], Liu et al. [25], Marano and Winkert [27], Papageorgiou et al. [35], and Pucci and Temperini [38].

For double phase systems, there are only the works of Guarnotta et al. [19], Liu et al. [26] and Marino and Winkert [28]. The methods used in these papers are based on appropriate enclosure results in terms of trapping regions formed by pairs of sup- and supersolutions and pseudomonotonicity arguments. But also without double phase operator, there exists only few works for coupled elliptic systems with convection terms, we refer to the papers of Faria et al. [15], Guarnotta and Marano [20,21], Guarnotta et al. [22], Motreanu et al. [31,32], and Toscano et al. [39]. In the direction of coupled systems without gradient dependence, we mention the works of Alves and Soares [2], Boccardo and de Figueiredo [3], Carl and Motreanu [4,5], Chabrowski [6], and de Godoi et al. [11].

Our paper is motivated by the works of Marino and Winkert [28] and El Manouni et al. [12]. In [28], existence and uniqueness results for coupled systems with homogeneous Dirichlet boundary condition have been proven for the first time. On the other hand, in [12], the authors prove existence results for single-valued equations with nonlinear boundary condition via suitable eigenvalue problems. In our paper, we combine the ideas from both papers to show existence and uniqueness results for the coupled system with nonlinear boundary conditions given in (1.1).

The paper is organized as follows. In Section 2, we recall the properties of the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, introduce our function space for problem (1.1), and present an equivalent definition of pseudomonotonicity for bounded operators (Proposition 2.3). In Section 3, we consider system (1.1) with a certain growth and prove that any weak solution of this system is bounded by applying a suitable version of Moser's iteration method (Theorem 3.2). Finally, in Section 4, we state and prove our main existence result (Theorem 4.2) using a surjectivity result for pseudomonotone operators and we give sufficient conditions for the uniqueness of this solution (Theorem 4.3).

2 Preliminaries

Throughout this paper, we denote by $|\cdot|$ the Euclidean norm and by $\cdot$ the inner product in $\mathbb{R}^N$. If X is a real Banach space, then X^* stands for its dual space and $\langle\cdot,\cdot\rangle_X$ for the dual pairing between X and X^*. Furthermore, we will use the notation $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$ for $s \in \mathbb{R}$. Therefore, the positive and the negative part of a function $u : \Omega \to \mathbb{R}$ are described by $u^+(\cdot) = u(\cdot)^+$ and $u^-(\cdot) = u(\cdot)^-$, respectively.

Let $1 \le r < \infty$. Then, we denote by $(L^r(\Omega), \|\cdot\|_r)$ and $(L^r(\Omega;\mathbb{R}^N), \|\cdot\|_r)$ the Lebesgue spaces on Ω and by $(W^{1,r}(\Omega), \|\cdot\|_{1,r})$ the corresponding Sobolev spaces, while the norms are given by

$$\|u\|_r = \left(\int_\Omega |u|^r \mathrm{d}x\right)^{\frac{1}{r}} \quad \text{and} \quad \|v\|_{1,r} = (\|v\|_r^r + \|\nabla v\|_r^r)^{\frac{1}{r}}$$

for $u \in L^r(\Omega)$ and $v \in W^{1,r}(\Omega)$. In addition, we will consider the space $L^\infty(\Omega)$ with the norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_\Omega |u|$$

for $u \in L^\infty(\Omega)$.

Moreover, we will use the boundary Lebesgue spaces $(L^r(\partial\Omega), \|\cdot\|_{r,\partial\Omega})$ for $1 \le r \le \infty$, which are defined by the $(N-1)$-dimensional Hausdorff surface measure on the boundary $\partial\Omega$ of Ω. We have

$$\|u\|_{r,\partial\Omega} = \left(\int_{\partial\Omega} |u|^r \mathrm{d}\sigma\right)^{\frac{1}{r}} \quad \text{and} \quad \|v\|_{\infty,\partial\Omega} = \operatorname*{ess\,sup}_{\partial\Omega} |v|$$

for $u \in L^r(\partial\Omega)$ with $1 \le r < \infty$ and $v \in L^\infty(\partial\Omega)$. In the following, we write $|\partial\Omega|$ for the Hausdorff measure of $\partial\Omega$, whereas $|\Omega|$ stands for the Lebesgue measure of Ω. If we use Hölder's inequality throughout this paper, we will write $s' \coloneqq \frac{s}{s-1}$ for the conjugated exponent of $s \in \mathbb{R}$ with $1 < s < \infty$.

From now on, let $i \in \{1, 2\}$. We choose $1 < p_i < N$ and define

$$p_i^* \coloneqq \frac{Np_i}{N - p_i}$$

as the critical exponent of p_i, i.e., the embedding

$$W^{1,p_i}(\Omega) \hookrightarrow L^r(\Omega)$$

is compact for $1 \le r < p_i^*$ and continuous for $r = p_i^*$. Analogously, let

$$p_{i*} \coloneqq \frac{(N-1)p_i}{N - p_i}$$

be the critical exponent for the embedding into $L^r(\partial\Omega)$. This holds, since the linear trace operator $\gamma : W^{1,p_i}(\Omega) \to L^r(\partial\Omega)$ is continuous for $1 \le r \le p_{i*}$ and compact for $1 \le r < p_{i*}$, see Nečas [34, Section 2.4]. For simplicity of the notation, we will write u instead of $\gamma(u)$, if we consider $u \in W^{1,p_i}(\Omega)$ on $\partial\Omega$.

Now, we suppose the following hypotheses:

(H0) $1 < p_i < N$, $p_i < q_i < p_i^*$ and $0 \le \mu_i(\cdot) \in L^\infty(\Omega)$ for $i = 1, 2$.

Under the conditions in (H0), we introduce the mapping

$$\mathcal{H}_i : \Omega \times [0,\infty) \to [0,\infty), \quad (x,t) \mapsto t^{p_i} + \mu_i(x)t^{q_i}$$

and consider the modular function

$$\rho_{\mathcal{H}_i}(u) := \int_\Omega \mathcal{H}_i(x,|u|)\mathrm{d}x = \int_\Omega (|u|^{p_i} + \mu_i(x)|u|^{q_i})\mathrm{d}x$$

for a function $u \in M(\Omega)$ with $M(\Omega)$ being the set of all measurable functions on Ω. The Musielak-Orlicz space $L^{\mathcal{H}_i}(\Omega)$ is defined by

$$L^{\mathcal{H}_i}(\Omega) := \{u \in M(\Omega) : \rho_{\mathcal{H}_i}(u) < \infty\},$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_i} := \inf\left\{\tau > 0 : \rho_{\mathcal{H}_i}\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Now, the corresponding Musielak-Orlicz Sobolev space is defined by

$$W^{1,\mathcal{H}_i}(\Omega) := \{u \in L^{\mathcal{H}_i}(\Omega) : |\nabla u| \in L^{\mathcal{H}_i}(\Omega)\}$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_i} := \|\nabla u\|_{\mathcal{H}_i} + \|u\|_{\mathcal{H}_i},$$

where $\|\nabla u\|_{\mathcal{H}_i} = \|\,|\nabla u|\,\|_{\mathcal{H}_i}$. From Colasuonno and Squassina [7, Proposition 2.14], we know that $L^{\mathcal{H}_i}(\Omega)$ and $W^{1,\mathcal{H}_i}(\Omega)$ are reflexive Banach spaces, see also the monographs of Harjulehto and Hästö [23] and Musielak [33]. Furthermore, we have the following useful embeddings, which are proved in Crespo-Blanco et al. [8, Proposition 2.16].

Proposition 2.1. *Let assumptions* (H0) *be satisfied. Then the following embeddings hold:*

(*i*) $L^{\mathcal{H}_i}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}_i}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *are continuous for* $1 \le r \le p_i$;

(*ii*) $W^{1,\mathcal{H}_i}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for* $1 \le r \le p_i^*$ *and compact for* $1 \le r < p_i^*$;

(*iii*) $W^{1,\mathcal{H}_i}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is continuous for* $1 \le r \le p_{i*}$ *and compact for* $1 \le r < p_{i*}$.

We equip the space $W^{1,\mathcal{H}_i}(\Omega)$ with the norm defined by

$$\|u\|_{1,\mathcal{H}_i}^* := \inf\left\{\tau > 0 : \int_\Omega \left[\left(\frac{|\nabla u|}{\tau}\right)^{p_i} + \mu_i(x)\left(\frac{|\nabla u|}{\tau}\right)^{q_i} + \left(\frac{|u|}{\tau}\right)^{p_i}\right]\mathrm{d}x \le 1\right\}$$

for $u \in W^{1,\mathcal{H}_i}(\Omega)$, which is equivalent to the standard one defined above, see Crespo-Blanco et al. [9, Proposition 2.2]. The corresponding modular is given by

$$\rho_{1,\mathcal{H}_i}^*(u) := \int_\Omega (|\nabla u|^{p_i} + \mu_i(x)|\nabla u|^{q_i} + |u|^{p_i})\mathrm{d}x.$$

Taking Proposition 2.3 in Crespo-Blanco et al. [9] into account, we obtain the estimates

$$\min\{(\|u\|_{1,\mathcal{H}_i}^*)^{p_i}, (\|u\|_{1,\mathcal{H}_i}^*)^{q_i}\} \le \rho_{1,\mathcal{H}_i}^*(u) \le \max\{(\|u\|_{1,\mathcal{H}_i}^*)^{p_i}, (\|u\|_{1,\mathcal{H}_i}^*)^{q_i}\} \tag{2.1}$$

for all $u \in W^{1,\mathcal{H}_i}(\Omega)$

Next, we recall some basis properties of the Robin and Steklov eigenvalue problems for the p-Laplacian for $p \in (1, \infty)$. The Robin eigenvalue problem is given by

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda\, |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu &= -\beta\, |u|^{p-2}u \text{ on } \partial\Omega, \end{aligned} \tag{2.2}$$

where $\beta > 0$ is a fixed number. Referring to Lê [24], the first (i.e., the smallest) eigenvalue $\lambda^R_{1,p,\beta}$ of (2.2) is positive, isolated, simple and can be characterized by

$$\lambda^R_{1,p,\beta} = \inf\left\{\int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma : \int_\Omega |u|^p \mathrm{d}x = 1\right\}.$$

This leads to

$$\lambda^R_{1,p,\beta}\|u\|_p^p \le \|\nabla u\|_p^p + \beta\|u\|^p_{p,\partial\Omega},$$

which can be equivalently written as

$$\|u\|_p^p \le (\lambda^R_{1,p,\beta})^{-1}(\|\nabla u\|_p^p + \beta\|u\|^p_{p,\partial\Omega}), \tag{2.3}$$

for $u \in W^{1,p}(\Omega)$.

Furthermore, we consider the Steklov eigenvalue problem for the p-Laplacian, which is defined by

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= -\, |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu &= \lambda\, |u|^{p-2}u \quad \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

Again, the first eigenvalue $\lambda^S_{1,p}$ of (2.4) is positive, isolated, simple and can be represented by

$$\lambda^S_{1,p} = \inf\left\{\int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Omega |u|^p \mathrm{d}x : \int_{\partial\Omega} |u|^p \mathrm{d}\sigma = 1\right\},$$

see Chapter 5 in Lê [24]. From that we obtain the inequality

$$\|u\|^p_{p,\partial\Omega} \le (\lambda^S_{1,p})^{-1}(\|\nabla u\|_p^p + \|u\|_p^p) \tag{2.5}$$

for $u \in W^{1,p}(\Omega)$.

Let us now recall the definition of pseudomonotonicity and the surjectivity result for pseudomonotone operators.

Definition 2.2. Let X be a reflexive Banach space and $\mathbf{A} : X \to X^*$. Then $\mathbf{A}$ is called

(i) bounded if $\mathbf{A}$ maps bounded sets to bounded sets;

(ii) pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle \mathbf{A}u_n, u_n - u\rangle \le 0$ imply $\liminf_{n\to\infty} \langle \mathbf{A}u_n, u_n - w\rangle \ge \langle \mathbf{A}u, u - w\rangle$ for all $w \in X$;

(iii) to satisfy the (S₊)-property if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle \mathbf{A}u_n, u_n - u\rangle \le 0$ imply $u_n \to u$ in X;

(iv) coercive if

$$\lim_{\|u\|\to\infty} \frac{\langle \mathbf{A}u, u\rangle}{\|u\|} = \infty$$

holds.

Since we consider bounded operators, we can use the following equivalent definition of pseudomonotonicity. As we did not find any reference for this known result, we present the proof here.

Proposition 2.3. *Let* $\mathbf{A} : X \to X^*$ *be bounded. Then the following two conditions are equivalent:*
(*i*) $\mathbf{A}$ *is pseudomonotone.*
(*ii*) *From* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} \langle \mathbf{A}u_n, u_n - u\rangle \le 0$, *we have* $\mathbf{A}u_n \rightharpoonup \mathbf{A}u$ *in* X^* *and* $\langle \mathbf{A}u_n, u_n\rangle \to \langle \mathbf{A}u, u\rangle$.

Proof. Let $\mathbf{A}$ be pseudomonotone and $\{u_n\}_{n\in\mathbb{N}} \subset X$ with $u_n \rightharpoonup u \in X$ and $\limsup_{n\to\infty}\langle \mathbf{A}u_n, u_n - u\rangle \le 0$. Due to the pseudomonotonicity of $\mathbf{A}$, we have

$$\lim_{n\to\infty} \langle \mathbf{A}u_n, u_n - u\rangle = 0. \tag{2.6}$$

By the boundedness of $\mathbf{A}$, we obtain the boundedness of $\{\mathbf{A}u_n\}_{n\in\mathbb{N}}$. Hence, there exists a subsequence $\{\mathbf{A}u_{n'}\}_{n'\in\mathbb{N}}$ of $\{\mathbf{A}u_n\}_{n\in\mathbb{N}}$ and $b \in X^*$ with

$$\mathbf{A}u_{n'} \rightharpoonup b \quad \text{in } X^*.$$

By the definition of pseudomonotonicity and (2.6), it follows that

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \langle \mathbf{A}u_{n'}, u_{n'} - w\rangle - \langle \mathbf{A}u, u - w\rangle \\ &= \liminf_{n\to\infty} (\langle \mathbf{A}u_{n'}, u_{n'} - u\rangle + \langle \mathbf{A}u_{n'}, u - w\rangle - \langle \mathbf{A}u, u - w\rangle) \\ &= \liminf_{n\to\infty} \langle \mathbf{A}u_{n'} - \mathbf{A}u, u - w\rangle \\ &= \langle b - \mathbf{A}u, u - w\rangle \quad \text{for all } w \in X. \end{aligned}$$

Therefore, we have $\mathbf{A}u = b$ and $\mathbf{A}u_n \rightharpoonup \mathbf{A}u$ for the original sequence. Furthermore, this implies

$$\langle \mathbf{A}u_n, u_n\rangle = \langle \mathbf{A}u_n, u_n - u\rangle + \langle \mathbf{A}u_n, u\rangle \to \langle \mathbf{A}u, u\rangle \quad \text{as } n \to \infty,$$

where we have used (2.6) again.

Now, let $\mathbf{A}$ satisfy the condition (ii). We consider $\{u_n\}_{n\in\mathbb{N}} \subset X$ with $u_n \rightharpoonup u \in X$ and $\limsup_{n\to\infty} \langle \mathbf{A}u_n, u_n - u\rangle \le 0$. Then we have $\mathbf{A}u_n \rightharpoonup \mathbf{A}u$ and $\langle \mathbf{A}u_n, u_n\rangle \to \langle \mathbf{A}u, u\rangle$, which yields

$$\liminf_{n\to\infty} \langle \mathbf{A}u_n, u_n - w\rangle = \liminf_{n\to\infty} \langle \mathbf{A}u_n, u_n\rangle - \liminf_{n\to\infty} \langle \mathbf{A}u_n, w\rangle = \langle \mathbf{A}u, u - w\rangle,$$

for all $w \in X$. Hence, $\mathbf{A}$ is pseudomonotone. □

The following surjectivity result for pseudomonotone operators will be useful for our existence result in Section 4. The proof can be found, for example, in Papageorgiou and Winkert [36, Theorem 6.1.57].

Theorem 2.4. *Let* X *be a real, reflexive Banach space, let* $\mathbf{A} : X \to X^*$ *be a pseudomonotone, bounded, and coercive operator, and let* $b \in X^*$. *Then, a solution of the equation* $\mathbf{A}u = b$ *exists.*

Finally, let $V := W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega)$ be equipped with the norm

$$\|(u_1, u_2)\|_V := \|u_1\|_{1,\mathcal{H}_1}^* + \|u_2\|_{1,\mathcal{H}_2}^*$$

for $(u_1, u_2) \in V$. Obviously, V is a reflexive Banach space. Moreover, let $A : V \to V^*$ be the operator given by

$$\begin{aligned} \langle A(u_1, u_2), (\varphi_1, \varphi_2)\rangle_V = &\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot\nabla\varphi_1 \mathrm{d}x \\ &+ \int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + \mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\cdot\nabla\varphi_2 \mathrm{d}x \\ &+ \int_\Omega |u_1|^{p_1-2}u_1\varphi_1 \mathrm{d}x + \int_\Omega |u_2|^{p_2-2}u_2\varphi_2 \mathrm{d}x \end{aligned} \tag{2.7}$$

for $(u_1, u_2), (\varphi_1, \varphi_2) \in V$.

Then, A fulfills the following useful properties.

Lemma 2.5. *Let* (H0) *be satisfied and let* $A : V \to V^*$ *be the operator defined by* (2.7). *Then,* A *is well-defined, bounded, continuous, monotone, and of type* (S₊).

This can be proved analogously to the proof of Proposition 3.4 in Crespo-Blanco et al. [8] with slight modifications.

3 Boundedness results

In this section, we show that weak solutions of systems like (1.1) are bounded, i.e., for any weak solution $(u_1, u_2) \in V$ of (1.1), we already have $u_i \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ for $i \in \{1, 2\}$. First, we state the following assumptions on the data of (1.1):

(H1) The functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $g_i : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i \in \{1, 2\})$ are Carathéodory functions such that the following hold:

(i) there exist constants $\tilde{A}_j, \tilde{B}_j \geq 0$ for $j \in \{1, \dots, 7\}$ and exponents $\tilde{a}_l, \tilde{b}_l \geq 0$ for $l \in \{1, \dots, 8\}$ with

$$|f_1(x, s, t, \xi, \eta)| \leq \tilde{A}_1\, |s|^{\tilde{a}_1} + \tilde{A}_2\, |t|^{\tilde{a}_2} + \tilde{A}_3\, |s|^{\tilde{a}_3}|t|^{\tilde{a}_4} + \tilde{A}_4\, |\xi|^{\tilde{a}_5} + \tilde{A}_5\, |\eta|^{\tilde{a}_6} + \tilde{A}_6\, |\xi|^{\tilde{a}_7}|\eta|^{\tilde{a}_8} + \tilde{A}_7,$$
$$|f_2(x, s, t, \xi, \eta)| \leq \tilde{B}_1\, |s|^{\tilde{b}_1} + \tilde{B}_2\, |t|^{\tilde{b}_2} + \tilde{B}_3\, |s|^{\tilde{b}_3}|t|^{\tilde{b}_4} + \tilde{B}_4\, |\xi|^{\tilde{b}_5} + \tilde{B}_5\, |\eta|^{\tilde{b}_6} + \tilde{B}_6\, |\xi|^{\tilde{b}_7}|\eta|^{\tilde{b}_8} + \tilde{B}_7$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}$ and for all $\xi, \eta \in \mathbb{R}^N$ such that the conditions below are satisfied:

(E1) $\tilde{a}_1 \leq p_1^* - 1$, (E2) $\tilde{a}_2 < \frac{p_1^* - p_1}{p_1^*} p_2^*$,

(E3) $\frac{\tilde{a}_3}{p_1^*} + \frac{\tilde{a}_4}{p_2^*} < \frac{p_1^* - p_1}{p_1^*}$, (E4) $\tilde{a}_5 \leq p_1 - 1$,

(E5) $\tilde{a}_6 < \frac{p_1^* - p_1}{p_1^*} p_2$, (E6) $\frac{\tilde{a}_7}{p_1} + \frac{\tilde{a}_8}{p_2} < \frac{p_1^* - p_1}{p_1^*}$,

(E7) $\tilde{b}_1 < \frac{p_2^* - p_2}{p_2^*} p_1^*$, (E8) $\tilde{b}_2 \leq p_2^* - 1$,

(E9) $\frac{\tilde{b}_3}{p_1^*} + \frac{\tilde{b}_4}{p_2^*} < \frac{p_2^* - p_2}{p_2^*}$, (E10) $\tilde{b}_5 < \frac{p_2^* - p_2}{p_2^*} p_1$,

(E11) $\tilde{b}_6 \leq p_2 - 1$, (E12) $\frac{\tilde{b}_7}{p_1} + \frac{\tilde{b}_8}{p_2} < \frac{p_2^* - p_2}{p_2^*}$;

(ii) there exist constants $\tilde{C}_j, \tilde{D}_j \geq 0$ for $j \in \{1, \dots, 4\}$ and exponents $\tilde{c}_l, \tilde{d}_l \geq 0$ for $l \in \{1, \dots, 4\}$ with

$$|g_1(x, s, t)| \leq \tilde{C}_1\, |s|^{\tilde{c}_1} + \tilde{C}_2\, |t|^{\tilde{c}_2} + \tilde{C}_3\, |s|^{\tilde{c}_3}|t|^{\tilde{c}_4} + \tilde{C}_4$$
$$|g_2(x, s, t)| \leq \tilde{D}_1\, |s|^{\tilde{d}_1} + \tilde{D}_2\, |t|^{\tilde{d}_2} + \tilde{D}_3\, |s|^{\tilde{d}_3}|t|^{\tilde{d}_4} + \tilde{D}_4,$$

for a.a. $x \in \partial\Omega$ and for all $s, t \in \mathbb{R}$ such that the conditions below are satisfied:

(F1) $\tilde{c}_1 \leq p_{1*} - 1$, (F2) $\tilde{c}_2 < \frac{p_{1*} - p_1}{p_{1*}} p_{2*}$,

(F3) $\frac{\tilde{c}_3}{p_{1*}} + \frac{\tilde{c}_4}{p_{2*}} < \frac{p_{1*} - p_1}{p_{1*}}$, (F4) $\tilde{d}_1 < \frac{p_{2*} - p_2}{p_{2*}} p_{1*}$,

(F5) $\tilde{d}_2 \leq p_{2*} - 1$, (F6) $\frac{\tilde{d}_3}{p_{1*}} + \frac{\tilde{d}_4}{p_{2*}} < \frac{p_{2*} - p_2}{p_{2*}}$.

A solution of (1.1) is understood in the weak sense in the following way.

Definition 3.1. A function $(u_1, u_2) \in V$ is called a weak solution of problem (1.1) if

$$\begin{aligned}\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot\nabla\varphi_1 \mathrm{d}x \\ = \int_\Omega f_1(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_1 \mathrm{d}x + \int_{\partial\Omega} g_1(x, u_1, u_2)\varphi_1 \mathrm{d}\sigma\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + \mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\cdot\nabla\varphi_2 \mathrm{d}x \\ = \int_\Omega f_2(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_2 \mathrm{d}x + \int_{\partial\Omega} g_2(x, u_1, u_2)\varphi_2 \mathrm{d}\sigma\end{aligned} \tag{3.2}$$

are satisfied for all $(\varphi_1, \varphi_2) \in V$.

Based on the assumptions in (H1), it is clear that the definition of a weak solution is well defined since all integrals in (3.1) and (3.2) are finite.

The following result, which gives us the boundedness of weak solutions, is mainly based on ideas in Theorems 3.1 and 3.2 of Marino and Winkert [29], where Moser's iteration scheme is used for the proof.

Theorem 3.2. *Let hypotheses* (H0) *and* (H1) *be satisfied. Then, we have the following assertions:*

(*i*) $(u_1, u_2) \in (L^r(\Omega) \cap L^r(\partial\Omega)) \times (L^r(\Omega) \cap L^r(\partial\Omega))$ *for all* $r \in (1, \infty)$;
(*ii*) $(u_1, u_2) \in (L^\infty(\Omega) \cap L^\infty(\partial\Omega)) \times (L^\infty(\Omega) \cap L^\infty(\partial\Omega))$

for every weak solution $(u_1, u_2) \in V$ *of problem* (1.1).

Proof. (i) Let $(u_1, u_2) \in V$ be a weak solution of (1.1) and $r \in (1, \infty)$. We will prove only $u_1 \in L^r(\Omega) \cap L^r(\partial\Omega)$, since the claim for u_2 follows analogously. Without loss of generality, we assume $u_1, u_2 \geq 0$, because otherwise we would consider u_i^+, u_i^- for $i \in \{1, 2\}$. Furthermore, we will denote by M_i, $i \in \mathbb{N}_{>0}$ constants, which will be used throughout the proof and may depend on the Lebesgue norms of $u_1, u_2, \nabla u_1, \nabla u_2$.

Let $h \geq 0$ and $\kappa > 0$. We define $u_{1,h} \coloneqq \min\{u_1, h\}$ and fix $\varphi_1 \coloneqq u_1 u_{1,h}^{\kappa p_1}$. First, we compute

$$\nabla\varphi_1 = u_{1,h}^{\kappa p_1}\nabla u_1 + \kappa p_1 u_1 u_{1,h}^{\kappa(p_1-1)}\nabla u_{1,h}.$$

Since $\varphi_1 \in W^{1,\mathcal{H}_1}(\Omega)$, we can choose φ_1 as test function in (3.1) and obtain

$$\begin{aligned}\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot(u_{1,h}^{\kappa p_1}\nabla u_1 + \kappa p_1 u_1 u_{1,h}^{\kappa(p_1-1)}\nabla u_{1,h})\mathrm{d}x \\ = \int_\Omega f_1(x, u_1, u_2, \nabla u_1, \nabla u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}x + \int_{\partial\Omega} g_1(x, u_1, u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}\sigma.\end{aligned}$$

As $\mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1 \cdot(u_{1,h}^{\kappa p_1}\nabla u_1 + \kappa p_1 u_1 u_{1,h}^{\kappa(p_1-1)}\nabla u_{1,h}) \geq 0$ for a.a. $x \in \Omega$, we can estimate

$$\begin{aligned}\int_\Omega |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1}\mathrm{d}x + \kappa p_1 \int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 \cdot\nabla u_{1,h})u_1 u_{1,h}^{\kappa(p_1-1)}\mathrm{d}x \\ \leq \int_\Omega f_1(x, u_1, u_2, \nabla u_1, \nabla u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}x + \int_{\partial\Omega} g_1(x, u_1, u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}\sigma.\end{aligned}$$

Next, since $u_{1,h} = u_1$ and $\nabla u_{1,h} = \nabla u_1$ on $\{x \in \Omega : u(x) \leq h\}$, this leads to

$$\begin{aligned}\int_\Omega |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1}\mathrm{d}x + \kappa p_1 \int_{\{x\in\Omega : u(x)\leq h\}} |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1}\mathrm{d}x \\ \leq \int_\Omega f_1(x, u_1, u_2, \nabla u_1, \nabla u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}x + \int_{\partial\Omega} g_1(x, u_1, u_2)u_1 u_{1,h}^{\kappa p_1}\mathrm{d}\sigma.\end{aligned}$$

According to (H1)(i), we can estimate the right-hand side and achieve

$$\begin{aligned}&\int_\Omega f_1(x,u_1,u_2,\nabla u_1,\nabla u_2)u_1u_{1,h}^{\kappa p_1}\mathrm{d}x\\ &\quad\le \int_\Omega(\tilde{A}_1\,|u_1|^{\tilde{a}_1}+\tilde{A}_2\,|u_2|^{\tilde{a}_2}+\tilde{A}_3\,|u_1|^{\tilde{a}_3}|u_2|^{\tilde{a}_4}+\tilde{A}_4\,|\nabla u_1|^{\tilde{a}_5}+\tilde{A}_5\,|\nabla u_2|^{\tilde{a}_6}+\tilde{A}_6\,|\nabla u_1|^{\tilde{a}_7}|\nabla u_2|^{\tilde{a}_8}+\tilde{A}_7)u_1u_{1,h}^{\kappa p_1}\mathrm{d}x.\end{aligned}\tag{3.3}$$

Analogously, we have

$$\int_{\partial\Omega} g_1(x,u_1,u_2)u_1u_{1,h}^{\kappa p_1}\mathrm{d}\sigma\le\int_{\partial\Omega}(\tilde{C}_1\,|u_1|^{\tilde{c}_1}+\tilde{C}_2\,|u_2|^{\tilde{c}_2}+\tilde{C}_3\,|u_1|^{\tilde{c}_3}|u_2|^{\tilde{c}_4}+\tilde{C}_4)u_1u_{1,h}^{\kappa p_1}\mathrm{d}\sigma\tag{3.4}$$

for the second term by using hypothesis (H1)(ii).

Now, we examine (3.3) more precisely. By (E1), the exponent $\tilde{a}_1$ satisfies $\tilde{a}_1\le p_1^*-1$, and therefore, we know that $u_1^{\tilde{a}_1+1}u_{1,h}^{\kappa p_1}\le 1+u_1^{p_1^*}u_{1,h}^{\kappa p_1}$ a.e. in Ω, and finally,

$$\tilde{A}_1\int_\Omega u_1^{\tilde{a}_1+1}u_{1,h}^{\kappa p_1}\mathrm{d}x\le\tilde{A}_1\int_\Omega(1+u_1^{p_1^*}u_{1,h}^{\kappa p_1})\mathrm{d}x=\tilde{A}_1\int_\Omega u_1^{p_1^*}u_{1,h}^{\kappa p_1}\mathrm{d}x+\tilde{A}_1|\Omega|.$$

Moreover, Hölder's inequality yields

$$\begin{aligned}\tilde{A}_2\int_\Omega u_2^{\tilde{a}_2}u_1u_{1,h}^{\kappa p_1}\mathrm{d}x&\le\tilde{A}_2\left(\int_\Omega u_2^{\tilde{a}_2 s_1}\mathrm{d}x\right)^{\frac{1}{s_1}}\left(\int_\Omega(u_1u_{1,h}^{\kappa p_1})^{s_1'}\mathrm{d}x\right)^{\frac{1}{s_1'}}\\ &=\tilde{A}_2\|u_2\|_{p_2^*}^{\tilde{a}_2}\|u_1u_{1,h}^{\kappa p_1}\|_{s_1'}\\ &\le M_1(1+\|u_1u_{1,h}^{\kappa}\|_{p_1s_1'}^{p_1}),\end{aligned}$$

where we choose $s_1=\frac{p_2^*}{\tilde{a}_2}$ and $s_1>1$ holds due to (E2). Note, that we have $u_1u_{1,h}^{\kappa p_1}\le(u_1u_{1,h}^{\kappa})^{p_1}+1$ a.e. in Ω. Similarly, Hölder's inequality with three components leads to

$$\begin{aligned}&\tilde{A}_3\int_\Omega u_1^{\tilde{a}_3}u_2^{\tilde{a}_4}u_1u_{1,h}^{\kappa p_1}\mathrm{d}x\\ &\le\ \tilde{A}_3\left(\int_\Omega u_1^{\tilde{a}_3x_1}\mathrm{d}x\right)^{\frac{1}{x_1}}\left(\int_\Omega u_2^{\tilde{a}_4y_1}\mathrm{d}x\right)^{\frac{1}{y_1}}\left(\int_\Omega(u_1u_{1,h}^{\kappa p_1})^{z_1}\mathrm{d}x\right)^{\frac{1}{z_1}}\\ &=\ \tilde{A}_3\|u_1\|_{p_1^*}^{\tilde{a}_3}\|u_2\|_{p_2^*}^{\tilde{a}_4}\|u_1u_{1,h}^{\kappa p_1}\|_{z_1}\\ &\le\ M_2(1+\|u_1u_{1,h}^{\kappa}\|_{p_1z_1}^{p_1}),\end{aligned}$$

where $x_1=\frac{p_1^*}{\tilde{a}_3},y_1=\frac{p_2^*}{\tilde{a}_4}$ and $\frac{1}{z_1}=1-\frac{1}{x_1}-\frac{1}{y_1}$. Thanks to (E3), this choice is admissible. By applying Young's inequality with $\frac{p_1}{\tilde{a}_5}>1$, we obtain

$$\begin{aligned}\tilde{A}_4\int_\Omega|\nabla u_1|^{\tilde{a}_5}u_1u_{1,h}^{\kappa p_1}\mathrm{d}x&=\tilde{A}_4\int_\Omega\left(\left(\frac{1}{2\tilde{A}_4}\right)^{\frac{\tilde{a}_5}{p_1}}|\nabla u_1|^{\tilde{a}_5}u_{1,h}^{\kappa\tilde{a}_5}\right)\left(\left(\frac{1}{2\tilde{A}_4}\right)^{-\frac{\tilde{a}_5}{p_1}}u_1u_{1,h}^{\kappa(p_1-\tilde{a}_5)}\right)\mathrm{d}x\\ &\le\tilde{A}_4\int_\Omega\frac{1}{2\tilde{A}_4}|\nabla u_1|^{p_1}u_{1,h}^{\kappa p_1}\mathrm{d}x+\tilde{A}_4\int_\Omega\left(\frac{1}{2\tilde{A}_4}\right)^{-\frac{\tilde{a}_5}{p_1-\tilde{a}_5}}u_1^{\frac{p_1}{p_1-\tilde{a}_5}}u_{1,h}^{\kappa p_1}\mathrm{d}x\\ &\le\frac{1}{2}\int_\Omega|\nabla u_1|^{p_1}u_{1,h}^{\kappa p_1}\mathrm{d}x+\tilde{A}_4\left(\frac{1}{2\tilde{A}_4}\right)^{-\frac{\tilde{a}_5}{p_1-\tilde{a}_5}}\left(|\Omega|+\int_\Omega u_1^{p_1^*}u_{1,h}^{\kappa p_1}\mathrm{d}x\right)\\ &\le\frac{1}{2}\int_\Omega|\nabla u_1|^{p_1}u_{1,h}^{\kappa p_1}\mathrm{d}x+M_3\left(1+\int_\Omega u_1^{p_1^*}u_{1,h}^{\kappa p_1}\mathrm{d}x\right).\end{aligned}$$

Next, Hölder's inequality gives us

$$\begin{aligned}\tilde{A}_5\int_\Omega |\nabla u_2|^{\tilde{a}_6}u_1u_{1,h}^{\kappa p_1}\mathrm{d}x &\le \tilde{A}_5\left(\int_\Omega |\nabla u_2|^{\tilde{a}_6 s_2}\mathrm{d}x\right)^{\frac{1}{s_2}}\left(\int_\Omega (u_1u_{1,h}^{\kappa p_1})^{s_2'}\mathrm{d}x\right)^{\frac{1}{s_2'}}\\ &= \tilde{A}_5\|\nabla u_2\|_{p_2}^{\tilde{a}_6}\|u_1u_{1,h}^{\kappa p_1}\|_{s_2'}\\ &\le M_4(1+\|u_1u_{1,h}^{\kappa}\|_{p_1s_2'}^{p_1}),\end{aligned}$$

where $s_2=\frac{p_2}{\tilde{a}_6}>1$ by (E5). Again, we use Hölder's inequality such that $x_2=\frac{p_1}{\tilde{a}_7}$, $y_2=\frac{p_2}{\tilde{a}_8}$ and $\frac{1}{z_2}=1-\frac{1}{x_2}-\frac{1}{y_2}$, so we obtain

$$\begin{aligned}&\tilde{A}_6\int_\Omega |\nabla u_1|^{\tilde{a}_7}|\nabla u_2|^{\tilde{a}_8}u_1u_{1,h}^{\kappa p_1}\mathrm{d}x\\ &\le \tilde{A}_6\left(\int_\Omega |\nabla u_1|^{\tilde{a}_7x_2}\mathrm{d}x\right)^{\frac{1}{x_2}}\left(\int_\Omega |\nabla u_2|^{\tilde{a}_8y_2}\mathrm{d}x\right)^{\frac{1}{y_2}}\left(\int_\Omega (u_1u_{1,h}^{\kappa p_1})^{z_2}\mathrm{d}x\right)^{\frac{1}{z_2}}\\ &= \tilde{A}_6\|\nabla u_1\|_{p_1}^{\tilde{a}_7}\|\nabla u_2\|_{p_2}^{\tilde{a}_8}\|u_1u_{1,h}^{\kappa p_1}\|_{z_2}\\ &\le M_5(1+\|u_1u_{1,h}^{\kappa}\|_{p_1z_2}^{p_1}).\end{aligned}$$

Note that $z_2>0$ by (E6). Finally, estimating the constant term yields

$$\tilde{A}_7\int_\Omega u_1u_{1,h}^{\kappa p_1}\mathrm{d}x\le\tilde{A}_7\int_\Omega u_1^{p_1^*}u_1^{\kappa p_1}\mathrm{d}x+\tilde{A}_7|\Omega|.$$

Now, we can turn to (3.4). By (F1), we have for the first term

$$\tilde{C}_1\int_{\partial\Omega}u_1^{\tilde{c}_1+1}u_{1,h}^{\kappa p_1}\mathrm{d}\sigma\le\tilde{C}_1\int_{\partial\Omega}(u_1^{p_{1*}}u_{1,h}^{\kappa p_1}+1)\mathrm{d}\sigma=\tilde{C}_1\int_{\partial\Omega}u_1^{p_{1*}}u_{1,h}^{\kappa p_1}\mathrm{d}\sigma+\tilde{C}_1|\partial\Omega|$$

Applying Hölder's inequality with $t_1=\frac{p_{2*}}{\tilde{c}_2}>1$ according to (F2) leads to

$$\begin{aligned}\tilde{C}_2\int_{\partial\Omega}u_2^{\tilde{c}_2}u_1u_{1,h}^{\kappa p_1}\mathrm{d}\sigma &\le\left(\int_{\partial\Omega}u_2^{\tilde{c}_2t_1}\mathrm{d}\sigma\right)^{\frac{1}{t_1}}\left(\int_{\partial\Omega}(u_1u_{1,h}^{\kappa p_1})^{t_1'}\mathrm{d}\sigma\right)^{\frac{1}{t_1'}}\\ &=\tilde{C}_2\|u_2\|_{p_{2*},\partial\Omega}^{\tilde{c}_2}\|u_1u_{1,h}^{\kappa p_1}\|_{t_1',\partial\Omega}\\ &\le M_6(1+\|u_1u_{1,h}^{\kappa}\|_{p_1t_1',\partial\Omega}^{p_1}).\end{aligned}$$

To deal with the next term, we need Hölder's inequality with three components again to obtain

$$\begin{aligned}\tilde{C}_3\int_{\partial\Omega}u_1^{\tilde{c}_3}u_2^{\tilde{c}_4}u_1u_{1,h}^{\kappa p_1}\mathrm{d}\sigma &\le\tilde{C}_3\left(\int_{\partial\Omega}u_1^{\tilde{c}_3x_3}\mathrm{d}\sigma\right)^{\frac{1}{x_3}}\left(\int_{\partial\Omega}u_2^{\tilde{c}_4y_3}\mathrm{d}\sigma\right)^{\frac{1}{y_3}}\left(\int_{\partial\Omega}(u_1u_{1,h}^{\kappa p_1})^{z_3}\mathrm{d}\sigma\right)^{\frac{1}{z_3}}\\ &=\tilde{C}_3\|u_1\|_{p_{1*},\partial\Omega}^{\tilde{c}_3}\|u_2\|_{p_{2*},\partial\Omega}^{\tilde{c}_4}\|u_1u_{1,h}^{\kappa p_1}\|_{z_3,\partial\Omega}\\ &\le M_7(1+\|u_1u_{1,h}^{\kappa}\|_{p_1z_3,\partial\Omega}^{p_1}),\end{aligned}$$

where the choice $x_3=\frac{p_{1*}}{\tilde{c}_3}$, $y_3=\frac{p_{2*}}{\tilde{c}_4}$ and $\frac{1}{z_3}=1-\frac{1}{x_3}-\frac{1}{y_3}$, is permitted in (F3). Finally, we get

$$\tilde{C}_4\int_{\partial\Omega}u_1u_{1,h}^{\kappa p_1}\mathrm{d}\sigma\le\tilde{C}_4\int_{\partial\Omega}u_1^{p_{1*}}u_{1,h}^{\kappa p_1}\mathrm{d}\sigma+\tilde{C}_4|\partial\Omega|.$$

According to (E2), (E5), and (F2), we know that

$$s_1',s_2'<\frac{p_1^*}{p_1}\quad\text{and}\quad t_1'<\frac{p_{1*}}{p_1}$$

and by (E3), (E6), and (F3) also

$$z_1, z_2 < \frac{p_1^*}{p_1} \quad \text{and} \quad z_3 < \frac{p_{1*}}{p_1}.$$

Therefore, we can set

$$s := \max\{s_1', s_2', z_1, z_2\} \in \left(1, \frac{p_1^*}{p_1}\right)$$
$$t := \max\{t_1', z_3\} \in \left(1, \frac{p_{1*}}{p_1}\right).$$

Summarizing, these estimates result in

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1} \mathrm{d}x + \kappa p_1 \int_{\{x\in\Omega: u_1(x)\le h\}} |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1} \mathrm{d}x \\ &\le (\tilde{A}_1 + M_3 + \tilde{A}_7)\int_\Omega u_1^{p_1^*} u_{1,h}^{\kappa p_1} \mathrm{d}x + (\tilde{C}_1 + \tilde{C}_4)\int_{\partial\Omega} u_1^{p_{1*}} u_{1,h}^{\kappa p_1} \mathrm{d}\sigma \\ &+ M_8 \|u_1 u_{1,h}^\kappa\|_{p_1 s}^{p_1} + M_9 \|u_1 u_{1,h}^\kappa\|_{p_1 t,\partial\Omega}^{p_1} + M_{10}(\kappa + 1). \end{aligned} \tag{3.5}$$

The left-hand side of (3.5) can be simplified to

$$\frac{1}{2}\int_\Omega |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1} \mathrm{d}x + \kappa p_1 \int_{\{x\in\Omega: u_1(x)\le h\}} |\nabla u_1|^{p_1} u_{1,h}^{\kappa p_1} \mathrm{d}x \ge \frac{\kappa p_1 + 1}{2(\kappa+1)^{p_1}} \int_\Omega |\nabla(u_1 u_{1,h}^\kappa)|^{p_1} \mathrm{d}x.$$

This can be found in Marino and Winkert [30], see the inequality after (3.7). Now, using this in (3.5) yields

$$\begin{aligned} \frac{\kappa p_1 + 1}{2(\kappa+1)^{p_1}} \|\nabla(u_1 u_{1,h}^\kappa)\|_{p_1}^{p_1} &\le M_{11}(\kappa p_1 + 1)\int_\Omega u_1^{p_1^*} u_{1,h}^{\kappa p_1} \mathrm{d}x + M_{12}\int_{\partial\Omega} u_1^{p_{1*}} u_{1,h}^{\kappa p_1} \mathrm{d}\sigma \\ &+ M_8 \|u_1 u_{1,h}^\kappa\|_{p_1 s}^{p_1} + M_9 \|u_1 u_{1,h}^\kappa\|_{p_1 t,\partial\Omega}^{p_1} + M_{10}(\kappa + 1). \end{aligned}$$

From now on, we can argue analogously to Marino and Winkert [29, Theorem 3.1] starting with (3.12). Note the slightly different notation. Here, the exponents are denoted by p, q instead of p_1, p_2 and the variables by (u, v) in place of (u_1, u_2).

(ii) The statement can be proved analogous to Marino and Winkert [29, Theorem 3.2]. □

4 Existence and uniqueness results

In this section, we are going to prove that problem (1.1) has at least one nontrivial weak solution under certain conditions for the right-hand sides of (1.1).

To this end, let $h_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $k_i : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions for $i \in \{1, 2\}$ such that

$$\begin{aligned} f_1(x, s, t, \xi, \eta) &= h_1(x, s, t, \xi, \eta) - |s|^{p_1-2}s \quad \text{for a.a.} \quad x \in \Omega, \\ f_2(x, s, t, \xi, \eta) &= h_2(x, s, t, \xi, \eta) - |t|^{p_2-2}t \quad \text{for a.a.} \quad x \in \Omega, \\ g_1(x, s, t) &= k_1(x, s, t) - \beta_1 |s|^{p_1-2}s \quad \text{for a.a.} \quad x \in \partial\Omega, \\ g_2(x, s, t) &= k_2(x, s, t) - \beta_2 |t|^{p_2-2}t \quad \text{for a.a.} \quad x \in \partial\Omega, \end{aligned}$$

for all $s, t \in \mathbb{R}$, for all $\xi, \eta \in \mathbb{R}^N$ and for $\beta_1, \beta_2 > 0$. Then, problem (1.1) becomes

$$\begin{aligned} -\operatorname{div}(\mathcal{D}_{p_1,q_1,\mu_1}(u_1)) &= h_1(x, u_1, u_2, \nabla u_1, \nabla u_2) - |u_1|^{p_1-2}u_1 \quad \text{in } \Omega, \\ -\operatorname{div}(\mathcal{D}_{p_2,q_2,\mu_2}(u_2)) &= h_2(x, u_1, u_2, \nabla u_1, \nabla u_2) - |u_2|^{p_2-2}u_2 \quad \text{in } \Omega, \\ \mathcal{D}_{p_1,q_1,\mu_1}(u_1)\cdot \vartheta &= k_1(x, u_1, u_2) - \beta_1 |u_1|^{p_1-2}u_1 \quad \text{on } \partial\Omega, \\ \mathcal{D}_{p_2,q_2,\mu_2}(u_2)\cdot \vartheta &= k_2(x, u_1, u_2) - \beta_2 |u_2|^{p_2-2}u_2 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

We assume the following assumptions on the functions h_1, h_2, k_1, k_2.

(H2) $h_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $k_i : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions with $h_i(x, 0, 0, 0, 0) \neq 0$ for a.a. $x \in \Omega$ and $i \in \{1, 2\}$ such that the following hold:

(i) There exists $\alpha_i \in L^{r'}(\Omega)$, $\alpha_i \geq 0$, $i \in \{1, 2\}$ with

$$|h_1(x, s, t, \xi, \eta)| \leq \hat{A}_1 \, |s|^{\hat{a}_1} + \hat{A}_2 \, |t|^{\hat{a}_2} + \hat{A}_3 \, |s|^{\hat{a}_3}|t|^{\hat{a}_4} + \hat{A}_4 \, |\xi|^{\hat{a}_5} + \hat{A}_5 \, |\eta|^{\hat{a}_6} + \hat{A}_6 \, |\xi|^{\hat{a}_7}|\eta|^{\hat{a}_8} + \alpha_1(x),$$
$$|h_2(x, s, t, \xi, \eta)| \leq \hat{B}_1 \, |s|^{\hat{b}_1} + \hat{B}_2 \, |t|^{\hat{b}_2} + \hat{B}_3 \, |s|^{\hat{b}_3}|t|^{\hat{b}_4} + \hat{B}_4 \, |\xi|^{\hat{b}_5} + \hat{B}_5 \, |\eta|^{\hat{b}_6} + \hat{B}_6 \, |\xi|^{\hat{b}_7}|\eta|^{\hat{b}_8} + \alpha_2(x)$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}$, for all $\xi, \eta \in \mathbb{R}^N$ with constants $\hat{A}_j, \hat{B}_j \geq 0$ for $j \in \{1, \dots, 6\}$ and $1 < r_i < p_i^*$ for $i \in \{1, 2\}$. The exponents $\hat{a}_l, \hat{b}_l \geq 0$ for $l \in \{1, \dots 8\}$ fulfill the conditions listed below:

$$\text{(E1')} \quad \hat{a}_1 \leq r_1 - 1, \qquad \text{(E2')} \quad \hat{a}_2 \leq \frac{r_1 - 1}{r_1} r_2,$$
$$\text{(E3')} \quad \frac{\hat{a}_3}{r_1} + \frac{\hat{a}_4}{r_2} \leq \frac{r_1 - 1}{r_1}, \qquad \text{(E4')} \quad \hat{a}_5 \leq \frac{r_1 - 1}{r_1} p_1,$$
$$\text{(E5')} \quad \hat{a}_6 \leq \frac{r_1 - 1}{r_1} p_2, \qquad \text{(E6')} \quad \frac{\hat{a}_7}{p_1} + \frac{\hat{a}_8}{p_2} \leq \frac{r_1 - 1}{r_1},$$
$$\text{(E7')} \quad \hat{b}_1 \leq \frac{r_2 - 1}{r_2} r_1, \qquad \text{(E8')} \quad \hat{b}_2 \leq r_2 - 1,$$
$$\text{(E9')} \quad \frac{\hat{b}_3}{r_1} + \frac{\hat{b}_4}{r_2} \leq \frac{r_2 - 1}{r_2}, \qquad \text{(E10')} \quad \hat{b}_5 \leq \frac{r_2 - 1}{r_2} p_1,$$
$$\text{(E11')} \quad \hat{b}_6 \leq \frac{r_2 - 1}{r_2} p_2, \qquad \text{(E12')} \quad \frac{\hat{b}_7}{p_1} + \frac{\hat{b}_8}{p_2} \leq \frac{r_2 - 1}{r_2};$$

(ii) There exists $\tilde{\alpha}_i \in L^{\tilde{r}'}(\partial\Omega)$, $\tilde{\alpha}_i \geq 0$, $i \in \{1, 2\}$ with

$$|k_1(x, s, t)| \leq \hat{C}_1 \, |s|^{\hat{c}_1} + \hat{C}_2 \, |t|^{\hat{c}_2} + \hat{C}_3 \, |s|^{\hat{c}_3}|t|^{\hat{c}_4} + \tilde{\alpha}_1(x)$$
$$|k_2(x, s, t)| \leq \hat{D}_1 \, |s|^{\hat{d}_1} + \hat{D}_2 \, |t|^{\hat{d}_2} + \hat{D}_3 \, |s|^{\hat{d}_3}|t|^{\hat{d}_4} + \tilde{\alpha}_2(x)$$

for a.a. $x \in \partial\Omega$, for all $s, t \in \mathbb{R}$ with constants $\hat{C}_j, \hat{D}_j \geq 0$, $j \in \{1, 2, 3\}$ and $1 < \tilde{r}_i < p_{i*}$ for $i \in \{1, 2\}$. The exponents $\hat{c}_j, \hat{d}_j \geq 0$ for $j \in \{1, 2, 3\}$ satisfy the following conditions:

$$\text{(F1')} \quad \hat{c}_1 \leq \tilde{r}_1 - 1, \qquad \text{(F2')} \quad \hat{c}_2 \leq \frac{\tilde{r}_1 - 1}{\tilde{r}_1} \tilde{r}_2,$$
$$\text{(F3')} \quad \frac{\hat{c}_3}{\tilde{r}_1} + \frac{\hat{c}_4}{\tilde{r}_2} \leq \frac{\tilde{r}_1 - 1}{\tilde{r}_1}, \qquad \text{(F4')} \quad \hat{d}_1 \leq \frac{\tilde{r}_2 - 1}{\tilde{r}_2} \tilde{r}_1,$$
$$\text{(F5')} \quad \hat{d}_2 \leq \tilde{r}_2 - 1, \qquad \text{(F6')} \quad \frac{\hat{d}_3}{\tilde{r}_1} + \frac{\hat{d}_4}{\tilde{r}_2} \leq \frac{\tilde{r}_2 - 1}{\tilde{r}_2};$$

(iii) There exists $\omega_1 \in L^1(\Omega)$ and $\omega_2 \in L^1(\partial\Omega)$, such that

$$h_1(x, s, t, \xi, \eta)s + h_2(x, s, t, \xi, \eta)t \leq \Lambda(|\xi|^{p_1} + |\eta|^{p_2}) + \Gamma(|s|^{p_1} + |t|^{p_2}) + \omega_1(x)$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}$, for all $\xi, \eta \in \mathbb{R}^N$ and

$$k_1(x, s, t)s + k_2(x, s, t)t \leq \Phi(|s|^{p_1} + |t|^{p_2}) + \omega_2(x)$$

for a.a. $x \in \partial\Omega$ and for all $s, t \in \mathbb{R}$, whereby $\Lambda, \Gamma, \Phi \geq 0$ are constants that satisfy **one** of the two conditions (A) or (B):

(A) There holds
  (i) $\Lambda + \Gamma \max\{(\lambda_{1,p_1,\zeta_1}^R)^{-1}, (\lambda_{1,p_2,\zeta_2}^R)^{-1}\} < 1$,
  (ii) $\Gamma(\lambda_{1,p_1,\zeta_1}^R)^{-1}\zeta_1 + \Phi < \beta_1$,
  (iii) $\Gamma(\lambda_{1,p_2,\zeta_2}^R)^{-1}\zeta_2 + \Phi < \beta_2$;

(B) There holds

$$\max\{\Lambda, \Gamma\} + \Phi \max\{(\lambda_{1,p_1}^S)^{-1}, (\lambda_{1,p_2}^S)^{-1}\} < 1.$$

Here, $\lambda^R_{1,p_1,\zeta_1}, \lambda^R_{1,p_2,\zeta_2}$ denote the first eigenvalue of the p_i-Laplacian with Robin boundary condition given in (2.2) and constants $\zeta_1, \zeta_2 > 0$, while $\lambda^S_{1,p_1}, \lambda^S_{1,p_2}$ stand for the first eigenvalues of the p_i-Laplacian with Steklov boundary condition given in (2.4).

Next, we state the definition of weak solutions of the system (4.1).

Definition 4.1. A function $(u_1, u_2) \in V$ is called a weak solution of (4.1) if

$$\begin{aligned}&\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot\nabla\varphi_1 \mathrm{d}x + \int_\Omega |u_1|^{p_1-2}u_1\varphi_1 \mathrm{d}x\\ &\quad = \int_\Omega h_1(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_1 \mathrm{d}x + \int_{\partial\Omega} k_1(x, u_1, u_2)\varphi_1 \mathrm{d}\sigma - \beta_1 \int_{\partial\Omega} |u_1|^{p_1-2}u_1\varphi_1 \mathrm{d}\sigma\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + \mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\cdot\nabla\varphi_2 \mathrm{d}x + \int_\Omega |u_2|^{p_2-2}u_2\varphi_2 \mathrm{d}x\\ &\quad = \int_\Omega h_2(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_2 \mathrm{d}x + \int_{\partial\Omega} k_2(x, u_1, u_2)\varphi_2 \mathrm{d}\sigma - \beta_2 \int_{\partial\Omega} |u_2|^{p_2-2}u_2\varphi_2 \mathrm{d}\sigma\end{aligned} \tag{4.3}$$

are satisfied for all $(\varphi_1, \varphi_2) \in V$.

Obviously, the terms in (4.2) and (4.3) are well-defined under hypotheses (H0) and (H2). Our main existence result reads as follows.

Theorem 4.2. *Let hypotheses* (H0) *and* (H2) *be satisfied. Then, there exists a nontrivial weak solution* $(u_1, u_2) \in V$ *of problem* (4.1).

Proof. We first consider the Nemytskij operators associated to $h := (h_1, h_2)$ and to $k := (k_1, k_2)$. They are defined by

$$\begin{aligned}&\tilde{N}_h : W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega) \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to L^{r_1'}(\Omega) \times L^{r_2'}(\Omega),\\ &\tilde{N}_k : L^{\tilde{r}_1}(\partial\Omega) \times L^{\tilde{r}_2}(\partial\Omega) \to L^{\tilde{r}_1'}(\partial\Omega) \times L^{\tilde{r}_2'}(\partial\Omega)\end{aligned}$$

with

$$\begin{aligned}&\tilde{N}_h(u_1, u_2) = (h_1(\cdot\ , u_1, u_2, \nabla u_1, \nabla u_2), h_2(\cdot\ , u_1, u_2, \nabla u_1, \nabla u_2)),\\ &\tilde{N}_k(u_1, u_2) = (k_1(\cdot\ , u_1, u_2), k_2(\cdot\ , u_1, u_2)).\end{aligned}$$

Due to the fact that $1 < r_i < p_i^*$ and $1 < \tilde{r}_i < p_{i*}$ hold for $i \in \{1, 2\}$, we have the compact embeddings

$$\begin{aligned}&j : W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega) \to L^{r_1}(\Omega) \times L^{r_2}(\Omega),\\ &\tilde{j} : W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega) \to L^{\tilde{r}_1}(\partial\Omega) \times L^{\tilde{r}_2}(\partial\Omega).\end{aligned}$$

Furthermore, the adjoint operators are given by

$$\begin{aligned}&j^* : L^{r_1'}(\Omega) \times L^{r_2'}(\Omega) \to (W^{1,\mathcal{H}_1}(\Omega))^* \times (W^{1,\mathcal{H}_2}(\Omega))^*,\\ &\tilde{j}^* : L^{\tilde{r}_1'}(\partial\Omega) \times L^{\tilde{r}_2'}(\partial\Omega) \to (W^{1,\mathcal{H}_1}(\Omega))^* \times (W^{1,\mathcal{H}_2}(\Omega))^*.\end{aligned}$$

Now, we define $N_h := j^* \circ \tilde{N}_h$ and $N_k := \tilde{j}^* \circ \tilde{N}_k \circ \tilde{j}$. Taking the isomorphism

$$V^* = (W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega))^* \cong (W^{1,\mathcal{H}_1}(\Omega))^* \times (W^{1,\mathcal{H}_2}(\Omega))^*$$

into account, we get

$$\langle N_h(u_1, u_2), (\varphi_1, \varphi_2)\rangle_V = \int_\Omega h_1(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_1 \mathrm{d}x + \int_\Omega h_2(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_2 \mathrm{d}x$$

and

$$\langle N_k(u_1,u_2),(\varphi_1,\varphi_2)\rangle_V = \int_{\partial\Omega} k_1(x,u_1,u_2)\varphi_1 \mathrm{d}\sigma + \int_{\partial\Omega} k_2(x,u_1,u_2)\varphi_2 \mathrm{d}\sigma$$

for $N_h, N_k : V \to V^*$. Moreover, we define the operator

$$\tilde{N}_\beta : L^{p_1}(\partial\Omega) \times L^{p_2}(\partial\Omega) \to L^{p_1'}(\partial\Omega) \times L^{p_2'}(\partial\Omega)$$

by

$$\tilde{N}_\beta(u_1,u_2) \coloneqq (\beta_1 |u_1|^{p_1-2}u_1, \beta_2 |u_2|^{p_2-2}u_2).$$

In the same way as above, we define $N_\beta \coloneqq l^* \circ B \circ l$, where

$$l : W^{1,\mathcal{H}_1}(\Omega) \times W^{1,\mathcal{H}_2}(\Omega) \to L^{p_1}(\partial\Omega) \times L^{p_2}(\partial\Omega)$$

is the compact embedding and

$$l^* : L^{p_1'}(\partial\Omega) \times L^{p_2'}(\partial\Omega) \to (W^{1,\mathcal{H}_1}(\Omega))^* \times (W^{1,\mathcal{H}_2}(\Omega))^*$$

its adjoint map. Then, $N_\beta : V \to V^*$ is given by

$$\langle N_\beta(u_1,u_2),(\varphi_1,\varphi_2)\rangle_V = \int_{\partial\Omega} \beta_1 |u_1|^{p_1-2}u_1\varphi_1 \mathrm{d}\sigma + \int_{\partial\Omega} \beta_2 |u_2|^{p_2-2}u_2\varphi_2 \mathrm{d}\sigma.$$

Now, we consider the operator $\mathcal{A} : V \to V^*$ given by

$$\mathcal{A} \coloneqq A - N_h - N_k + N_\beta,$$

where A was defined in (2.7). By construction, a function $(u_1,u_2) \in V$ is a weak solution of (4.1) if and only if $\mathcal{A}(u_1,u_2) = 0$ holds.

In the following, we will apply Theorem 2.4 and show that $\mathcal{A}$ satisfies the assumptions given in the theorem.

First, the operator $\mathcal{A}$ is bounded. This follows directly from Lemma 2.5 and hypotheses (H2)(i) and (ii).

Next, we are going to show that $\mathcal{A}$ is pseudomonotone. For this purpose, let $\{(u_1^{(n)},u_2^{(n)})\}_{n\in\mathbb{N}}$ be an arbitrary sequence in V such that

$$(u_1^{(n)},u_2^{(n)}) \rightharpoonup (u_1,u_2) \quad \text{in } V$$

and

$$\limsup_{n\to\infty} \langle \mathcal{A}(u_1^{(n)},u_2^{(n)}),(u_1^{(n)}-u_1,u_2^{(n)}-u_2)\rangle_V \leq 0.$$

In particular, we obtain $u_1^{(n)} \rightharpoonup u_1$ in $W^{1,\mathcal{H}_1}(\Omega)$ and $u_2^{(n)} \rightharpoonup u_2$ in $W^{1,\mathcal{H}_2}(\Omega)$. As $r_i < p_i^*$ and $\tilde{r}_i < p_{i*}$ holds, we know that

$$W^{1,\mathcal{H}_i}(\Omega) \to L^{r_i}(\Omega) \quad \text{and} \quad W^{1,\mathcal{H}_i}(\Omega) \to L^{\tilde{r}_i}(\partial\Omega)$$

compactly for $i \in \{1,2\}$ due to Proposition 2.1(ii), (iii). Hence, there are subsequences, not relabeled, such that

$$u_1^{(n)} \to u_1 \quad \text{in } L^{r_1}(\Omega) \quad \text{and} \quad u_2^{(n)} \to u_2 \quad \text{in } L^{r_2}(\Omega), \tag{4.4}$$

$$u_1^{(n)} \to u_1 \quad \text{in } L^{\tilde{r}_1}(\partial\Omega) \quad \text{and} \quad u_2^{(n)} \to u_2 \quad \text{in } L^{\tilde{r}_2}(\partial\Omega). \tag{4.5}$$

By a similar argument, we have

$$u_1^{(n)} \to u_1 \quad \text{in } L^{p_1}(\partial\Omega) \quad \text{and} \quad u_2^{(n)} \to u_2 \quad \text{in } L^{p_2}(\partial\Omega),$$

since $p_i < p_{i*}$ holds for $i \in \{1,2\}$.

We now prove that

$$\limsup_{n\to\infty} \langle A(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)} - u_1, u_2^{(n)} - u_2)\rangle_V = \limsup_{n\to\infty} \langle \mathcal{A}(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)} - u_1, u_2^{(n)} - u_2)\rangle_V \le 0, \tag{4.6}$$

to use the (S+)-property of A, see Lemma 2.5. Taking the growth condition in (H2)(i) into account yields the estimate

$$\begin{aligned}
&\left|\int_\Omega h_1(x, u_1^{(n)}, u_2^{(n)}, \nabla u_1^{(n)}, \nabla u_2^{(n)})(u_1^{(n)} - u_1)\mathrm{d}x\right| \\
&\le \int_\Omega (\hat{A}_1\, |u_1^{(n)}|^{\hat{a}_1} + \hat{A}_2\, |u_2^{(n)}|^{\hat{a}_2} + \hat{A}_3\, |u_1^{(n)}|^{\hat{a}_3}|u_2^{(n)}|^{\hat{a}_4} + \hat{A}_4\, |\nabla u_1^{(n)}|^{\hat{a}_5} \\
&\quad + \hat{A}_5\, |\nabla u_2^{(n)}|^{\hat{a}_6} + \hat{A}_6\, |\nabla u_1^{(n)}|^{\hat{a}_7}|\nabla u_2^{(n)}|^{\hat{a}_8} + \alpha_1(x))|u_1^{(n)} - u_1|\mathrm{d}x,
\end{aligned}$$

where we are going to show that the right-hand side converges to zero by (4.4). We consider every term separately. Applying Hölder's inequality gives

$$\begin{aligned}
\hat{A}_1 \int_\Omega |u_1^{(n)}|^{\hat{a}_1}|u_1^{(n)} - u_1|\mathrm{d}x &\le \hat{A}_1\left(\int_\Omega |u_1^{(n)}|^{\hat{a}_1 r_1'}\mathrm{d}x\right)^{\frac{1}{r_1'}}\left(\int_\Omega |u_1^{(n)} - u_1|^{r_1}\right)^{\frac{1}{r_1}} \\
&= \hat{A}_1\|u_1^{(n)}\|_{\hat{a}_1 r_1'}^{\hat{a}_1}\|u_1^{(n)} - u_1\|_{r_1}
\end{aligned}$$

for the first term. By (E1'), we have $\hat{a}_1 r_1' \le r_1$, so

$$\|u_1^{(n)}\|_{\hat{a}_1 r_1'}^{\hat{a}_1} \le \left(\int_\Omega (1 + |u_1^{(n)}|^{r_1})\mathrm{d}x\right)^{\frac{1}{r_1'}} \le C(1 + \|u_1^{(n)}\|_{r_1}^{r_1 - 1})$$

with $C > 0$ is bounded for $n \in \mathbb{N}$, and finally,

$$\hat{A}_1\int_\Omega |u_1^{(n)}|^{\hat{a}_1}|u_1^{(n)} - u_1|\mathrm{d}x \le \hat{A}_1\|u_1^{(n)}\|_{\hat{a}_1 r_1'}^{\hat{a}_1}\|u_1^{(n)} - u_1\|_{r_1} \to 0$$

as $n \to \infty$ due to (4.4). Analogously, the condition (E2') and (4.4) yield

$$\hat{A}_2\int_\Omega |u_2^{(n)}|^{\hat{a}_2}|u_1^{(n)} - u_1|\mathrm{d}x \le \hat{A}_2\|u_2^{(n)}\|_{\hat{a}_2 r_1'}^{\hat{a}_2}\|u_1^{(n)} - u_1\|_{r_1} \to 0$$

as $n \to \infty$. Furthermore, Hölder's inequality with three components and (E3') gives us

$$\hat{A}_3\int_\Omega |u_1^{(n)}|^{\hat{a}_3}|u_2^{(n)}|^{\hat{a}_4}|u_1^{(n)} - u_1|\mathrm{d}x \le \hat{A}_3\|u_1^{(n)}\|_{\hat{a}_3 x_1}^{\hat{a}_3}\|u_2^{(n)}\|_{\hat{a}_4 y_1}^{\hat{a}_4}\|u_1^{(n)} - u_1\|_{r_1} \to 0 \quad \text{as } n \to \infty,$$

where we choose $x_1, y_1, z_1 > 1$ with

$$\frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{z_1} = 1, \quad z_1 = r_1, \quad \hat{a}_3 x_1 \le r_1, \quad \hat{a}_4 y_1 \le r_2.$$

To estimate the next term, we note that $W^{1,\mathcal{H}_i}(\Omega) \hookrightarrow W^{1,p_i}(\Omega)$ holds for $i \in \{1, 2\}$. Consequently, (E4') and (E5') as well as Hölder's inequality and (4.4) imply

$$\begin{aligned}
\hat{A}_4 \int_\Omega |\nabla u_1^{(n)}|^{\hat{a}_5}|u_1^{(n)} - u_1|\mathrm{d}x &\le \hat{A}_4\|\nabla u_1^{(n)}\|_{\hat{a}_5 r_1'}^{\hat{a}_5}\|u_1^{(n)} - u_1\|_{r_1} \to 0, \\
\hat{A}_5 \int_\Omega |\nabla u_2^{(n)}|^{\hat{a}_6}|u_1^{(n)} - u_1|\mathrm{d}x &\le \hat{A}_5\|\nabla u_2^{(n)}\|_{\hat{a}_6 r_1'}^{\hat{a}_6}\|u_1^{(n)} - u_1\|_{r_1} \to 0
\end{aligned}$$

as $n \to \infty$. In addition, we obtain

$$\begin{aligned}&\hat{A}_6 \int_\Omega |\nabla u_1^{(n)}|^{\hat{a}_7} |\nabla u_2^{(n)}|^{\hat{a}_8} |u_1^{(n)} - u_1| \mathrm{d}x \\ &\leq \hat{A}_6 \|\nabla u_1^{(n)}\|_{\hat{a}_7 x_2}^{\hat{a}_7} \|\nabla u_2^{(n)}\|_{\hat{a}_8 y_2}^{\hat{a}_8} \|u_1^{(n)} - u_1\|_{r_1} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

where we choose $x_2, y_2, z_2 > 1$ with (E6') such that

$$\frac{1}{x_2} + \frac{1}{y_2} + \frac{1}{z_2} = 1, \quad z_2 = r_1, \quad \hat{a}_7 x_2 \leq p_1, \quad \hat{a}_8 y_2 \leq p_2$$

is satisfied. Finally, we obtain

$$\int_\Omega |\alpha_1(x)||u_1^{(n)} - u_1| \mathrm{d}x \leq \|\alpha_1\|_{r_1'} \|u_1^{(n)} - u_1\|_{r_1} \to 0 \quad \text{as } n \to \infty,$$

since $\alpha_1 \in L^{r_1'}(\Omega)$. All in all, it follows that

$$\lim_{n\to\infty} \int_\Omega h_1(x, u_1^{(n)}, u_2^{(n)}, \nabla u_1^{(n)}, \nabla u_2^{(n)})(u_1^{(n)} - u_1) \mathrm{d}x = 0.$$

The same conclusion can be done for h_2 by the help of (E7') to (E12') and (4.4), so that we know

$$\lim_{n\to\infty} \int_\Omega h_2(x, u_1^{(n)}, u_2^{(n)}, \nabla u_1^{(n)}, \nabla u_2^{(n)})(u_2^{(n)} - u_2) \mathrm{d}x = 0.$$

We conclude that

$$\lim_{n\to\infty} \langle N_h(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)} - u_1, u_2^{(n)} - u_2)\rangle_V = 0.$$

The boundary term can be handled in the same way, the only difference being in the application of (H3)(ii) and (4.5). This leads to

$$\begin{aligned}&\langle N_k(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)} - u_1, u_2^{(n)} - u_2)\rangle_V \\ &= \int_{\partial\Omega} k_1(x, u_1^{(n)}, u_2^{(n)})(u_1^{(n)} - u_1) \mathrm{d}\sigma + \int_{\partial\Omega} k_2(x, u_1^{(n)}, u_2^{(n)})(u_2^{(n)} - u_2) \mathrm{d}\sigma \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\langle N_\beta(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)} - u_1, u_2^{(n)} - u_2)\rangle_V \\ &= \int_{\partial\Omega} \beta_1 |u_1^{(n)}|^{p_1-2} u_1^{(n)}(u_1^{(n)} - u_1) \mathrm{d}\sigma + \int_{\partial\Omega} \beta_2 |u_2^{(n)}|^{p_2-2} u_2^{(n)}(u_2^{(n)} - u_2) \mathrm{d}\sigma \\ &\leq \beta_1 \left(\int_{\partial\Omega} |u_1^{(n)}|^{p_1} \mathrm{d}\sigma\right)^{\frac{p_1-1}{p_1}} \left(\int_{\partial\Omega} |u_1^{(n)} - u_1|^{p_1} \mathrm{d}\sigma\right)^{\frac{1}{p_1}} + \beta_2 \left(\int_{\partial\Omega} |u_2^{(n)}|^{p_2} \mathrm{d}\sigma\right)^{\frac{p_2-1}{p_2}} \left(\int_{\partial\Omega} |u_2^{(n)} - u_2|^{p_2} \mathrm{d}\sigma\right)^{\frac{1}{p_2}} \\ &= \beta_1 \|u_1^{(n)}\|_{p_1,\partial\Omega}^{p_1-1} \|u_1^{(n)} - u_1\|_{p_1,\partial\Omega} + \beta_2 \|u_2^{(n)}\|_{p_2,\partial\Omega}^{p_2-1} \|u_2^{(n)} - u_2\|_{p_2,\partial\Omega} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, (4.6) holds. Since A fulfills the (S₊)-property, we conclude $(u_1^{(n)}, u_2^{(n)}) \to (u_1, u_2)$ in V. Thus, the original sequence converges as well. By the continuity of $\mathcal{A}$, we have $\mathcal{A}(u_1^{(n)}, u_2^{(n)}) \to \mathcal{A}(u_1, u_2)$ in V^*. In particular, this implies

$$\mathcal{A}(u_1^{(n)}, u_2^{(n)}) \rightharpoonup \mathcal{A}(u_1, u_2)$$

and

$$\langle \mathcal{A}(u_1^{(n)}, u_2^{(n)}), (u_1^{(n)}, u_2^{(n)})\rangle_V \to \langle \mathcal{A}(u_1, u_2), (u_1, u_2)\rangle_V.$$

So $\mathcal{A}$ is pseudomonotone.

It remains to show that $\mathcal{A} : V \to V^*$ is coercive. For $(u_1, u_2) \in V$, we use (H2)(iii) to estimate

$$\begin{aligned}
&\langle \mathcal{A}(u_1, u_2), (u_1, u_2)\rangle_V \\
&= \int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot\nabla u_1 \mathrm{d}x + \int_\Omega |u_1|^{p_1}\mathrm{d}x \\
&\quad + \int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + \mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\cdot\nabla u_2 \mathrm{d}x + \int_\Omega |u_2|^{p_2}\mathrm{d}x \\
&\quad - \int_\Omega h_1(x, u_1, u_2, \nabla u_1, \nabla u_2)u_1\mathrm{d}x - \int_\Omega h_2(x, u_1, u_2, \nabla u_1, \nabla u_2)u_2\mathrm{d}x \\
&\quad - \int_{\partial\Omega} k_1(x, u_1, u_2)u_1\mathrm{d}\sigma - \int_{\partial\Omega} k_2(x, u_1, u_2)u_2\mathrm{d}\sigma + \beta_1\int_{\partial\Omega}|u_1|^{p_1}\mathrm{d}\sigma + \beta_2\int_{\partial\Omega}|u_2|^{p_2}\mathrm{d}\sigma \\
&\geq \int_\Omega (|\nabla u_1|^{p_1} + \mu_1(x)|\nabla u_1|^{q_1})\mathrm{d}x + \int_\Omega (|\nabla u_2|^{p_2} + \mu_2(x)|\nabla u_2|^{q_2})\mathrm{d}x + \|u_1\|_{p_1}^{p_1} + \|u_2\|_{p_2}^{p_2} \\
&\quad - \int_\Omega (\Lambda(|\nabla u_1|^{p_1} + |\nabla u_2|^{p_2}) + \Gamma(|u_1|^{p_1} + |u_2|^{p_2}) + \omega_1(x))\mathrm{d}x \\
&\quad - \int_{\partial\Omega} (\Phi(|u_1|^{p_1} + |u_2|^{p_2}) + \omega_2(x))\mathrm{d}\sigma + \beta_1\|u_1\|_{p_1,\partial\Omega}^{p_1} + \beta_2\|u_2\|_{p_2,\partial\Omega}^{p_2} \\
&= \|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_1\|_{q_1,\mu_1}^{q_1} + \|\nabla u_2\|_{p_2}^{p_2} + \|\nabla u_2\|_{q_2,\mu_2}^{q_2} + \|u_1\|_{p_1}^{p_1} + \|u_2\|_{p_2}^{p_2} \\
&\quad - \Lambda\|\nabla u_1\|_{p_1}^{p_1} - \Lambda\|\nabla u_2\|_{p_2}^{p_2} - \Gamma\|u_1\|_{p_1}^{p_1} - \Gamma\|u_2\|_{p_2}^{p_2} - \|\omega_1\|_1 \\
&\quad - \Phi\|u_1\|_{p_1,\partial\Omega}^{p_1} - \Phi\|u_2\|_{p_2,\partial\Omega}^{p_2} - \|\omega_2\|_{1,\partial\Omega} + \beta_1\|u_1\|_{p_1,\partial\Omega}^{p_1} + \beta_2\|u_2\|_{p_2,\partial\Omega}^{p_2}.
\end{aligned} \tag{4.7}$$

The following observation is divided into two different cases.

Case 1: Condition (A) of (H2)(iii) is satisfied. In this case, we use the Robin eigenvalue problem for the p_1- and p_2-Laplacian given in (2.2), that is, inequality (2.3) for p_1 and p_2 reads as follows:

$$\|u_1\|_{p_1}^{p_1} \leq (\lambda_{1,p_1,\zeta_1}^R)^{-1}(\|\nabla u_1\|_{p_1}^{p_1} + \zeta_1\|u_1\|_{p_1,\partial\Omega}^{p_1}), \quad u_1 \in W^{1,p_1}(\Omega), \tag{4.8}$$

$$\|u_2\|_{p_2}^{p_2} \leq (\lambda_{1,p_2,\zeta_2}^R)^{-1}(\|\nabla u_2\|_{p_2}^{p_2} + \zeta_2\|u_2\|_{p_2,\partial\Omega}^{p_2}), \quad u_2 \in W^{1,p_2}(\Omega). \tag{4.9}$$

Combining (4.7) with (4.8) and (4.9), we obtain

$$\begin{aligned}
&\langle \mathcal{A}(u_1, u_2), (u_1, u_2)\rangle_V \\
&\geq \|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_1\|_{q_1,\mu_1}^{q_1} + \|\nabla u_2\|_{p_2}^{p_2} + \|\nabla u_2\|_{q_2,\mu_2}^{q_2} + \|u_1\|_{p_1}^{p_1} + \|u_2\|_{p_2}^{p_2} \\
&\quad - \Lambda\|\nabla u_1\|_{p_1}^{p_1} - \Lambda\|\nabla u_2\|_{p_2}^{p_2} - \Gamma(\lambda_{1,p_1,\zeta_1}^R)^{-1}(\|\nabla u_1\|_{p_1}^{p_1} + \zeta_1\|u_1\|_{p_1,\partial\Omega}^{p_1}) \\
&\quad - \Gamma(\lambda_{1,p_2,\zeta_2}^R)^{-1}(\|\nabla u_2\|_{p_2}^{p_2} + \zeta_2\|u_2\|_{p_2,\partial\Omega}^{p_2}) - \|\omega_1\|_1 - \Phi\|u_1\|_{p_1,\partial\Omega}^{p_1} \\
&\quad - \Phi\|u_2\|_{p_2,\partial\Omega}^{p_2} - \|\omega_2\|_{1,\partial\Omega} + \beta_1\|u_1\|_{p_1,\partial\Omega}^{p_1} + \beta_2\|u_2\|_{p_2,\partial\Omega}^{p_2} \\
&\geq (\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_1\|_{q_1,\mu_1}^{q_1} + \|u_1\|_{p_1}^{p_1})(1 - \Lambda - \Gamma(\lambda_{1,p_1,\zeta_1}^R)^{-1}) \\
&\quad + (\|\nabla u_2\|_{p_2}^{p_2} + \|\nabla u_2\|_{q_2,\mu_2}^{q_2} + \|u_2\|_{p_2}^{p_2})(1 - \Lambda - \Gamma(\lambda_{1,p_2,\zeta_2}^R)^{-1}) \\
&\quad + (\beta_1 - \Gamma(\lambda_{1,p_1,\zeta_1}^R)^{-1}\zeta_1 - \Phi)\|u_1\|_{p_1\partial\Omega}^{p_1} \\
&\quad + (\beta_2 - \Gamma(\lambda_{1,p_2,\zeta_2}^R)^{-1}\zeta_2 - \Phi)\|u_2\|_{p_2,\partial\Omega}^{p_2} - \|\omega_1\|_1 - \|\omega_2\|_{1,\partial\Omega} \\
&\geq \rho_{1,\mathcal{H}_1}^*(u_1)(1 - \Lambda - \Gamma(\lambda_{1,p_1,\zeta_1}^R)^{-1}) + \rho_{1,\mathcal{H}_2}^*(u_2)(1 - \Lambda - \Gamma(\lambda_{1,p_2,\zeta_2}^R)^{-1}) - \|\omega_1\|_1 - \|\omega_2\|_{1,\partial\Omega},
\end{aligned}$$

since $\beta_i - \Gamma(\lambda_{1,p_i,\zeta_i}^R)^{-1}\zeta_i - \Phi > 0$ for $i \in \{1, 2\}$. We fix

$$C \coloneqq 1 - \Lambda - \Gamma\max\{(\lambda_{1,p_1,\zeta_1}^R)^{-1}, (\lambda_{1,p_2,\zeta_2}^R)^{-1}\}.$$

Then, we know from (H2)(iii)(A) that $C > 0$, and hence (2.1) yields

$$\begin{aligned}\frac{\langle \mathcal{A}(u_1,u_2),(u_1,u_2)\rangle_V}{\|(u_1,u_2)\|_V} &\geq \frac{C(\rho^*_{1,\mathcal{H}_1}(u_1)+\rho^*_{1,\mathcal{H}_2}(u_2))}{\|(u_1,u_2)\|_V} - \frac{\|\omega_1\|_1 + \|\omega_2\|_{1,\partial\Omega}}{\|(u_1,u_2)\|_V}\\ &\geq \frac{C(\min\{(\|u_1\|^*_{1,\mathcal{H}_1})^{p_1},(\|u_1\|^*_{1,\mathcal{H}_1})^{q_1}\} + \min\{(\|u_2\|^*_{1,\mathcal{H}_2})^{p_2},(\|u_2\|^*_{1,\mathcal{H}_2})^{q_2}\})}{\|u_1\|^*_{1,\mathcal{H}_1}+\|u_2\|^*_{1,\mathcal{H}_2}}\\ &\quad - \frac{\|\omega_1\|_1 + \|\omega_2\|_{1,\partial\Omega}}{\|(u_1,u_2)\|_V} \to \infty \quad \text{for } \|(u_1,u_2)\|_V \to \infty,\end{aligned}$$

because

$$\frac{\|\omega_1\|_1 + \|\omega_2\|_{1,\partial\Omega}}{\|(u_1,u_2)\|_V} \to 0$$

and $p_i, q_i > 1$ for $i \in \{1,2\}$.

Case 2: Condition (B) of (H2)(iii) is satisfied. Now we consider the Steklov eigenvalue problem (2.4) and use inequality (2.5) for p_1, p_2 which gives

$$\begin{aligned}\|u_1\|^{p_1}_{p_1,\partial\Omega} &\leq (\lambda^S_{1,p_1})^{-1}(\|\nabla u_1\|^{p_1}_{p_1} + \|u_1\|^{p_1}_{p_1}), \quad u_1 \in W^{1,p_1}(\Omega),\\ \|u_2\|^{p_2}_{p_2,\partial\Omega} &\leq (\lambda^S_{1,p_2})^{-1}(\|\nabla u_2\|^{p_2}_{p_2} + \|u_2\|^{p_2}_{p_2}), \quad u_2 \in W^{1,p_2}(\Omega).\end{aligned}$$

As in the proof of Case 1, equation (4.7) gives

$$\begin{aligned}&\langle \mathcal{A}(u_1,u_2),(u_1,u_2)\rangle_V\\ &\geq \|\nabla u_1\|^{p_1}_{p_1} + \|\nabla u_1\|^{q_1}_{q_1,\mu_1} + \|\nabla u_2\|^{p_2}_{p_2} + \|\nabla u_2\|^{q_2}_{q_2,\mu_2} + \|u_1\|^{p_1}_{p_1} + \|u_2\|^{p_2}_{p_2}\\ &\quad - \Lambda\|\nabla u_1\|^{p_1}_{p_1} - \Lambda\|\nabla u_2\|^{p_2}_{p_2} - \Gamma\|u_1\|^{p_1}_{p_1} - \Gamma\|u_2\|^{p_2}_{p_2}\\ &\quad - \Phi(\lambda^S_{1,p_1})^{-1}(\|\nabla u_1\|^{p_1}_{p_1} + \|u_1\|^{p_1}_{p_1}) - \Phi(\lambda^S_{1,p_2})^{-1}(\|\nabla u_2\|^{p_2}_{p_2} + \|u_2\|^{p_2}_{p_2}) - \|\omega_1\|_1 - \|\omega_2\|_{1,\partial\Omega}\\ &\geq (\|\nabla u_1\|^{p_1}_{p_1} + \|\nabla u_1\|^{q_1}_{q_1,\mu_1} + \|u_1\|^{p_1}_{p_1})(1 - \max\{\Lambda,\Gamma\} - \Phi(\lambda^S_{1,p_1})^{-1})\\ &\quad + (\|\nabla u_2\|^{p_2}_{p_2} + \|\nabla u_2\|^{q_2}_{q_2,\mu_2} + \|u_2\|^{p_2}_{p_2})(1 - \max\{\Lambda,\Gamma\} - \Phi(\lambda^S_{1,p_2})^{-1}) - \|\omega_1\|_1 - \|\omega_2\|_{1,\partial\Omega}\\ &\geq (1 - \max\{\Lambda,\Gamma\} - \Phi\max\{(\lambda^S_{1,p_1})^{-1},(\lambda^S_{1,p_2})^{-1}\})(\rho^*_{1,\mathcal{H}_1}(u_1) + \rho^*_{1,\mathcal{H}_2}(u_2)) - \|\omega_1\|_1 - \|\omega_2\|_{1,\partial\Omega},\end{aligned}$$

where

$$1 - \max\{\Lambda,\Gamma\} - \Phi\max\{(\lambda^S_{1,p_1})^{-1},(\lambda^S_{1,p_2})^{-1}\} > 0$$

holds by assumption. Again, the result is

$$\frac{\langle \mathcal{A}(u_1,u_2),(u_1,u_2)\rangle_V}{\|(u_1,u_2)\|_V} \to \infty \quad \text{as } \|(u_1,u_2)\|_V \to \infty.$$

Hence, the operator $\mathcal{A}$ is coercive. By Theorem 2.4, there exists $(u_1,u_2) \in V$ with $\mathcal{A}(u_1,u_2) = 0$. Moreover, $h_i(x,0,0,0,0) \neq 0$ for a.a. $x \in \Omega, i \in \{1,2\}$ implies $(u_1,u_2) \neq 0$. This finishes the proof of the theorem. □

Next, we prove a uniqueness result for problem (4.1) under some stronger assumptions. To this end, we define

$$\mathbf{h} : \Omega \times \mathbb{R}^2 \times (\mathbb{R}^N)^2 \to \mathbb{R}^2, \; \mathbf{h}(x,s,\xi) := (h_1(x,s,\xi), h_2(x,s,\xi))$$

and

$$\mathbf{k} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2, \; \mathbf{k}(x,s) := (k_1(x,s), k_2(x,s))$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$. From now on, we make the following assumptions in addition to (H2):

(U1) There exists $G_1 \geq 0$ such that

$$(\mathbf{h}(x, s, \xi) - \mathbf{h}(x, t, \xi))\cdot(s - t) \leq G_1\, |s - t|^2$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$.

(U2) There exists $G_2 \geq 0$ such that

$$(\mathbf{k}(x, s) - \mathbf{k}(x, t))\cdot(s - t) \leq G_2\, |s - t|^2$$

for a.a. $x \in \partial\Omega$ and all $s, t \in \mathbb{R}^2$.

(U3) There exists $\rho = (\rho_1, \rho_2)$ with $\rho_i \in L^{s_i}(\Omega)$ for $1 < s_i < p_i^*$, $i \in \{1, 2\}$ and $G_3 \geq 0$ such that $\mathbf{h}(x, s, \cdot\,) - \rho(x)$ is linear on $(\mathbb{R}^N)^2$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}^2$ and in addition, we have

$$|\mathbf{h}(x, s, \xi) - \rho(x)| \leq G_3|\xi|$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$.

We obtain the following uniqueness results.

Theorem 4.3. *Let hypotheses* (H0), (H2), *and* (U1)–(U3) *be satisfied with* $p_1 = p_2 = 2$. *If*

$$G_1\tilde{\lambda} + G_2 \max\{\beta_1^{-1}, \beta_2^{-1}\} + G_3\sqrt{2\tilde{\lambda}} < 1 \tag{4.10}$$

with $\tilde{\lambda} \coloneqq \max\{(\lambda^R_{1,2,\beta_1})^{-1}, (\lambda^R_{1,2,\beta_2})^{-1}\}$ *is satisfied, then there exists a unique nontrivial weak solution of problem* (4.1).

Proof. By Theorem 4.2 and hypotheses (H2), there exists at least one nontrivial weak solution of problem (4.1). Now, let $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$ be two weak solutions of (4.1). According to the weak formulation of the problem, we have

$$\begin{aligned}
&\int_\Omega (\nabla u_1 + \mu_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\cdot\nabla\varphi_1 \mathrm{d}x + \int_\Omega (\nabla u_2 + \mu_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\cdot\nabla\varphi_2 \mathrm{d}x + \int_\Omega u_1\varphi_1 \mathrm{d}x + \int_\Omega u_2\varphi_2 \mathrm{d}x\\
&\quad = \int_\Omega \mathbf{h}(x, u, \nabla u)\cdot \varphi \mathrm{d}x + \int_{\partial\Omega} \mathbf{k}(x, u)\cdot \varphi \mathrm{d}\sigma - \beta_1 \int_{\partial\Omega} u_1\varphi_1 \mathrm{d}\sigma - \beta_2 \int_{\partial\Omega} u_2\varphi_2 \mathrm{d}\sigma,
\end{aligned}$$

for $\varphi = (\varphi_1, \varphi_2) \in V$. Replacing u by v yields an analogous equation. Next, we choose $\varphi \coloneqq u - v$ and subtract both equations, which leads to

$$\begin{aligned}
&\int_\Omega |\nabla(u_1 - v_1)|^2 \mathrm{d}x + \int_\Omega \mu_1(x)(|\nabla u_1|^{q_1-2}\nabla u_1 - |\nabla v_1|^{q_1-2}\nabla v_1)\cdot\nabla(u_1 - v_1)\mathrm{d}x + \int_\Omega |\nabla(u_2 - v_2)|^2 \mathrm{d}x\\
&\quad + \int_\Omega \mu_2(x)(|\nabla u_2|^{q_2-2}\nabla u_2 - |\nabla v_2|^{q_2-2}\nabla v_2)\cdot\nabla(u_2 - v_2)\mathrm{d}x + \int_\Omega |u_1 - v_1|^2 \mathrm{d}x + \int_\Omega |u_2 - v_2|^2 \mathrm{d}x\\
&\quad = \int_\Omega (\mathbf{h}(x, u, \nabla u) - \mathbf{h}(x, v, \nabla v))\cdot(u - v)\mathrm{d}x + \int_{\partial\Omega} (\mathbf{k}(x, u) - \mathbf{k}(x, v))\cdot(u - v)\mathrm{d}\sigma\\
&\qquad - \beta_1 \int_{\partial\Omega} |u_1 - v_1|^2 \mathrm{d}\sigma - \beta_2 \int_{\partial\Omega} |u_2 - v_2|^2 \mathrm{d}\sigma.
\end{aligned}$$

Since $\xi \mapsto |\xi|^{q_i-2}\xi$ is monotone and $\int_\Omega |u_i - v_i|^2 \mathrm{d}x \geq 0$ for $i \in \{1, 2\}$, we can estimate

$$\begin{aligned}
&\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2 + \beta_1\|u_1 - v_1\|_{2,\partial\Omega}^2 + \beta_2\|u_2 - v_2\|_{2,\partial\Omega}^2\\
&\quad = \int_\Omega |\nabla(u_1 - v_1)|^2 \mathrm{d}x + \int_\Omega |\nabla(u_2 - v_2)|^2 \mathrm{d}x + \beta_1 \int_{\partial\Omega} |u_1 - v_1|^2 \mathrm{d}\sigma + \beta_2 \int_{\partial\Omega} |u_2 - v_2|^2 \mathrm{d}\sigma\\
&\quad \leq \int_\Omega (\mathbf{h}(x, u, \nabla u) - \mathbf{h}(x, v, \nabla v))\cdot(u - v)\mathrm{d}x + \int_{\partial\Omega} (\mathbf{k}(x, u) - \mathbf{k}(x, v))\cdot(u - v)\mathrm{d}\sigma.
\end{aligned} \tag{4.11}$$

Applying (U1)–(U3) to the right-hand side of (4.11) implies

$$\begin{aligned}
&\int_\Omega (\mathbf{h}(x,u,\nabla u)-\mathbf{h}(x,v,\nabla v))\cdot(u-v)\mathrm{d}x+\int_{\partial\Omega}(\mathbf{k}(x,u)-\mathbf{k}(x,v))\cdot(u-v)\mathrm{d}\sigma\\
&=\int_\Omega (\mathbf{h}(x,u,\nabla u)-\mathbf{h}(x,v,\nabla u))\cdot(u-v)\mathrm{d}x\\
&\quad+\int_\Omega (\mathbf{h}(x,v,\nabla u)-\rho(x)-\mathbf{h}(x,v,\nabla v)+\rho(x))\cdot(u-v)\mathrm{d}x+\int_{\partial\Omega}(\mathbf{k}(x,u)-\mathbf{k}(x,v))\cdot(u-v)\mathrm{d}\sigma\\
&\le \int_\Omega G_1\,|u-v|^2\mathrm{d}x+\int_{\partial\Omega}G_2\,|u-v|^2\mathrm{d}\sigma+\int_\Omega h_1(x,v_1,v_2,(u_1-v_1)\nabla(u_1-v_1),(u_1-v_1)\nabla(u_2-v_2))-\rho_1(x)\mathrm{d}x\\
&\quad+\int_\Omega h_2(x,v_1,v_2,(u_2-v_2)\nabla(u_1-u_2),(u_2-v_2)\nabla(u_2-v_2))-\rho_2(x)\mathrm{d}x\\
&\le G_1\|u-v\|_2^2+G_2\|u-v\|_{\partial\Omega,2}^2+G_3\int_\Omega(|u_1-v_1|+|u_2-v_2|)(|\nabla(u_1-v_1)|^2+|\nabla(u_2-v_2)|^2)^{\frac12}\mathrm{d}x\\
&\le G_1\|u-v\|_2^2+G_2\|u-v\|_{\partial\Omega,2}^2+G_3\left(\int_\Omega(|u_1-v_1|+|u_2-v_2|)^2\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|\nabla(u_1-v_1)|^2+|\nabla(u_2-v_2)|^2\mathrm{d}x\right)^{\frac12}\\
&\le G_1\|u-v\|_2^2+G_2\|u-v\|_{\partial\Omega,2}^2\\
&\quad+G_3\sqrt{2}(\|u_1-v_1\|_2^2+\|u_2-v_2\|_2^2)^{\frac12}(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2)^{\frac12},
\end{aligned}$$

where we have used Hölder's inequality. In summary, the aforementioned estimates result in

$$\begin{aligned}
&\int_\Omega (\mathbf{h}(x,u,\nabla u)-\mathbf{h}(x,v,\nabla v))\cdot(u-v)\mathrm{d}x+\int_{\partial\Omega}(\mathbf{k}(x,u)-\mathbf{k}(x,v))\cdot(u-v)\mathrm{d}\sigma\\
&\quad\le G_1\|u-v\|_2^2+G_2\|u-v\|_{\partial\Omega,2}^2+G_3\sqrt{2}(\|u_1-v_1\|_2^2+\|u_2-v_2\|_2^2)^{\frac12}\\
&\qquad\times(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2)^{\frac12}.
\end{aligned}\tag{4.12}$$

Moreover, choosing $p_1=p_2=2$ and $\zeta_i=\beta_i>0$ for $i\in\{1,2\}$ in (4.8) and (4.9) leads to

$$\|z_1\|_2^2\le(\lambda^R_{1,2,\beta_1})^{-1}(\|\nabla z_1\|_2^2+\beta_1\|z_1\|_{2,\partial\Omega}^2),\quad z_1\in W^{1,\mathcal{H}_1}(\Omega),\tag{4.13}$$

$$\|z_2\|_2^2\le(\lambda^R_{1,2,\beta_2})^{-1}(\|\nabla z_2\|_2^2+\beta_2\|z_2\|_{2,\partial\Omega}^2),\quad z_2\in W^{1,\mathcal{H}_2}(\Omega).\tag{4.14}$$

Now, we combine (4.11), (4.12) with (4.13), (4.14) and obtain

$$\begin{aligned}
&\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2\\
&\le G_1(\|u_1-v_1\|_2^2+\|u_2-v_2\|_2^2)+G_2(\|u_1-v_1\|_{\partial\Omega,2}^2+\|u_2-v_2\|_{\partial\Omega,2}^2)\\
&\quad+G_3\sqrt{2}(\|u_1-v_1\|_2^2+\|u_2-v_2\|_2^2)^{\frac12}(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2)^{\frac12}\\
&\le G_1((\lambda^R_{1,2,\beta_1})^{-1}(\|\nabla(u_1-v_1)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2)\\
&\quad+(\lambda^R_{1,2,\beta_2})^{-1}(\|\nabla(u_2-v_2)\|_2^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2))\\
&\quad+G_2\max\{\beta_1^{-1},\beta_2^{-1}\}(\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2)\\
&\quad+G_3\sqrt{2}((\lambda^R_{1,2,\beta_1})^{-1}(\|\nabla(u_1-v_1)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2)\\
&\quad+(\lambda^R_{1,2,\beta_2})^{-1}(\|\nabla(u_2-v_2)\|_2^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2))^{\frac12}\\
&\quad\times(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2)^{\frac12}\\
&=\Big(G_1\max\{(\lambda^R_{1,2,\beta_1})^{-1},(\lambda^R_{1,2,\beta_2})^{-1}\}+G_2\max\{\beta_1^{-1},\beta_2^{-1}\}\\
&\quad+G_3\sqrt{2}\max\{(\lambda^R_{1,2,\beta_1})^{-1},(\lambda^R_{1,2,\beta_2})^{-1}\}^{\frac12}\Big)\\
&\quad\times(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2)\\
&=(G_1\tilde{\lambda}+G_2\max\{\beta_1^{-1},\beta_2^{-1}\}+G_3\sqrt{2\tilde{\lambda}})\\
&\quad\times(\|\nabla(u_1-v_1)\|_2^2+\|\nabla(u_2-v_2)\|_2^2+\beta_1\|u_1-v_1\|_{2,\partial\Omega}^2+\beta_2\|u_2-v_2\|_{2,\partial\Omega}^2),
\end{aligned}$$

where $\tilde{\lambda}$ is defined as above. Finally, we conclude

$$\begin{aligned}&(\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2 + \beta_1\|u_1 - v_1\|_{2,\partial\Omega}^2 + \beta_2\|u_2 - v_2\|_{2,\partial\Omega}^2)\\ &\quad\times (1 - (G_1\tilde{\lambda} + G_2 \max\{\beta_1^{-1}, \beta_2^{-1}\} + G_3\sqrt{2\tilde{\lambda}}\,)) \leq 0.\end{aligned}$$

Taking (4.10) into account yields

$$\|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2 + \beta_1\|u_1 - v_1\|_{2,\partial\Omega}^2 + \beta_2\|u_2 - v_2\|_{2,\partial\Omega}^2 = 0.$$

We know from Pagageorgiou and Winkert [37] that $\|\cdot\|_{\beta_i,2}$ given by

$$\|u\|_{\beta_i,2} := (\|\nabla u\|_2^2 + \beta_i\|u\|_{2,\partial\Omega}^2)^{\frac{1}{2}}, \;\; i \in \{1, 2\}$$

and $\|\cdot\|_{1,2}$ are equivalent norms on $W^{1,2}(\Omega)$. Hence, we have $u_1 = v_1$ and $u_2 = v_2$, which completes the proof. □

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