Advances in Differential Equations Volume 15, Numbers 5–6 (2010), 561–599

# CONSTANT-SIGN AND SIGN-CHANGING SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY VALUES

Patrick Winkert
Martin-Luther-University Halle-Wittenberg
Department of Mathematics, 06099 Halle, Germany

(Submitted by: Reza Aftabizadeh)

**Abstract.** In this paper we study the existence of multiple solutions to the equation

$$-\Delta_p u = f(x,u) - |u|^{p-2}u$$

with the nonlinear boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + g(x,u).$$

We establish the existence of a smallest positive solution, a greatest negative solution, and a nontrivial sign-changing solution when the parameter $\lambda$ is greater than the second eigenvalue of the Steklov eigenvalue problem. Our approach is based on truncation techniques and comparison principles for nonlinear elliptic differential inequalities. In particular, we make use of variational and topological tools, such as critical point theory, the mountain-pass theorem, the second deformation lemma and variational characterizations of the second eigenvalue of the Steklov eigenvalue problem.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following boundary-value problem:

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u), 1 < p < \infty$, is the negative $p-$Laplacian, $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of $u$ with respect to $\partial\Omega$, $\lambda$ is a real parameter and the nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are

Accepted for publication: November 2009.
AMS Subject Classifications: 35B38, 35J20, 47J10.

some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation $\tau(u)$ and write for short $u$.

Neumann boundary-value problems in the form (1.1) arise in different areas of pure and applied mathematics, for example in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [29],[60]), in the study of optimal constants for the Sobolev trace embedding (see [18],[32], [33],[34]) or at non-Newtonian fluids, flow through porous media, nonlinear elasticity, reaction diffusion problems, glaciology and so on (see [4],[5],[6],[19]).

Our main goal is to provide the existence of multiple solutions of (1.1) meaning that, for all values $\lambda > \lambda_2$, where $\lambda_2$ denotes the second eigenvalue of $(-\Delta_p, W^{1,p}(\Omega))$ known as the Steklov eigenvalue problem (see, e.g., [36, 49, 56]) given by

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

we show that there exist at least three nontrivial solutions. More precisely, we obtain two constant-sign solutions and one sign-changing solution of problem (1.1). This is the main result of the present paper and it is formulated in Theorems 4.3 and 6.3, respectively. In our consideration, the nonlinearities $f$ and $g$ only need to be Carathéodory functions which are bounded on bounded sets whereby their growth does not need to be necessarily polynomial. We only require some growth properties at zero and infinity given by

$$\lim_{s\to 0}\frac{f(x,s)}{|s|^{p-2}s} = \lim_{s\to 0}\frac{g(x,s)}{|s|^{p-2}s} = 0, \qquad \lim_{|s|\to\infty}\frac{f(x,s)}{|s|^{p-2}s} = \lim_{|s|\to\infty}\frac{g(x,s)}{|s|^{p-2}s} = -\infty$$

and we suppose the existence of $\delta_f > 0$ such that $f(x,s)/|s|^{p-2}s \geq 0$ for all $0 < |s| \leq \delta_f$.

In the last years many papers about the existence of Neumann problems similar to the form (1.1) were developed (see, e.g., [3, 17, 31, 35, 48, 69]). Martínez et al [48] proved the existence of weak solutions of the Neumann boundary problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}u + f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u - h(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where the perturbations $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Carathéodory functions satisfying an integral condition of Landesmann - Lazer type. Their main result is given in [48, Theorem 1.2] which yields the existence of a weak solution of (1.3) with $\lambda = \lambda_1$, where $\lambda_1$ is the first eigenvalue of the Steklov eigenvalue problem (see (1.2)). Moreover, they supposed in their main theorem the boundedness of $f(x,t)$ and $h(x,t)$ by functions $\overline{f} \in L^q(\Omega)$ and $\overline{h} \in L^q(\partial\Omega)$ for all $(x,t) \in \Omega\times\mathbb{R}$ and $(x,t) \in \partial\Omega\times\mathbb{R}$, respectively. A similar work regarding (1.1) can be found in [32]. There the authors get as well three nontrivial solutions for the nonlinear boundary-value problem

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u &= f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where they assume among other things that the Carathéodory functions $f$ and $g$ are also continuously differentiable in the second argument. The proof is based on the Lusternik-Schnirelmann method for non-compact manifolds. If the Neumann boundary values are defined by a function $f : \mathbb{R} \to \mathbb{R}$, meaning the problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(u) && \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

we refer to the results of J. Fernández Bonder and J.D. Rossi in [35]. They consider various cases where $f$ has subcritical growth, critical growth and supercritical growth, respectively. In the first two cases the existence of infinitely many solutions under some conditions on the exponents of the growth was demonstrated.

Another result obtaining multiple solutions with nonlinear boundary conditions can be found in the paper of J.H. Zhao and P.-H. Zhao [69]. They study the equation

$$\begin{aligned} -\Delta_p u + \lambda(x)|u|^{p-2}u &= f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \eta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $\lambda(x) \in L^\infty(\Omega)$ satisfies ess $\inf_{x\in\overline{\Omega}}\lambda(x) > 0$ and $\eta$ is a real parameter. They prove the existence of infinitely many solutions when $f$ is superlinear and subcritical with respect to $u$ by using the fountain theorem and the dual fountain theorem, respectively. In the case that $f$ has the form $f(x,u) =$

$|u|^{p^*-2}u + |u|^{r-2}u$ they get at least one nontrivial solution when $p < r < p^*$ and infinitely many solutions when $1 < r < p$ by using the mountain-pass theorem and the "concentration-compactness principle," respectively. A similar result of the same authors is also developed in [68]. The existence of multiple solutions and sign-changing solutions for zero Neumann boundary values have been proven in [44, 54, 55, 66] and [69], respectively. Analogous results for the Dirichlet problem have been recently obtained in [10, 11, 12, 13, 27, 50, 52]. An interesting problem about the existence of multiple solutions, for both the Dirichlet problem and the Neumann problem, can be found in [15]. The authors study the existence of multiple solutions to the abstract equation $J_p u = N_f u$, where $J_p$ is the duality mapping on a real reflexive and smooth Banach space $X$, corresponding to the gauge function $\varphi(t) = t^{p-1}, 1 < p < \infty$, and $N_f : L^q(\Omega) \to L^{q'}(\Omega), 1/q + 1/q' = 1$, is the Nemytskij operator generated by a function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

The novelty of our paper is the fact that we do not need differentiability, polynomial growth or some integral conditions on the mappings $f$ and $g$. In order to prove our main results we make use of variational and topological tools, e.g. critical point theory, the mountain-pass theorem, the second deformation lemma and variational characterizations of the second eigenvalue of the Steklov eigenvalue problem. This paper is motivated by recent publications of S. Carl and D. Motreanu in [12] and [11], respectively. In [12] the authors consider the Dirichlet problem $-\Delta_p u = \lambda |u|^{p-2}u + g(x, u)$ in $\Omega$, $u = 0$ on $\partial\Omega$, and show the existence of at least three nontrivial solutions for all values $\lambda > \lambda_2$, where $\lambda_2$ denotes the second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Therein, the main theorem about the existence of a sign-changing solution is also based on the mountain-pass theorem and the second deformation lemma. These results have been extended by themselves to the equation $-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + g(x, u)$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ denote the positive and negative part of $u$, respectively. Carl et al have shown that at least three nontrivial solutions exist provided the value $(a, b)$ is above the first nontrivial curve $\mathcal{C}$ of the Fǔcik spectrum constructed by Cuesta et al in [16].

The rest of the paper is organized as follows. In Section 2 and Section 3, we recall some preliminaries and formulate our notation and hypotheses, respectively. In Section 4, we will show the existence of specific sub- and supersolutions of problem (1.1), then we will prove that every solution between these pairs of sub- and supersolutions belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$ and finally we will provide the existence of extremal constant-sign solutions. A variational characterization of these extremal solutions is given in Section 5 and our main result about the existence of a nontrivial sign-changing solution is proven in the last section by using the mountain-pass theorem.

## 2. Preliminaries

Let us consider some nonlinear boundary-value problems with Neumann conditions involving the $p$-Laplacian. In [47] the authors study the Steklov problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u \qquad && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is linear and bounded (and even compact), thus a best constant $\lambda_1$ exists such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

The best Sobolev trace constant $\lambda_1$ can be characterized as

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)}\Big\{ \int_\Omega [|\nabla u|^p + |u|^p]dx : \int_{\partial\Omega} |u|^p d\sigma = 1\Big\},$$

and $\lambda_1$ is the first eigenvalue of (2.1). Martínez et al. showed that the first eigenvalue $\lambda_1 > 0$ is isolated and simple. The corresponding eigenfunction $\varphi_1$ is strictly positive in $\overline{\Omega}$ and belongs to $L^\infty(\Omega)$ (cf. [43, Lemma 5.6 and Theorem 4.3]). Applying the results of Lieberman in [45, Theorem 2] implies $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$. This fact along with $\varphi_1(x) > 0$ in $\overline{\Omega}$ yields $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$, where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \ge 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, given by

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \big\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\big\}.$$

The study of Neumann eigenvalue problems with or without weights are also considered in [17, 28, 41, 43, 61]. Analogous to the results for the Dirichlet eigenvalue problem (see [16]), there also exists a variational characterization of the second eigenvalue of (2.1) meaning that $\lambda_2$ can be represented as follows:

$$\lambda_2 = \inf_{g\in\Gamma} \max_{u\in g([-1,1])} \int_\Omega \Big(|\nabla u|^p + |u|^p\Big)dx, \tag{2.2}$$

where

$$\Gamma = \{g \in C([-1,1], S) : g(-1) = -\varphi_1, g(1) = \varphi_1\}, \tag{2.3}$$

and

$$S = \Big\{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1 \Big\}. \tag{2.4}$$

The proof of this result can be found in [49]. Now we consider solutions of the Neumann boundary-value problem

$$\begin{aligned} -\Delta_p u &= -\varsigma |u|^{p-2} u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where $\varsigma > 1$ is a constant. Let $B : L^p(\Omega) \to L^q(\Omega)$ be the Nemytskij operator defined by $Bu(x) := \varsigma |u(x)|^{p-2} u(x)$. It is well known that $B : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ is the adjoint operator of the compact embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$. The operator $\widehat{B}$ is bounded, continuous, completely continuous and thus, also pseudomonotone. We denote by $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ the trace operator and by $\tau^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ its adjoint operator. The weak formulation of (2.5) is given by

$$u \in W^{1,p}(\Omega) : \ \langle -\Delta_p u + \widehat{B} u - i^*(1) - \tau^*(1), \varphi \rangle = 0, \ \forall \varphi \in W^{1,p}(\Omega), \tag{2.6}$$

meaning, for all $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \varsigma \int_\Omega |u|^{p-2} u \varphi dx - \int_\Omega \varphi dx - \int_{\partial\Omega} \varphi d\sigma = 0,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and its dual space $(W^{1,p}(\Omega))^*$. The negative $p$-Laplacian $-\Delta_p$ is pseudomonotone and, therefore, the sum $-\Delta_p + \widehat{B}$ is pseudomonotone. The coercivity of $-\Delta_p + \widehat{B}$ follows directly and thus, using classical existence results, implies the existence of a solution of problem (2.5). Let $e_1, e_2$ be solutions of (2.5) satisfying $e_1 \neq e_2$. Subtracting the corresponding weak formulation of (2.5) with respect to $e_1, e_2$ and taking $\varphi = e_1 - e_2$ yields

$$\begin{aligned} &\int_\Omega [|\nabla e_1|^{p-2} \nabla e_1 - |\nabla e_2|^{p-2} \nabla e_2] \nabla (e_1 - e_2) dx \\ &+ \varsigma \int_\Omega [|e_1|^{p-2} e_1 - |e_2|^{p-2} e_2](e_1 - e_2) dx = 0. \end{aligned}$$

As the left-hand side is strictly positive for $e_1 \neq e_2$, we obtain a contradiction, and thus, $e_1 = e_2$. Let $e$ be the unique solution of (2.5) in the weak sense.

Choosing the test function $\varphi = e^- = \max\{-e, 0\} \in W^{1,p}(\Omega)$ results in

$$-\int_{\{x\in\Omega:e(x)<0\}} |\nabla e|^p dx - \varsigma \int_{\{x\in\Omega:e(x)<0\}} |e|^p dx = \int_\Omega e^- dx + \int_{\partial\Omega} e^- d\sigma \geq 0,$$

which proves that $e$ is nonnegative. Notice that $e$ is not identically zero. Applying the Moser iteration (cf. [26], [43] or see the proof of Proposition 5.2) yields $e \in L^\infty(\Omega)$ and thus, the regularity results of Lieberman (see [45, Theorem 2]) ensure $e \in C^{1,\alpha}(\overline{\Omega})$. From (2.5) we conclude

$$\Delta_p e = \varsigma |e|^{p-2} e - 1 \leq \varsigma e^{p-1} \text{ a.e. in } \Omega.$$

Setting $\beta(s) = \varsigma s^{p-1}$ for $s > 0$ allows us to apply Vázguez's strong maximum principle (see [64]) which is possible since $\int_{0^+} \frac{1}{(s\beta(s))^{1/p}} ds = +\infty$. This shows that $e(x) > 0$ for almost all $x \in \Omega$. If there exists $x_0 \in \partial\Omega$ such that $e(x_0) = 0$, we obtain by applying again Vázguez's strong maximum principle that $\frac{\partial e}{\partial \nu}(x_0) < 0$, which is a contradiction since $|\nabla e|^{p-2} \frac{\partial e}{\partial \nu}(x_0) = 1$. Hence, $e(x) > 0$ in $\overline{\Omega}$ and therefore, we get $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

## 3. Notation and hypotheses

We impose the following conditions on the nonlinearities $f$ and $g$ in problem (1.1). The mappings $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (that is, measurable in the first argument and continuous in the second argument) such that

- (f1) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.
- (f2) $\displaystyle\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.
- (f3) $f$ is bounded on bounded sets.
- (f4) There exists $\delta_f > 0$ such that $\displaystyle\frac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.
- (g1) $\displaystyle\lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.
- (g2) $\displaystyle\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.
- (g3) $g$ is bounded on bounded sets.
- (g4) $g$ is locally Hölder continuous in $\partial\Omega \times \mathbb{R}$; that is,

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\big],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where $M_0$ is a positive constant and $\alpha \in (0, 1]$.

Note that the function $s \mapsto |s|^{p-2}s$ is locally Hölder continuous in $\mathbb{R}$. This implies in view of (g4) that the mapping $\Phi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x, s) := \lambda|s|^{p-2}s + g(x, s)$ is locally Hölder continuous in $\partial\Omega \times \mathbb{R}$. Recall that we write $g(x, u(x)) := g(x, \tau(u(x)))$ for $u \in W^{1,p}(\Omega)$, where $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ stands for the trace operator. With a view to the conditions (f1) and (g1), we see at once that $f(x, 0) = g(x, 0) = 0$, and thus, $u = 0$ is a trivial solution of problem (1.1).

**Corollary 3.1.** *Let* (f1),(f3) *and* (g1),(g3) *be satisfied. Then, for each* $a > 0$, *there exist constants* $b_1, b_2 > 0$ *such that*

$$\begin{aligned} &|f(x, s)| \le b_1|s|^{p-1}, \ \forall s : 0 \le |s| \le a, \\ &|g(x, s)| \le b_2|s|^{p-1}, \ \forall s : 0 \le |s| \le a. \end{aligned} \tag{3.1}$$

**Proof.** The assumption (f1) implies that for each $c_1 > 0$ there exists $\delta > 0$ such that

$$|f(x, s)| \le c_1|s|^{p-1}, \ \forall s : 0 \le |s| \le \delta. \tag{3.2}$$

Due to condition (f3), there exists a constant $c_2 > 0$ such that, for a given $a > 0$,

$$|f(x, s)| \le c_2, \ \forall s : 0 \le |s| \le a. \tag{3.3}$$

If $\delta > a$, then inequality (3.2), in particular, implies

$$|f(x, s)| \le b_1|s|^{p-1}, \ \forall s : 0 \le |s| \le a,$$

where $b_1 := c_1$. Let us assume $\delta < a$. From (3.3) we obtain

$$|f(x, s)| \le \frac{c_2}{\delta^{p-1}}|s|^{p-1}, \ \forall s : \delta \le |s| \le a, \tag{3.4}$$

and thus, combining (3.2) and (3.4) yields

$$|f(x, s)| \le (c_1 + \frac{c_2}{\delta^{p-1}})|s|^{p-1}, \ \forall s : 0 \le |s| \le a,$$

where setting $b_1 := c_1 + \frac{c_2}{\delta^{p-1}}$ proves (3.1). In the same way, one shows the assertion for $g$. □

**Example 3.2.** Consider the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,s) = \begin{cases} |s|^{p-2}s(1+(s+1)e^{-s}) & \text{if } \ s \le -1 \\ \operatorname{sgn}(s)\dfrac{|s|^p}{2}(|(s-1)\cos(s+1)|+s+1) & \text{if } \ -1 \le s \le 1 \\ s^{p-1}e^{1-s} - |x|(s-1)s^{p-1}e^s & \text{if } s \ge 1, \end{cases}$$

and

$$g(x,s) = \begin{cases} |s|^{p-2}s(s+1+e^{s+1}) & \text{if } \ s \le -1 \\ |s|^{p-1}se^{(s^2+1)\sqrt{|x|}} & \text{if } \ -1 \le s \le 1 \\ s^{p-1}(\cos(1-s)+(1-s)e^s) & \text{if } s \ge 1. \end{cases}$$

One verifies that all assumptions (f1)-(f4) and (g1)-(g4) are satisfied.

The definition of a solution of problem (1.1) in the weak sense is defined as follows.

**Definition 3.3.** *A function $u \in W^{1,p}(\Omega)$ is called a solution of* (1.1) *if the following holds:*

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx \\ &= \int_\Omega (f(x,u) - |u|^{p-2}u)\varphi dx + \int_{\partial\Omega} (\lambda |u|^{p-2}u + g(x,u))\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \end{aligned}$$

Next, we recall the notations of sub- and supersolutions of problem (1.1).

**Definition 3.4.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of* (1.1) *if the following holds:*

$$\begin{aligned} &\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla \varphi dx \\ &\le \int_\Omega (f(x,\underline{u}) - |\underline{u}|^{p-2}\underline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\underline{u}|^{p-2}\underline{u} + g(x,\underline{u}))\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega)_+. \end{aligned}$$

**Definition 3.5.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of* (1.1) *if the following holds:*

$$\begin{aligned} &\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla \varphi dx \\ &\ge \int_\Omega (f(x,\overline{u}) - |\overline{u}|^{p-2}\overline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\overline{u}|^{p-2}\overline{u} + g(x,\overline{u}))\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega)_+. \end{aligned}$$

Here, $W^{1,p}(\Omega)_+ := \{\varphi \in W^{1,p}(\Omega) : \varphi \geq 0\}$ stands for all nonnegative functions of $W^{1,p}(\Omega)$. Recall that if $u \in W^{1,p}(\Omega)$ satisfies $v \leq u \leq w$, where $v, w$ are some functions in $W^{1,p}(\Omega)$, then $\tau(v) \leq \tau(u) \leq \tau(w)$, where $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

## 4. Extremal constant-sign solutions

We start by generating two ordered pairs of sub- and supersolutions of problem (1.1) having constant signs. Here and in the following we denote by $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ the first eigenfunction of the Steklov eigenvalue problem (2.1) corresponding to the first eigenvalue $\lambda_1$.

**Lemma 4.1.** *Assume* (f1)–(f4), (g1)–(g4) *and* $\lambda > \lambda_1$ *and let* $e$ *be the unique solution of problem* (2.5). *Then there exists a constant* $\vartheta > 0$ *such that* $\vartheta e$ *and* $-\vartheta e$ *are supersolution and subsolution, respectively, of problem* (1.1). *In addition,* $\varepsilon\varphi_1$ *is a subsolution and* $-\varepsilon\varphi_1$ *is a supersolution of problem* (1.1) *provided the number* $\varepsilon > 0$ *is sufficiently small.*

**Proof.** Let $\underline{u} = \varepsilon\varphi_1$, where $\varepsilon$ is a positive constant specified later. In view of the Steklov eigenvalue problem (2.1) we obtain,

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\ &= -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi dx + \int_{\partial\Omega} \lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega).\end{aligned} \tag{4.1}$$

We are going to prove that Definition 3.4 is satisfied for $\underline{u} = \varepsilon\varphi_1$ meaning that the inequality

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\ &\leq \int_\Omega (f(x,\varepsilon\varphi_1) - (\varepsilon\varphi_1)^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda(\varepsilon\varphi_1)^{p-1} + g(x,\varepsilon\varphi_1))\varphi d\sigma\end{aligned} \tag{4.2}$$

is valid for all $\varphi \in W^{1,p}(\Omega)_+$. Therefore, (4.2) is fulfilled provided the following holds true, for all $\varphi \in W^{1,p}(\Omega)_+$:

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx + \int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma \leq 0.$$

Condition (f4) implies, for $\varepsilon \in (0, \delta_f/\|\varphi_1\|_\infty]$,

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx = \int_\Omega -\frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi dx \leq 0,$$

where $\|\cdot\|_\infty$ stands for the supremum norm. Due to assumption (g1) there exists a number $\delta_\lambda > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} < \lambda - \lambda_1 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \le \delta_\lambda.$$

If $\varepsilon \in (0, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}]$, we get

$$\begin{aligned}
&\int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - g(x, \varepsilon\varphi_1))\varphi d\sigma \\
&\le \int_{\partial\Omega} \left(\lambda_1 - \lambda + \frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma \\
&< \int_{\partial\Omega} (\lambda_1 - \lambda + \lambda - \lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma = 0.
\end{aligned}$$

Choosing $0 < \varepsilon \le \min\{\delta_f/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$ proves that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution. In a similar way one proves that $\overline{u} = -\varepsilon\varphi_1$ is a negative supersolution.

Let $\overline{u} = \vartheta e$, where $\vartheta$ is a positive constant specified later. From the auxiliary problem (2.5) we conclude

$$\begin{aligned}
&\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx \\
&= -\varsigma \int_\Omega (\vartheta e)^{p-1}\varphi dx + \int_\Omega \vartheta^{p-1}\varphi dx + \int_{\partial\Omega} \vartheta^{p-1}\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega).
\end{aligned} \tag{4.3}$$

In order to fulfill the assertion of the lemma, we have to show the validity of Definition 3.5 for $\overline{u} = \vartheta e$, meaning that, for all $\varphi \in W^{1,p}(\Omega)_+$,

$$\begin{aligned}
&\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx \\
&\ge \int_\Omega (f(x,\vartheta e) - (\vartheta e)^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda(\vartheta e)^{p-1} + g(x,\vartheta e))\varphi d\sigma.
\end{aligned} \tag{4.4}$$

With a view to (4.3) we see at once that inequality (4.4) is satisfied if the following holds:

$$\begin{aligned}
&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x,\vartheta e))\varphi dx \\
&\qquad + \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x,\vartheta e))\varphi d\sigma \ge 0,
\end{aligned} \tag{4.5}$$

where $\widetilde{c} = \varsigma - 1$ with $\widetilde{c} > 0$. By (f2) there exists $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -\widetilde{c}, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma,$$

and by (f3) we have

$$|-f(x,s) - \widetilde{c}s^{p-1}| \leq |f(x,s)| + \widetilde{c}s^{p-1} \leq c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\varsigma].$$

Thus, we get

$$f(x,s) \leq -\widetilde{c}s^{p-1} + c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{4.6}$$

Applying (4.6) to the first integral in (4.5) yields

$$\begin{aligned}
&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x, \vartheta e))\varphi dx \\
&\geq \int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} + \widetilde{c}(\vartheta e)^{p-1} - c_\varsigma)\varphi dx = \int_\Omega (\vartheta^{p-1} - c_\varsigma)\varphi dx,
\end{aligned}$$

which shows that for $\vartheta \geq c_\varsigma^{\frac{1}{p-1}}$ the integral is nonnegative. Due to hypothesis (g2) there is $s_\lambda > 0$ such that

$$\frac{g(x,s)}{s^{p-1}} < -\lambda, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s > s_\lambda.$$

Assumption (g3) ensures the existence of a constant $c_\lambda > 0$ such that

$$|-g(x,s) - \lambda s^{p-1}| \leq |g(x,s)| + \lambda s^{p-1} \leq c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \in [0, s_\lambda].$$

We obtain

$$g(x,s) \leq -\lambda s^{p-1} + c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{4.7}$$

Using (4.7) in the second integral in (4.5) provides

$$\begin{aligned}
&\int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x, \vartheta e))\varphi d\sigma \\
&\geq \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} + \lambda(\vartheta e)^{p-1} - c_\lambda)\varphi d\sigma \geq \int_{\partial\Omega} (\vartheta^{p-1} - c_\lambda)\varphi d\sigma.
\end{aligned}$$

Choosing $\vartheta := \max\{c_\varsigma^{\frac{1}{p-1}}, c_\lambda^{\frac{1}{p-1}}\}$ proves that both integrals in (4.5) are nonnegative, and thus, $\overline{u} = \vartheta e$ is a positive supersolution of problem (1.1). In order to prove that $\underline{u} = -\vartheta e$ is a negative subsolution we make use of the following estimates:

$$\begin{aligned}
f(x,s) &\geq -\widetilde{c}s^{p-1} - c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \leq 0, \\
g(x,s) &\geq -\lambda s^{p-1} - c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \leq 0,
\end{aligned} \tag{4.8}$$

which can be derived as stated above. With the aid of (4.8) one verifies that $\underline{u} = -\vartheta e$ is a negative subsolution of problem (1.1). $\square$

According to Lemma 4.1 we obtain a positive pair $[\varepsilon\varphi_1, \vartheta e]$ and a negative pair $[-\vartheta e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (1.1) assuming $\varepsilon > 0$ is sufficiently small.

The next lemma will prove the $C^{1,\alpha}$ regularity of solutions of problem (1.1) lying in the order interval $[0, \vartheta e]$ and $[-\vartheta e, 0]$, respectively. Note that $\underline{u} = \overline{u} = 0$ is both a subsolution and a supersolution due to the assumptions (f1) and (g1). In the following proof we make use of the regularity results of Lieberman (see [45]) and Vázguez in [64]. To obtain regularity results, in particular for elliptic Neumann problems, we refer also to the papers of Tolksdorf in [60] and DiBenedetto in [20].

**Lemma 4.2.** *Let the conditions* (f1)–(f4) *and* (g1)–(g4) *be satisfied and let* $\lambda > \lambda_1$. *If* $u \in [0, \vartheta e]$ *(respectively,* $u \in [-\vartheta e, 0]$*) is a solution of problem* (1.1) *satisfying* $u \not\equiv 0$ *in* $\Omega$*, then* $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *(respectively,* $u \in -\operatorname{int}(C^1(\overline{\Omega})_+)$*).*

**Proof.** Let $u$ be a solution of (1.1) such that $0 \le u \le \vartheta e$. Then it follows that $u \in L^\infty(\Omega)$, and thus, $u \in C^{1,\alpha}(\overline{\Omega})$ by Lieberman [45, Theorem 2] (see also Fan [30]). The conditions (f1), (f3), (g1) and (g3) (cf. Corollary 3.1) imply the existence of constants $c_f, c_g > 0$ such that

$$\begin{aligned} |f(x,s)| &\le c_f s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \le s \le \vartheta \|e\|_\infty, \\ |g(x,s)| &\le c_g s^{p-1} \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 \le s \le \vartheta \|e\|_\infty. \end{aligned} \tag{4.9}$$

Applying the first line in (4.9) along with (1.1) yields $\Delta_p u \le \widetilde{c} u^{p-1}$ almost everywhere in $\Omega$, where $\widetilde{c}$ is a positive constant. This allows us to apply Vázguez's strong maximum principle (see [64, Theorem 5]). We take $\beta(s) = \widetilde{c} s^{p-1}$ for all $s > 0$ which is possible because $\int_{0^+} \frac{1}{(s\beta(s))^{\frac{1}{p}}} ds = +\infty$. We get $u > 0$ in $\Omega$. Let us assume there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$. By applying again the maximum principle we obtain $\frac{\partial u}{\partial \nu}(x_0) < 0$. But taking into account $g(x_0, u(x_0)) = g(x_0, 0) = 0$ along with the Neumann condition in (1.1) yields $\frac{\partial u}{\partial \nu}(x_0) = 0$, which is a contradiction. Thus, $u > 0$ in $\overline{\Omega}$ which proves $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. The proof in case $u \in [-\vartheta e, 0]$ can be shown in an analogous manner. $\square$

The result of the existence of extremal constant-sign solutions is read as follows.

**Theorem 4.3.** *Assume* (f1)–(f4) *and* (g1)–(g4). *Then for every* $\lambda > \lambda_1$ *there exists a smallest positive solution* $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *in the order interval* $[0, \vartheta e]$ *and a greatest negative solution* $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ *in the order interval* $[-\vartheta e, 0]$ *with* $\vartheta > 0$ *stated in Lemma* 4.1.

**Proof.** We fix $\lambda > \lambda_1$. On the basis of Lemma 4.1, there exists an ordered pair of a positive supersolution $\overline{u} = \vartheta e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a positive subsolution $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) assuming $\varepsilon > 0$ is sufficiently small such that $\varepsilon\varphi_1 \le \vartheta e$. The method of sub- and supersolution (see [9]) with respect to the order interval $[\varepsilon\varphi_1, \vartheta e]$ implies the existence of a smallest positive solution $u_\varepsilon = u_\varepsilon(\lambda)$ of problem (1.1) satisfying $\varepsilon\varphi_1 \le u_\varepsilon \le \vartheta e$ which ensures $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Lemma 4.2). Hence, for every positive integer $n$ sufficiently large there exists a smallest solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) in the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$, and therefore, we have

$$u_n \downarrow u_+ \text{ for a.a. } x \in \Omega, \tag{4.10}$$

where $u_+ : \Omega \to \mathbb{R}$ is some function satisfying $0 \le u_+ \le \vartheta e$. We are going to show that $u_+$ is a solution of problem (1.1). Since $u_n$ belongs to the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$, it follows that $u_n$ is bounded in $L^p(\Omega)$. Moreover, we obtain the boundedness of $u_n$ in $L^p(\partial\Omega)$ because $\tau(u_n) \le \tau(\vartheta e)$. As $u_n$ solves (1.1) in the weak sense, one has by setting $\varphi = u_n$ in Definition 3.3

$$\begin{aligned}
\|\nabla u_n\|^p_{L^p(\Omega)} \le & \int_\Omega |f(x,u_n)| u_n dx + \|u_n\|^p_{L^p(\Omega)} \\
& + \lambda \|u_n\|^p_{L^p(\partial\Omega)} + \int_\Omega |g(x,u_n)| u_n d\sigma \\
\le & \|u_n\|^p_{L^p(\Omega)} + a_1 \|u_n\|_{L^p(\Omega)} + \lambda \|u_n\|^p_{L^p(\partial\Omega)} + a_2 \|u_n\|_{L^p(\partial\Omega)} \le a_3,
\end{aligned}$$

where $a_i, i = 1, \dots, 3$ are some positive constants independent of $n$. Thus, $u_n$ is bounded in $W^{1,p}(\Omega)$. The reflexivity of $W^{1,p}(\Omega), 1 < p < \infty$, ensures the existence of a weak convergent subsequence of $u_n$. Due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotonicity of $u_n$ and the compactness of the trace operator $\tau$, we get for the entire sequence $u_n$

$$\begin{aligned}
& u_n \rightharpoonup u_+ \text{ in } W^{1,p}(\Omega), \\
& u_n \to u_+ \text{ in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\
& u_n \to u_+ \text{ in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega.
\end{aligned} \tag{4.11}$$

Due to the fact that $u_n$ solves problem (1.1), one has for all $\varphi \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx \tag{4.12}$$

$$= \int_\Omega (f(x,u_n) - u_n^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))\varphi d\sigma.$$

The choice $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ is admissible in equation (4.12) which implies

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx \tag{4.13}$$
$$= \int_\Omega (f(x,u_n) - u_n^{p-1})(u_n - u_+)dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))(u_n - u_+)d\sigma.$$

Applying (4.11) and the conditions (f3), (g3) results in

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx \le 0, \tag{4.14}$$

which ensures by the $S_+$−property of $-\Delta_p$ on $W^{1,p}(\Omega)$ combined with (4.11)

$$u_n \to u_+ \text{ in } W^{1,p}(\Omega). \tag{4.15}$$

Taking into account the uniform boundedness of the sequence $(u_n)$ in combination with the strong convergence in (4.15) and the assumptions (f3) and (g3) allows us to pass to the limit in (4.12) which proves that $u_+$ is a solution of problem (1.1).

As $u_+$ is a solution of (1.1) belonging to $[0, \vartheta e]$, we can use Lemma 4.2 provided $u_+ \not\equiv 0$. We argue by contradiction and assume that $u_+ \equiv 0$ which in view of (4.10) results in

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{4.16}$$

We set

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n. \tag{4.17}$$

Obviously, the sequence $(\widetilde{u}_n)$ is bounded in $W^{1,p}(\Omega)$ which implies the existence of a weakly convergent subsequence of $\widetilde{u}_n$, not relabeled, such that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } W^{1,p}(\Omega), \\ &\widetilde{u}_n \to \widetilde{u} \quad \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\ &\widetilde{u}_n \to \widetilde{u} \quad \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega, \end{aligned} \tag{4.18}$$

where $\widetilde{u} : \Omega \to \mathbb{R}$ is some function belonging to $W^{1,p}(\Omega)$. Moreover, we may suppose there are functions $z_1 \in L^p(\Omega)_+, z_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} &|\widetilde{u}_n(x)| \le z_1(x) \quad \text{for a.a. all } x \in \Omega, \\ &|\widetilde{u}_n(x)| \le z_2(x) \quad \text{for a.a. all } x \in \partial\Omega. \end{aligned} \tag{4.19}$$

By means of (4.12), we get for $\widetilde{u}_n$ the following variational equation:

$$\int_\Omega |\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n \nabla \varphi dx = \int_\Omega \left( \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1} \right) \varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1} \varphi d\sigma + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} \varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \tag{4.20}$$

Choosing $\varphi = \widetilde{u}_n - \widetilde{u} \in W^{1,p}(\Omega)$ in the last equality, we obtain

$$\int_\Omega |\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n \nabla (\widetilde{u}_n - \widetilde{u}) dx = \int_\Omega \left( \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1} \right) (\widetilde{u}_n - \widetilde{u}) dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma. \tag{4.21}$$

Using (4.9) along with (4.19) implies

$$\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \le c_f z_1(x)^{p-1} (z_1(x) + |\widetilde{u}(x)|), \tag{4.22}$$

respectively,

$$\frac{|g(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \le c_g z_2(x)^{p-1} (z_2(x) + |\widetilde{u}(x)|). \tag{4.23}$$

The right-hand sides of (4.22) and (4.23) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's dominated convergence theorem. This fact and the convergence properties in (4.18) show

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) dx = 0, \tag{4.24}$$
$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma = 0.$$

From (4.18), (4.21), (4.24) we conclude

$$\limsup_{n\to\infty} \int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla (\widetilde{u}_n - \widetilde{u}) dx = 0. \tag{4.25}$$

Taking into account the $S_+$−property of $-\Delta_p$ with respect to $W^{1,p}(\Omega)$, we have

$$\widetilde{u}_n \to \widetilde{u} \ \text{ in } W^{1,p}(\Omega). \tag{4.26}$$

Notice that $\|\widetilde{u}\|_{W^{1,p}(\Omega)} = 1$. The statements in (4.16), (4.26) and (4.20) yield along with the conditions (f1), (g1), for all $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \widetilde{u}|^{p-2} \nabla \widetilde{u} \nabla \varphi dx = - \int_\Omega \widetilde{u}^{p-1} \varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}^{p-1} \varphi d\sigma. \tag{4.27}$$

Due to $\widetilde{u} \not\equiv 0$, the equation (4.27) is the Steklov eigenvalue problem in (2.1), where $\widetilde{u} \geq 0$ is the eigenfunction corresponding to the eigenvalue $\lambda > \lambda_1$. The fact that $\widetilde{u} \geq 0$ is nonnegative in $\overline{\Omega}$ yields a contradiction to the results of Martínez et al. in [47, Lemma 2.4] because $\widetilde{u}$ must change sign on $\partial\Omega$. Thus, $u_+ \not\equiv 0$ and we obtain by applying Lemma 4.2 that $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Now we need to show that $u_+$ is the smallest positive solution of (1.1) within $[0, \vartheta e]$. Let $u \in W^{1,p}(\Omega)$ be a positive solution of (1.1) lying in the order interval $[0, \vartheta e]$. Lemma 4.2 implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Then there exists an integer $n$ sufficiently large such that $u \in [\frac{1}{n}\varphi_1, \vartheta e]$. Since $u_n$ is the smallest solution of (1.1) in $[\frac{1}{n}\varphi_1, \vartheta e]$, one gets $u_n \leq u$. This yields by passing to the limit $u_+ \leq u$. Hence, $u_+$ must be the smallest positive solution of (1.1). In a similar way one proves the existence of the greatest negative solution of (1.1) within $[-\vartheta e, 0]$. This completes the proof of the theorem. □

## 5. Variational characterization of extremal solutions

Theorem 4.3 implies the existence of extremal positive and negative solutions of (1.1) for all $\lambda > \lambda_1$ denoted by $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. Now, we introduce truncation functions $T_+, T_-, T_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $T_+^{\partial\Omega}, T_-^{\partial\Omega}, T_0^{\partial\Omega} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ as follows:

$$T_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}, \quad T_+^{\partial\Omega}(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}$$

$$T_-(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \geq 0 \end{cases}, \quad T_-^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \geq 0 \end{cases}$$

$$T_0(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}, \quad T_0^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x). \end{cases}$$

For $u \in W^{1,p}(\Omega)$ the truncation operators on $\partial\Omega$ apply to the corresponding traces $\tau(u)$. We just write for simplification $T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u)$

without $\tau$. Furthermore, the truncation operators are continuous and uniformly bounded on $\mathbb{R}$ and they are Lipschitz continuous with respect to the second argument (see, e.g. [40]). By means of these truncations, we define the following associated functionals given by

$$\begin{aligned} E_+(u) =& \frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_+(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda T_+^{\partial\Omega}(x,s)^{p-1} + g(x, T_+^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.1}$$

$$\begin{aligned} E_-(u) =& \frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_-(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_-^{\partial\Omega}(x,s)|^{p-2} T_-^{\partial\Omega}(x,s) + g(x, T_-^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.2}$$

$$\begin{aligned} E_0(u) =& \frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2} T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.3}$$

which are well defined and belong to $C^1(W^{1,p}(\Omega))$. Due to the truncations, one can easily show that these functionals are coercive and weakly lower semicontinuous which implies that their global minimizers exist.

**Lemma 5.1.** *Let $u_+$ and $u_-$ be the extremal constant-sign solutions of* (1.1). *Then the following holds:*

(i) *A critical point $v \in W^{1,p}(\Omega)$ of $E_+$ is a (nonnegative) solution of* (1.1) *satisfying $0 \le v \le u_+$.*
(ii) *A critical point $v \in W^{1,p}(\Omega)$ of $E_-$ is a (nonpositive) solution of* (1.1) *satisfying $u_- \le v \le 0$.*
(iii) *A critical point $v \in W^{1,p}(\Omega)$ of $E_0$ is a solution of* (1.1) *satisfying $u_- \le v \le u_+$.*

**Proof.** Let $v$ be a critical point of $E_+$; that is, $E'_+(v) = 0$. In view of (5.1) we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi dx =& \int_\Omega [f(x, T_+(x,v)) - |v|^{p-2}v]\varphi dx \\ & + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x,v)^{p-1} + g(x, T_+^{\partial\Omega}(x,v))]\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{5.4}$$

Since $u_+$ is a positive solution of (1.1) we have by using Definition 3.3

$$\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \nabla\varphi dx = \int_\Omega [f(x,u_+) - u_+^{p-1}]\varphi dx + \int_{\partial\Omega} [\lambda u_+^{p-1} + g(x,u_+)]\varphi d\sigma,\ \forall \varphi \in W^{1,p}(\Omega). \tag{5.5}$$

Choosing $\varphi = (v-u_+)^+ \in W^{1,p}(\Omega)$ in (5.5) and (5.4) and subtracting (5.5) from (5.4) results in

$$\begin{aligned}
&\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v-u_+)^+ dx \\
&\qquad + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx \\
&= \int_\Omega [f(x,T_+(x,v)) - f(x,u_+)](v-u_+)^+ dx \\
&\quad + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x,v)^{p-1} - \lambda u_+^{p-1} + g(x,T_+^{\partial\Omega}(x,v)) - g(x,u_+)](v-u_+)^+ d\sigma \\
&= 0,
\end{aligned}$$

by the definition of $T_+$ and $T_+^{\partial\Omega}$, respectively. We obtain for $v > u_+$

$$\begin{aligned}
0 = &\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v-u_+)^+ dx \\
&+ \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx > 0,
\end{aligned} \tag{5.6}$$

which is a contradiction. This implies $(v-u_+)^+ = 0$, and thus, $v \le u_+$. Taking $\varphi = v^- = \max(-v,0)$ in (5.4) yields

$$\int_{\{x:v(x)<0\}} |\nabla v|^p dx + \int_{\{x:v(x)<0\}} |v|^p dx = 0,$$

consequently, $\|v^-\|^p_{W^{1,p}(\Omega)} = 0$ and equivalently $v^- = 0$; that is, $v \ge 0$. By the definition of the truncation operator we see at once that $T_+(x,v) = v, T_+^{\partial\Omega}(x,v) = v$ and therefore, $v$ is a solution of (1.1) satisfying $0 \le v \le u_+$. The statements in (ii) and (iii) can be shown in a similar way. □

The next result matches $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$-local minimizers for a large class of $C^1-$ functionals. We will show that every local $C^1-$minimizer of $E_0$ is a local $W^{1,p}(\Omega)$-minimizer of $E_0$. This result was first proven for the Dirichlet problem by Brezis and Nirenberg [8] when $p = 2$ and was extended by García Azorero et al in [37] for $p \ne 2$ (see also [39] when $p > 2$). For the zero Neumann problem we refer to the recent results of Motreanu et al. in [51] for $1 < p < \infty$. In the case of nonsmooth functionals the authors in [53] and [7] proved the same result for the Dirichlet problem and the zero Neumann problem when $p \geq 2$. We give the proof for the nonlinear nonzero Neumann problem for any $1 < p < \infty$.

**Proposition 5.2.** *If $z_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})-$minimizer of $E_0$, meaning that there exists $r_1 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq r_1,$$

*then $z_0$ is a local minimizer of $E_0$ in $W^{1,p}(\Omega)$, meaning that there exists $r_2 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{W^{1,p}(\Omega)} \leq r_2.$$

**Proof.** Let $h \in C^1(\overline{\Omega})$. If $\beta > 0$ is small, we have

$$0 \leq \frac{E_0(z_0 + \beta h) - E_0(z_0)}{\beta},$$

meaning that the directional derivative of $E_0$ at $z_0$ in direction $h$ satisfies

$$0 \leq E_0'(z_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$

We recall that $h \mapsto E_0'(z_0; h)$ is continuous on $W^{1,p}(\Omega)$ and the density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ results in $0 \leq E_0'(z_0; h)$ for all $h \in W^{1,p}(\Omega)$. Therefore, setting $-h$ instead of $h$, we get $0 = E_0'(z_0)$, which yields

$$\begin{aligned} 0 = &\int_\Omega |\nabla z_0|^{p-2} \nabla z_0 \nabla \varphi dx - \int_\Omega (f(x, z_0) - |z_0|^{p-2} z_0) \varphi dx \\ &- \int_{\partial\Omega} \lambda |z_0|^{p-2} z_0 \varphi d\sigma - \int_{\partial\Omega} g(x, z_0) \varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{5.7}$$

By means of Lemma 5.1, we obtain $u_- \leq z_0 \leq u_+$, and thus, $z_0 \in L^\infty(\Omega)$. As before, via the regularity results of Lieberman [45] and Vázguez [64], it follows that $z_0 \in \operatorname{int}(C^1(\overline{\Omega}))$ (cf. Lemma 4.2). Let us assume the proposition is not valid. The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is weakly sequentially lower semicontinuous and the set $\overline{B}_\varepsilon = \{y \in W^{1,p}(\Omega) : \|y\|_{W^{1,p}(\Omega)} \leq \varepsilon\}$ is weakly compact in $W^{1,p}(\Omega)$. Thus, for any $\varepsilon > 0$ we can find $y_\varepsilon \in \overline{B}_\varepsilon$ such that

$$E_0(z_0 + y_\varepsilon) = \min\{E_0(z_0 + y) : y \in \overline{B}_\varepsilon)\} < E_0(z_0). \tag{5.8}$$

Obviously, $y_\varepsilon$ is a solution of the following minimum problem:

$$\begin{cases} \min E_0(z_0 + y) \\ y \in \overline{B}_\varepsilon, g_\varepsilon(y) := \frac{1}{p}(\|y\|^p_{W^{1,p}(\Omega)} - \varepsilon^p) \leq 0. \end{cases}$$

Applying the Lagrange multiplier rule (see, e.g., [46] or [14]) yields the existence of a multiplier $\lambda_\varepsilon > 0$ such that

$$E_0'(z_0 + y_\varepsilon) + \lambda_\varepsilon g_\varepsilon'(y_\varepsilon) = 0, \tag{5.9}$$

which results in

$$\begin{aligned}
&\int_\Omega |\nabla(z_0 + y_\varepsilon)|^{p-2}\nabla(z_0 + y_\varepsilon)\nabla\varphi dx \\
&- \int_\Omega (f(x, T_0(x, z_0 + y_\varepsilon)) - |z_0 + y_\varepsilon|^{p-2}(z_0 + y_\varepsilon))\varphi dx \\
&- \int_{\partial\Omega} (\lambda|T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)|^{p-2}T_0^{\partial\Omega}(x, z_0 + y_\varepsilon) + g(x, T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)))\varphi d\sigma \\
&+ \lambda_\varepsilon \int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |y_\varepsilon|^{p-2} y_\varepsilon \varphi dx = 0,
\end{aligned} \tag{5.10}$$

for all $\varphi \in W^{1,p}(\Omega)$. Notice that $\lambda_\varepsilon$ cannot be zero since the constraints guarantee that $y_\varepsilon$ belongs to $\overline{B}_\varepsilon$. Let $0 < \lambda_\varepsilon \le 1$ for all $\varepsilon \in (0, 1]$. We multiply (5.7) by $\lambda_\varepsilon$, set $v_\varepsilon = z_0 + y_\varepsilon$ in (5.10), and add these new equations. One obtains

$$\begin{aligned}
&\int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla\varphi dx \\
&+ \lambda_\varepsilon \int_\Omega |\nabla(v_\varepsilon - z_0)|^{p-2}\nabla(v_\varepsilon - z_0)\nabla\varphi dx \\
&= \int_\Omega (\lambda_\varepsilon f(x, z_0) + f(x, T_0(x, v_\varepsilon)))\varphi dx \\
&\quad - \int_\Omega (\lambda_\varepsilon |z_0|^{p-2} z_0 + |v_\varepsilon|^{p-2} v_\varepsilon + \lambda_\varepsilon |v_\varepsilon - z_0|^{p-2}(v_\varepsilon - z_0))\varphi dx \\
&\quad + \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon)|^{p-2} T_0^{\partial\Omega}(x, v_\varepsilon))\varphi d\sigma \\
&\quad + \int_{\partial\Omega} (\lambda_\varepsilon g(x, z_0) + g(x, T_0^{\partial\Omega}(x, v_\varepsilon)))\varphi d\sigma.
\end{aligned} \tag{5.11}$$

Now, we introduce the maps $\mathcal{A}_\varepsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\Phi_\varepsilon : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{A}_\varepsilon(x, \xi) &= |\xi|^{p-2}\xi + \lambda_\varepsilon |H|^{p-2} H + \lambda_\varepsilon |\xi - H|^{p-2}(\xi - H) \\
-\mathcal{B}_\varepsilon(x, \psi) &= \lambda_\varepsilon f(x, z_0) + f(x, T_0(x, \psi)) \\
&\qquad - (\lambda_\varepsilon |z_0|^{p-2} z_0 + |\psi|^{p-2}\psi + \lambda_\varepsilon |\psi - z_0|^{p-2}(\psi - z_0)) \\
\Phi_\varepsilon(x, \psi) &= \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, \psi)|^{p-2} T_0^{\partial\Omega}(x, \psi))
\end{aligned}$$

$$+ \lambda_\varepsilon g(x, z_0) + g(x, T_0^{\partial\Omega}(x, \psi)),$$

where $H(x) = \nabla z_0(x)$ and $H \in (C^\alpha(\overline{\Omega}))^N$ for some $\alpha \in (0, 1]$. Apparently, the operator $\mathcal{A}_\varepsilon(x, \xi)$ belongs to $C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$. For $x \in \Omega$ we have

$$\begin{aligned} &(\mathcal{A}_\varepsilon(x, \xi), \xi)_{\mathbb{R}^N} \\ &= \|\xi\|^p + \lambda_\varepsilon(|\xi - H|^{p-2}(\xi - H) - |-H|^{p-2}(-H), \xi - H - (-H))_{\mathbb{R}^N} \\ &\geq \|\xi\|^p \quad \text{for all } \xi \in \mathbb{R}^N, \end{aligned} \tag{5.12}$$

where $(\cdot, \cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. (5.12) shows that $\mathcal{A}_\varepsilon$ satisfies a strong ellipticity condition. Hence, the equation in (5.11) is the weak formulation of the elliptic Neumann problem

$$\begin{aligned} -\operatorname{div} \mathcal{A}_\varepsilon(x, \nabla v_\varepsilon) + \mathcal{B}_\varepsilon(x, v_\varepsilon) = 0 \qquad & \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial \nu} = \Phi_\varepsilon(x, v_\varepsilon) \qquad & \text{on } \partial\Omega, \end{aligned} \tag{5.13}$$

where $\frac{\partial v_\varepsilon}{\partial \nu}$ denotes the conormal derivative of $v_\varepsilon$. To prove the $L^\infty-$regularity of $v_\varepsilon$, we will use the Moser iteration technique (see e.g. [23],[24],[25], [26], [43]). It suffices to consider the proof in case $1 \leq p \leq N$, otherwise we would be done. First we are going to show that $v_\varepsilon^+ = \max\{v_\varepsilon, 0\}$ belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{v_\varepsilon^+(x), M\}$. Letting $K(t) = t$ if $t \leq M$ and $K(t) = M$ if $t > M$, it follows by [43, Theorem B.3] that $K \circ v_\varepsilon^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For real $k \geq 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1) v_M^{kp} \nabla v_M$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Notice that $v_\varepsilon(x) \leq 0$ implies directly $v_M(x) = 0$. Testing (5.11) with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned} &(kp+1) \int_\Omega |\nabla v_\varepsilon^+|^{p-2} \nabla v_\varepsilon^+ \nabla v_M v_M^{kp} dx + \int_\Omega |v_\varepsilon^+|^{p-2} v_\varepsilon^+ v_M^{kp+1} dx \\ &+ \lambda_\varepsilon (kp+1) \int_\Omega \Big[ |\nabla(v_\varepsilon^+ - z_0)|^{p-2} \nabla(v_\varepsilon^+ - z_0) - |-\nabla z_0|^{p-2}(-\nabla z_0) \Big] \\ &\quad \times (\nabla v_M - \nabla z_0 - (-\nabla z_0)) v_M^{kp} dx \\ &= \int_\Omega (\lambda_\varepsilon f(x, z_0) + f(x, T_0(x, v_\varepsilon^+))) v_M^{kp+1} dx \\ &\quad - \int_\Omega (\lambda_\varepsilon |z_0|^{p-2} z_0 + \lambda_\varepsilon |v_\varepsilon^+ - z_0|^{p-2}(v_\varepsilon^+ - z_0)) v_M^{kp+1} dx \\ &\quad + \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon^+)|^{p-2} T_0^{\partial\Omega}(x, v_\varepsilon^+)) v_M^{kp+1} d\sigma \end{aligned} \tag{5.14}$$

$$+\int_{\partial\Omega}(\lambda_\varepsilon g(x,z_0)+g(x,T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1}d\sigma.$$

Since $z_0\in[u_-,u_+]$, $\tau(z_0)\in[\tau(u_-),\tau(u_+)]$, $T_0(x,v_\varepsilon)\in[u_-,u_+]$ and $T_0^{\partial\Omega}(x,v_\varepsilon)\in[\tau(u_-),\tau(u_+)]$ we get for the right-hand side of (5.14) by using (f3) and (g3)

$$\begin{aligned}
&(1)\ \int_\Omega(\lambda_\varepsilon f(x,z_0)+f(x,T_0(x,v_\varepsilon^+)))v_M^{kp+1}dx\le e_1\int_\Omega(v_\varepsilon^+)^{kp+1}dx \qquad (5.15)\\
&(2)\ -\int_\Omega(\lambda_\varepsilon|z_0|^{p-2}z_0+\lambda_\varepsilon|v_\varepsilon^+-z_0|^{p-2}(v_\varepsilon^+-z_0))v_M^{kp+1}dx\\
&\qquad\le e_2\int_\Omega|v_\varepsilon^+|^{p-1}(v_\varepsilon^+)^{kp+1}dx+e_3\int_\Omega|z_0|^{p-1}(v_\varepsilon^+)^{kp+1}dx\\
&\qquad\le\int_\Omega e_2(v_\varepsilon^+)^{(k+1)p}dx+e_4\int_\Omega(v_\varepsilon^+)^{kp+1}dx\\
&(3)\ \int_{\partial\Omega}\lambda(\lambda_\varepsilon|z_0|^{p-2}z_0+|T_0^{\partial\Omega}(x,v_\varepsilon^+)|^{p-2}T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1}d\sigma\\
&\qquad\le e_5\int_{\partial\Omega}(v_\varepsilon^+)^{kp+1}d\sigma\\
&(4)\ \int_{\partial\Omega}(\lambda_\varepsilon g(x,z_0)+g(x,T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1}d\sigma\le e_6\int_{\partial\Omega}(v_\varepsilon^+)^{kp+1}d\sigma.
\end{aligned}$$

The left-hand-side of (5.14) can be estimated to obtain

$$\begin{aligned}
&(kp+1)\int_\Omega|\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+\nabla v_M v_M^{kp}dx+\int_\Omega|v_\varepsilon^+|^{p-2}v_\varepsilon^+v_M^{kp+1}dx \qquad (5.16)\\
&+\lambda_\varepsilon(kp+1)\int_\Omega\Big[|\nabla(v_\varepsilon^+-z_0)|^{p-2}\nabla(v_\varepsilon^+-z_0)-|-\nabla z_0|^{p-2}(-\nabla z_0)\Big]\\
&\quad\times(\nabla v_M-\nabla z_0-(-\nabla z_0))v_M^{kp}dx\\
&\ge(kp+1)\int_\Omega|\nabla v_M|^p v_M^{kp}dx+\int_\Omega(v_\varepsilon^+)^{p-1}v_M^{kp+1}dx\\
&\ge\frac{kp+1}{(k+1)^p}\Big[\int_\Omega|\nabla v_M^{k+1}|^p dx+\int_\Omega(v_\varepsilon^+)^{p-1}v_M^{kp+1}dx\Big].
\end{aligned}$$

Using the Hölder inequality we see at once

$$\int_\Omega 1\cdot(v_\varepsilon^+)^{kp+1}dx\le|\Omega|^{\frac{p-1}{(k+1)p}}\Big(\int_\Omega(v_\varepsilon^+)^{(k+1)p}dx\Big)^{\frac{kp+1}{(k+1)p}}, \qquad (5.17)$$

and analogously for the boundary integral

$$\int_{\partial\Omega} 1 \cdot (v_\varepsilon^+)^{kp+1} d\sigma \le |\partial\Omega|^{\frac{p-1}{(k+1)p}} \Big( \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma \Big)^{\frac{kp+1}{(k+1)p}}. \tag{5.18}$$

Applying the estimates (5.15)–(5.18) to (5.14) one gets

$$\begin{aligned} &\frac{kp+1}{(k+1)^p} \Big[ \int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega (v_\varepsilon^+)^{p-1} v_M^{kp+1} dx \Big] \\ &\le e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_7 \Big( \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx \Big)^{\frac{kp+1}{(k+1)p}} \\ &\qquad + e_8 \Big( \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma \Big)^{\frac{kp+1}{(k+1)p}}. \end{aligned}$$

We have $\lim_{M\to\infty} v_M(x) = v_\varepsilon^+(x)$ for almost all $x \in \Omega$ and can apply Fatou's lemma which results in

$$\begin{aligned} &\frac{kp+1}{(k+1)^p} \Big[ \int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx \Big] \\ &\le e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_7 \Big( \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx \Big)^{\frac{kp+1}{(k+1)p}} \\ &\qquad + e_8 \Big( \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma \Big)^{\frac{kp+1}{(k+1)p}}. \end{aligned} \tag{5.19}$$

We have either

$$\Big( \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx \Big)^{\frac{kp+1}{(k+1)p}} \le 1 \text{ or } \Big( \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx \Big)^{\frac{kp+1}{(k+1)p}} \le \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx,$$

respectively, either

$$\Big( \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma \Big)^{\frac{kp+1}{(k+1)p}} \le 1 \text{ or } \Big( \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma \Big)^{\frac{kp+1}{(k+1)p}} \le \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma.$$

Thus, we obtain from (5.19)

$$\begin{aligned} &\frac{kp+1}{(k+1)^p} \left[ \int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx \right] \\ &\le e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_{10} \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma + e_{11}. \end{aligned} \tag{5.20}$$

Next we want to estimate the boundary integral by an integral in the domain $\Omega$. To this end, we need the following continuous embeddings:

$$T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \tfrac{1}{p},$$

$$T_2 : B_{pp}^{s-\frac{1}{p}}(\partial\Omega) = F_{pp}^{s-\frac{1}{p}}(\partial\Omega) \to F_{p2}^{0}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \tfrac{1}{p},$$

where $\Omega$ is a bounded $C^\infty$-domain (see [58, page 75 and page 82], [62, 2.3.1 and 2.3.2] and [63, 3.3.1]). Let $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \le \iota < 1$. Then the embeddings are also valid if $\partial\Omega \in C^{m,1}$ ([59]). In [21, Satz 9.40] the proof is given for $p = 2$, however, it can be extended to $p \in (1,\infty)$ by using the Fourier transformation in $L^p(\Omega)$ ([22]).

Here $B_{pq}^s$ and $F_{pq}^s$ denote the Besov and Lizorkin-Triebel spaces, respectively, which are equal in case $p = q$ with $1 < p < \infty$ and $-\infty < s < \infty$. We set $s = \frac{1}{p} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is arbitrarily fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$. Thus the embeddings are valid for a Lipschitz boundary $\partial\Omega$. This yields the continuous embedding

$$T_3 : B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega) \to L^p(\partial\Omega). \tag{5.21}$$

The real interpolation theory implies

$$\big(F_{p2}^0(\Omega), F_{p2}^1(\Omega)\big)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \big(L^p(\Omega), W^{1,p}(\Omega)\big)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)$$

(for more details see [2], [62], [63]), which ensures the norm estimate

$$\|v\|_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)} \le e_{12}\|v\|_{W^{1,p}(\Omega)}^{\frac{1}{p}+\widetilde{\varepsilon}}\|v\|_{L^p(\Omega)}^{1-\frac{1}{p}-\widetilde{\varepsilon}}, \ \forall v \in W^{1,p}(\Omega) \tag{5.22}$$

with a positive constant $e_{12}$. Using (5.21), (5.22) and Young's inequality yields

$$\begin{aligned}
&\int_{\partial\Omega}((v_\varepsilon^+)^{k+1})^p d\sigma = \|(v_\varepsilon^+)^{k+1}\|_{L^p(\partial\Omega)}^p \le e_{13}^p\|(v_\varepsilon^+)^{k+1}\|_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)}^p \\
&\le e_{13}^p e_{12}^p \|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^{\left(\frac{1}{p}+\widetilde{\varepsilon}\right)p}\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^{\left(1-\frac{1}{p}-\widetilde{\varepsilon}\right)p} \\
&\le e_{13}^p e_{12}^p (\delta\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^{(1+\widetilde{\varepsilon}p)\widetilde{q}} + C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^{(p-1-\widetilde{\varepsilon}p)\widetilde{q}'}) \\
&= e_{13}^p e_{12}^p (\delta\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^p + C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^p),
\end{aligned} \tag{5.23}$$

where $\widetilde{q} = \frac{p}{1+\widetilde{\varepsilon}p}$ and $\widetilde{q}' = \frac{p}{p-1-\widetilde{\varepsilon}p}$ are chosen such that $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$ and $\delta$ is a free parameter specified later. Note that the positive constant $C(\delta)$ depends only on $\delta$. Applying (5.23) to (5.20) shows

$$\frac{kp+1}{(k+1)^p}\Big[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\Big]$$

$$\leq e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_{10} \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma + e_{11}$$
$$\leq e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + e_{14}\delta \|(v_\varepsilon^+)^{k+1}\|^p_{W^{1,p}(\Omega)} + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\Omega)} + e_{11},$$

where $e_{14} = e_{10}e_{13}^p e_{12}^p$ is a positive constant. We take $\delta = \frac{kp+1}{e_{14}2(k+1)^p}$ to get

$$\left(\frac{kp+1}{(k+1)^p} - e_{14}\frac{kp+1}{e_{14}2(k+1)^p}\right)\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p) dx\right]$$
$$\leq e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\Omega)} + e_{11}$$
$$\leq e_{15} \int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + e_{11},$$

and hence,

$$\|(v_\varepsilon^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \leq \frac{2(k+1)^p}{kp+1}\left[e_{15}\int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + e_{11}\right].$$

By Sobolev's embedding theorem a positive constant $e_{16}$ exists such that

$$\|(v_\varepsilon^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{16}\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)} \tag{5.24}$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We have

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} = \|(v_\varepsilon^+)^{k+1}\|^{\frac{1}{k+1}}_{L^{p^*}(\Omega)} \leq e_{16}^{\frac{1}{k+1}} \|(v_\varepsilon^+)^{k+1}\|^{\frac{1}{k+1}}_{W^{1,p}(\Omega)}$$
$$\leq e_{16}^{\frac{1}{k+1}} \left(\frac{2^{\frac{1}{p}}(k+1)}{(kp+1)^{1/p}}\right)^{\frac{1}{k+1}} \left[e_{15}\int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + e_{11}\right]^{\frac{1}{(k+1)p}}$$
$$\leq e_{16}^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}} \left(\frac{(k+1)}{(kp+1)^{1/p}}\right)^{\frac{1}{k+1}} \left[\int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + 1\right]^{\frac{1}{(k+1)p}},$$

where $e_{17} = 2\max\{e_{15}, e_{11}\}$.

Since $\left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{k+1}}} \geq 1$ and $\lim_{k\to\infty} \left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{18} > 1$ such that $\left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{k+1}} \leq e_{18}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{16}^{\frac{1}{k+1}} e_{18}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (v_\varepsilon^+)^{(k+1)p)} dx + 1\right]^{\frac{1}{(k+1)p}}. \tag{5.25}$$

Now, we will use the bootstrap arguments similarly as in the proof of [26, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get $\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \le c(k)$ for any finite number $k > 0$ which shows that $v_\varepsilon^+ \in L^r(\Omega)$ for any $r \in (1,\infty)$. To prove the uniform estimate with respect to $k$ we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k_n+1)p} dx \le 1,$$

we have immediately $\|v_\varepsilon^+\|_{L^\infty(\Omega)} \le 1$ (cf. the proof of [26, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx > 1$$

for any $k \ge k_0$. Then we conclude from (5.25)

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{16}^{\frac{1}{k+1}} e_{18}^{\frac{1}{\sqrt{k+1}}} e_{19}^{\frac{1}{(k+1)p}} \|v_\varepsilon^+\|_{L^{(k+1)p}}, \text{ for any } k \ge k_0, \tag{5.26}$$

where $e_{19} = 2e_{17}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$\|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{16}^{\frac{1}{k_1+1}} e_{18}^{\frac{1}{\sqrt{k_1+1}}} e_{19}^{\frac{1}{(k_1+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{5.27}$$

Next, we can choose $k_2$ in (5.26) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$\begin{aligned}\|v_\varepsilon^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{16}^{\frac{1}{k_2+1}} e_{18}^{\frac{1}{\sqrt{k_2+1}}} e_{19}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_2+1)p}(\Omega)} \\ &= e_{16}^{\frac{1}{k_2+1}} e_{18}^{\frac{1}{\sqrt{k_2+1}}} e_{19}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)}.\end{aligned} \tag{5.28}$$

By induction we obtain

$$\begin{aligned}\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} &\le e_{16}^{\frac{1}{k_n+1}} e_{18}^{\frac{1}{\sqrt{k_n+1}}} e_{19}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{16}^{\frac{1}{k_n+1}} e_{18}^{\frac{1}{\sqrt{k_n+1}}} e_{19}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)},\end{aligned} \tag{5.29}$$

where the sequence $(k_n)$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = (\frac{p^*}{p})^n$. Thus,

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} = e_{16}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{18}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{19}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)},$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{20} > 0$ such that

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \le e_{20} \|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{5.30}$$

Let us assume that $v_\varepsilon^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set $A$ of positive measure in $\Omega$ such that $v_\varepsilon^+(x) \geq e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \geq \left(\int_A |v_\varepsilon^+(x)|^{(k_n+1)p^*}\right)^{\frac{1}{(k_n+1)p^*}} \geq (e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta)|A|^{\frac{1}{(k_n+1)p^*}}.$$

Passing to the limits inferior in the above inequality yields

$$\liminf_{n\to\infty} \|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \geq e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (5.30), and hence, $v_\varepsilon^+ \in L^\infty(\Omega)$. In a similar way one shows that $v_\varepsilon^- = \max\{-v_\varepsilon, 0\} \in L^\infty(\Omega)$. This proves $v_\varepsilon = v_\varepsilon^+ - v_\varepsilon^- \in L^\infty(\Omega)$.

In order to show some structure properties of $\mathcal{A}_\varepsilon$ note that its derivative has the form

$$\begin{aligned} D_\xi \mathcal{A}_\varepsilon(x,\xi) =& |\xi|^{p-2}I + (p-2)|\xi|^{p-4}\xi\xi^T \\ &+ \lambda_\varepsilon|\xi - H|^{p-2}I + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H)(\xi - H)^T, \end{aligned} \tag{5.31}$$

where $I$ is the unit matrix and $\xi^T$ stands for the transpose of $\xi$. Using (5.31) implies

$$\|D_\xi \mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N} \leq a_1 + a_2|\xi|^{p-2}, \tag{5.32}$$

where $a_1, a_2$ are some positive constants. We also obtain

$$\begin{aligned} (D_\xi \mathcal{A}_\varepsilon(x,\xi)y, y)_{\mathbb{R}^N} &= |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi, y)^2_{\mathbb{R}^N} \\ &\quad + \lambda_\varepsilon|\xi - H|^{p-2}\|y\|^2_{\mathbb{R}^N} + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H, y)^2_{\mathbb{R}^N} \\ &\geq \begin{cases} |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } p \geq 2 \\ (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } 1 < p < 2 \end{cases} \\ &\geq \min\{1, p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}. \end{aligned} \tag{5.33}$$

For the case $1 < p < 2$ in (5.33) we have used the estimate $|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi, y)^2_{\mathbb{R}^N} \geq (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}$. Because of (5.32) and (5.33), the operators $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ and $\Phi_\varepsilon$ satisfy the assumptions (0.3a-d) and (0.6) of Lieberman in [45], and thus, Theorem 2 in [45] ensures the existence of $\alpha \in (0,1)$ and $M > 0$, both independent of $\varepsilon \in (0,1]$, such that

$$v_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M, \quad \text{for all } \varepsilon \in (0,1]. \tag{5.34}$$

Due to $y_\varepsilon = v_\varepsilon - z_0$ and the fact that $v_\varepsilon, z_0 \in C^{1,\alpha}(\overline{\Omega})$, one realizes immediately that $y_\varepsilon$ satisfies (5.34), too. Next, we assume $\lambda_\varepsilon > 1$ with $\varepsilon \in (0,1]$. Multiplying (5.7) by $-1$ and adding this new equation to (5.10) yields

$$\begin{aligned}
&\int_\Omega |\nabla(z_0+y_\varepsilon)|^{p-2}\nabla(z_0+y_\varepsilon)\nabla\varphi dx - \int_\Omega |\nabla z_0|^{p-2}\nabla z_0\nabla\varphi dx \qquad (5.35)\\
&+\lambda_\varepsilon \int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla\varphi dx\\
&= \int_\Omega (f(x,T_0(x,z_0+y_\varepsilon)) - f(x,z_0))\varphi dx\\
&\quad + \int_\Omega (|z_0|^{p-2}z_0 - |z_0+y_\varepsilon|^{p-2}(z_0+y_\varepsilon) - \lambda_\varepsilon |y_\varepsilon|^{p-2}y_\varepsilon)\varphi dx\\
&\quad + \int_{\partial\Omega} \lambda(|T_0^{\partial\Omega}(x,z_0+y_\varepsilon)|^{p-2}T_0^{\partial\Omega}(x,z_0+y_\varepsilon) - |z_0|^{p-2}z_0)\varphi d\sigma\\
&\quad + \int_{\partial\Omega} (g(x,T_0^{\partial\Omega}(x,z_0+y_\varepsilon)) - g(x,z_0))\varphi d\sigma dx.
\end{aligned}$$

Defining again

$$\begin{aligned}
\mathcal{A}_\varepsilon(x,\xi) &= \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}(H+\xi) - |H|^{p-2}H) + |\xi|^{p-2}\xi \qquad (5.36)\\
-\mathcal{B}_\varepsilon(x,\psi) &= f(x,T_0(x,z_0+\psi)) - f(x,z_0) + |z_0|^{p-2}z_0\\
&\quad - |z_0+\psi|^{p-2}(z_0+\psi) - \lambda_\varepsilon|\psi|^{p-2}y_\varepsilon\\
\Phi_\varepsilon(x,\psi) &= \lambda(|T_0^{\partial\Omega}(x,z_0+\psi)|^{p-2}T_0^{\partial\Omega}(x,z_0+\psi) - |z_0|^{p-2}z_0)\\
&\quad + g(x,T_0^{\partial\Omega}(x,z_0+\psi)) - g(x,z_0),
\end{aligned}$$

and rewriting (5.35) yields the Neumann equation

$$\begin{aligned}
-\operatorname{div}\mathcal{A}_\varepsilon(x,\nabla y_\varepsilon) + \frac{1}{\lambda_\varepsilon}\mathcal{B}_\varepsilon(x,y_\varepsilon) &= 0 && \text{in } \Omega\\
\frac{\partial v_\varepsilon}{\partial\nu} &= \frac{1}{\lambda_\varepsilon}\Phi_\varepsilon(x,y_\varepsilon) && \text{on } \partial\Omega,
\end{aligned} \qquad (5.37)$$

where $\frac{\partial v_\varepsilon}{\partial \nu}$ denotes the conormal derivative of $v_\varepsilon$. As above, we have the estimate

$$\begin{aligned}
(\mathcal{A}_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N} &= \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}(H+\xi) - |H|^{p-2}H, H+\xi-H)_{\mathbb{R}^N} + \|\xi\|^p\\
&\geq \|\xi\|^p \quad \text{for all } \xi \in \mathbb{R}^N, \qquad (5.38)
\end{aligned}$$

and can write the derivative $D_\xi \mathcal{A}_\varepsilon(x,\xi)$ as

$$D_\xi \mathcal{A}_\varepsilon(x,\xi) = \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}I + (p-2)|H+\xi|^{p-4}(H+\xi)(H+\xi)^T \\ |\xi|^{p-2}I + (p-2)|\xi|^{p-4}\xi\xi^T. \tag{5.39}$$

We have again the estimate

$$\|D_\xi \mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N} \le a_1 + a_2|\xi|^{p-2}, \tag{5.40}$$

where $a_1, a_2$ are some positive constants. One also gets

$$(D_\xi \mathcal{A}_\varepsilon(x,\xi)y, y)_{\mathbb{R}^N} \ge \min\{1, p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}. \tag{5.41}$$

As before, the nonlinear regularity theory implies the existence of $\alpha \in (0,1)$ and $M > 0$, both independent of $\varepsilon \in (0,1)$, such that (5.34) holds for $y_\varepsilon$.

Let $\varepsilon \downarrow 0$. Using the compact embedding $C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ (cf. [42, page 38] or [1, page 11]), we may assume for a subsequence $y_\varepsilon \to \widetilde{y}$ in $C^1(\overline{\Omega})$. By construction we have $y_\varepsilon \to 0$ in $W^{1,p}(\Omega)$, and thus, $\widetilde{y} = 0$ which implies for a subsequence $\|y_\varepsilon\|_{C^1(\overline{\Omega})} \le r_1$. Hence, one has $E_0(z_0) \le E_0(z_0 + y_\varepsilon)$, which is a contradiction to (5.8). This completes the proof of the proposition. □

**Lemma 5.3.** *Let $\lambda > \lambda_1$. Then the extremal positive solution $u_+$ (respectively, negative solution $u_-$) of (1.1) is the unique global minimizer of the functional $E_+$ (respectively, $E_-$). Moreover, $u_+$ and $u_-$ are local minimizers of $E_0$.*

**Proof.** We know that $E_+ : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Therefore, by [67, Theorem 25.D] there exists a global minimizer $v_+ \in W^{1,p}(\Omega)$ of $E_+$. Since $v_+$ is a critical point of $E_+$, Lemma 5.1 implies that $v_+$ is a nonnegative solution of (1.1) satisfying $0 \le v_+ \le u_+$. By (g1) we infer that

$$|g(x,s)| \le (\lambda - \lambda_1)s^{p-1}, \ \forall s : 0 < s \le \delta_\lambda. \tag{5.42}$$

Using (f4) and (5.42) and applying the Steklov eigenvalue problem in (2.1), we conclude for $\varepsilon < \min\{\frac{\delta_f}{\|\varphi_1\|_\infty}, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}\}$

$$E_+(\varepsilon\varphi_1) = -\int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\lambda_1 - \lambda}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} \\ - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} g(x,s)dsd\sigma$$

$$< \frac{\lambda_1 - \lambda}{p} \varepsilon^p \|\varphi_1\|_{L^p(\partial\Omega)} + \int_{\partial\Omega} \int_0^{\varepsilon\varphi_1(x)} (\lambda - \lambda_1) s^{p-1} ds d\sigma = 0.$$

This shows $E_+(v_+) < 0$ and we obtain $v_+ \neq 0$. Applying Lemma 4.2 implies $v_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Since $u_+$ is the smallest positive solution of (1.1) in $[0, \vartheta e]$ and $0 \leq v_+ \leq u_+$, it holds $v_+ = u_+$. Thus, $u_+$ is the unique global minimizer of $E_+$. In the same way one verifies that $u_-$ is the unique global minimizer of $E_-$. Now we want to show that $u_+$ and $u_-$ are local minimizers of the functional $E_0$. As $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ there exists a neighborhood $V_{u_+}$ of $u_+$ in the space $C^1(\overline{\Omega})$ such that $V_{u_+} \subset C^1(\overline{\Omega})_+$. Hence $E_+ = E_0$ on $V_{u_+}$ which ensures that $u_+$ is a local minimizer of $E_0$ on $C^1(\overline{\Omega})$. In view of Proposition 5.2, we obtain that $u_+$ is also a local minimizer of $E_0$ on the space $W^{1,p}(\Omega)$. By the same arguments as above we prove that $u_-$ is a local minimizer of $E_0$. □

**Lemma 5.4.** *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ has a global minimizer $v_0$ which is a nontrivial solution of* (1.1) *satisfying $u_- \leq v_0 \leq u_+$.*

**Proof.** The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Hence, a global minimizer $v_0$ of $E_0$ exists. Since $v_0$ is a critical point of $E_0$ we know by Lemma 5.1 that $v_0$ is a solution of (1.1) satisfying $u_- \leq v_0 \leq u_+$. Due to $E_0(u_+) = E_+(u_+) < 0$ (cf. the proof of Lemma 5.3) we obtain that $v_0$ is nontrivial meaning $v_0 \neq 0$. □

## 6. Existence of sign-changing solutions

First, we are going to show that our functionals introduced in Section 5 satisfy the Palais-Smale condition. In order to prove this result, we will need a preliminary lemma which can be found in [48, Lemma 2.1-Lemma 2.3] in a similar form.

**Lemma 6.1.** *Let $A, B, C : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be given by*

$$\langle A(u), v \rangle := \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv dx$$

$$\langle B(u), v \rangle := \int_{\partial\Omega} \lambda |T_0^{\partial\Omega}(x,u)|^{p-2} T_0^{\partial\Omega}(x,u) v dx$$

$$\langle C(u), v \rangle := \int_\Omega f(x, T_0(x,u)) v dx + \int_{\partial\Omega} g(x, T_0^{\partial\Omega}(x,u)) v dx;$$

*then $A$ is continuous and continuously invertible and the operators $B, C$ are continuous and compact.*

By means of this auxiliary lemma, we can prove the following.

**Lemma 6.2.** *The functionals $E_+, E_-, E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ satisfy the Palais-Smale condition.*

**Proof.** We show this lemma only for $E_0$. The proof for $E_+, E_-$ is very similar. Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence such that $E_0(u_n)$ is bounded and $E_0'(u_n) \to 0$ as $n$ tends to infinity. Since $|E_0(u_n)| \leq M$ for all $n$, we obtain by using Young's inequality and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$

$$\begin{aligned} M \geq E_0(u_n) &= \frac{1}{p}\left[\|\nabla u_n\|^p_{L^p(\Omega)} + \|u_n\|^p_{L^p(\Omega)}\right] - \int_\Omega \int_0^{u_n(x)} f(x, T_0(x,s)) ds dx \\ &\quad - \int_{\partial\Omega} \int_0^{u_n(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2} T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] ds d\sigma \\ &\geq (1/p - \varepsilon_1 - \varepsilon_2 - \varepsilon_3) ||u_n\|^p_{W^{1,p}(\Omega)} - C. \end{aligned}$$

Choosing $\varepsilon_i, i = 1, 2, 3$ sufficiently small yields the boundedness of $u_n$ in $W^{1,p}(\Omega)$, and thus, we get $u_n \rightharpoonup u$ for a subsequence of $u_n$ still denoted by $u_n$. We have

$$A(u_n) - \lambda B(u_n) - C(u_n) = E_0'(u_n) \to 0,$$

which implies the existence of a sequence $(\delta_n) \subset (W^{1,p}(\Omega))^*$ converging to zero such that

$$u_n = A^{-1}(\lambda B(u_n) + C(u_n) + \delta_n).$$

By Lemma 6.1 we know that $B$ and $C$ are compact and $A^{-1}$ is continuous. Passing to the limit in the previous equality yields $u_n \to A^{-1}(\lambda B(u) + C(u)) =: u$, meaning that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. □

Now, we can formulate our main result about the existence of a nontrivial solution of problem (1.1).

**Theorem 6.3.** *Under hypotheses* (f1)–(f4), (g1)–(g4) *and for every number $\lambda > \lambda_2$, problem* (1.1) *has a nontrivial sign-changing solution $u_0 \in C^1(\overline{\Omega})$.*

**Proof.** Lemma 5.1 implies that every critical point of $E_0$ is a solution of problem (1.1) in $[u_-, u_+]$. The coercivity and the weakly sequentially lower semicontinuity of $E_0$ ensure along with $\inf_{W^{1,p}(\Omega)} E_+(u) < 0$ (cf. the proof of Lemma 5.3) the existence of a global minimizer $v_0 \in W^{1,p}(\Omega)$ satisfying $v_0 \neq 0$. This means that $v_0$ is a nontrivial solution of (1.1) belonging to $[u_-, u_+]$. If $v_0 \neq u_-$ and $v_0 \neq u_+$, then $u_0 := v_0$ must be a sign-changing solution since $u_-$ is the greatest negative solution and $u_+$ is the smallest positive solution of (1.1) which proves the theorem in this case. So, we still have to show that the theorem is also true in case either $v_0 = u_-$ or $v_0 = u_+$. Without loss of generality we suppose $v_0 = u_+$. The function $u_-$ can be assumed to be a strict local minimizer. Otherwise we would be done. Now, we can find a $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{6.1}$$

where $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Assertion (6.1) along with the fact that $E_0$ satisfies the Palais-Smale condition (see Lemma 6.2) enables us to apply the mountain-pass theorem to $E_0$ (see [57]) which yields the existence of $u_0 \in W^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} E_0(\gamma(t)), \tag{6.2}$$

where $\Gamma = \{\gamma \in C([-1,1], W^{1,p}(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+\}$. We see at once that (6.1) and (6.2) show $u_0 \neq u_-$ and $u_0 \neq u_+$, and therefore, $u_0$ is a sign-changing solution provided $u_0 \neq 0$. In order to prove $u_0 \neq 0$ we are going to show that $E_0(u_0) < 0$ which is satisfied if there exists a path $\widetilde{\gamma} \in \Gamma$ such that

$$E_0(\widetilde{\gamma}(t)) < 0, \ \forall t \in [-1,1]. \tag{6.3}$$

Let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Gamma_0 = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\},$$
$$\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}.$$

In view of assumption (g1) there exists a constant $\delta_2 > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} \leq \mu, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leq \delta_2, \tag{6.4}$$

where $\mu \in (0, \lambda - \lambda_2)$. We select $\rho_0 \in (0, \lambda - \lambda_2 - \mu)$. Thanks to the results of Martínez and Rossi in [49] we have the following variational characterization of $\lambda_2$ given by (see (2.2)-(2.4) in Section 2)

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \int_\Omega \Big[|\nabla u|^p + |u|^p\Big] dx. \tag{6.5}$$

By (6.5) there exists a $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\gamma(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2 + \frac{\rho_0}{2}.$$

It is well known that $S_C$ is dense in $S$. This implies the density of $\Gamma_{0,C}$ in $\Gamma_0$ and thus, for a fixed number $r$ satisfying $0 < r \leq (\lambda_2 + \rho_0)^{\frac{1}{p}} - (\lambda_2 + \frac{\rho_0}{2})^{\frac{1}{p}}$, there is a $\gamma_0 \in \Gamma_{0,C}$ such that $\max_{t\in[-1,1]} \|\gamma(t) - \gamma_0(t)\|^p_{W^{1,p}(\Omega)} < r$. This yields $\max_{t\in[-1,1]} \|\gamma_0(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2 + \rho_0$. Let $\delta := \min\{\delta_f, \delta_2\}$, where $\delta_f$ is the constant in condition (f4). The boundedness of the set $\gamma_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of $\varepsilon_0 > 0$ such that

$$\varepsilon_0|u(x)| \leq \delta \quad \text{for all } x \in \Omega \text{ and all } u \in \gamma_0([-1,1]). \tag{6.6}$$

Lemma 4.3 ensures that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Thus, for every $u \in \gamma_0([-1,1])$ and any bounded neighborhood $V_u$ of $u$ in $C^1(\overline{\Omega})$ there exist positive numbers $h_u$ and $j_u$ satisfying

$$u_+ - \frac{1}{h}v \in \operatorname{int}(C^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + \frac{1}{j}v \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{6.7}$$

if $h \geq h_u, j \geq j_u, v \in V_u$. By a compactness argument from (6.7) we conclude the existence of $\varepsilon_1 > 0$ such that

$$u_-(x) \leq \tilde{\varepsilon}u(x) \leq u_+(x) \ \forall\ x \in \Omega, u \in \gamma_0([-1,1]), \forall\ \tilde{\varepsilon} \in (0, \varepsilon_1). \tag{6.8}$$

Let $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. Now, we consider the continuous path $\varepsilon\gamma_0$ in $C^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. We obtain by using hypothesis (f4)

$$-\int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x, T_0(x,s))dsdx \leq 0. \tag{6.9}$$

Applying (6.4), (6.6), (6.7), (6.8), (6.9), and the fact that $\gamma_0([-1,1]) \subset \partial B_1^{L^p(\partial\Omega)}$, we have

$$\begin{aligned} E_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}[\|\nabla\gamma_0(t)\|^p_{L^p(\Omega)} + \|\gamma_0(t)\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x, T_0(x,s))dsdx \\ &\quad - \int_{\partial\Omega} \int_0^{\varepsilon\gamma_0(t)(x)} \left[\lambda|T_0^{\partial\Omega}(x,s)|^{p-2}T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma \\ &< \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0) - \frac{\varepsilon^p}{p}\lambda - \int_{\partial\Omega}\int_0^{\varepsilon\gamma_0(t)(x)} g(x,s)dsd\sigma \\ &< \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0 - \lambda + \mu) < 0 \quad \text{for all } t \in [-1,1]. \end{aligned} \tag{6.10}$$

In the next step we are going to construct continuous paths $\gamma_+, \gamma_-$ which join $\varepsilon\varphi_1$ and $u_+$, respectively, $u_-$ and $-\varepsilon\varphi_1$. We denote

$$c_+ = c_+(\lambda) = E_+(\varepsilon\varphi_1), \quad m_+ = m_+(\lambda) = E_+(u_+),$$

$$E_+^{c_+} = \{u \in W^{1,p}(\Omega) : E_+(u) \le c_+\}.$$

Since $u_+$ is a global minimizer of $E_+$, we see at once that $m_+ < c_+$. Using Lemma 5.1 yields the nonexistence of critical values in the interval $(m_+, c_+]$. Due to the coercivity of $E_+$ along with its property of satisfying the Palais-Smale condition (see Lemma 6.2), we can apply the second deformation lemma (see, e.g. [38, page 366]) to $E_+$. This guarantees the existence of a continuous mapping $\eta \in C([0,1] \times E_+^{c_+}, E_+^{c_+})$ with the following properties:

(i) $\eta(0,u) = u$ for all $u \in E_+^{c_+}$
(ii) $\eta(1,u) = u_+$ for all $u \in E_+^{c_+}$
(iii) $E_+(\eta(t,u)) \le E_+(u),\ \forall t \in [0,1]$ and $\forall u \in E_+^{c_+}$.

We introduce the path $\gamma_+ : [0,1] \to W^{1,p}(\Omega)$ given by $\gamma_+(t) = \eta(t, \varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$. Apparently, $\gamma_+$ is continuous in $W^{1,p}(\Omega)$ and joins $\varepsilon\varphi_1$ and $u_+$. Moreover, we have, for all $t \in [0,1]$,

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \le E_+(\eta(t,\varepsilon\varphi_1)) \le E_+(\varepsilon\varphi_1) < 0. \tag{6.11}$$

Analogously, we can apply the second deformation lemma to the functional $E_-$ and obtain a continuous path $\gamma_- : [0,1] \to W^{1,p}(\Omega)$ between $-\varepsilon\varphi_1$ and $u_-$ such that

$$E_0(\gamma_-(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{6.12}$$

Putting the paths together, $\gamma_-, \varepsilon\gamma_0$ and $\gamma_+$ yield a continuous path $\widetilde{\gamma} \in \Gamma$ joining $u_-$ and $u_+$. In view of (6.10), (6.11) and (6.12), $u_0 \ne 0$. So, we have found a nontrivial sign-changing solution $u_0$ of problem (1.1) satisfying $u_- \le u_0 \le u_+$. This completes the proof. □

**Acknowledgement.** The author gratefully acknowledges the helpful suggestions of Professor S. Carl, Dr. M. Wilke and Dr. R. Zacher during the preparation of this paper.

## REFERENCES

[1] R. A. Adams, "Sobolev Spaces," Academic Press, New York/London, 1975.
[2] H. Amann, "Linear and Quasilinear Parabolic Problems", Birkhäuser Boston Inc., Boston, MA, 1995.
[3] C. C. Aranda and E. Lami Dozo, *Multiple solutions to a singular Lane-Emden-Fowler equation with convection term*, Electron. J. Differential Equations, Vol. 2007 (2007), 1–21.
[4] D. Arcoya, J. I. Diaz, and L. Tello, *S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology*, J. Differential Equations, 150 (1998), 215–225.
[5] C. Atkinson and C. R. Champion, *On some boundary value problems for the equation* $\nabla \cdot (F(|\nabla w|)\nabla w) = 0$, Proc. Roy. Soc. London Ser. A, 448 (1995), 269–279.

[6] C. Atkinson and K. El Kalli, *Some boundary value problems for the bingham model*, J. Differential Equations, 41 (1992), 339–363.
[7] G. Barletta and N. S. Papageorgiou, *A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential*, J. Global Optim., 39 (2007), 365–392.
[8] H. Brezis and L. Nirenberg, *$H^1$ versus $C^1$ local minimizers*, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), 465–472.
[9] S. Carl, *Existence and comparison results for noncoercive and nonmonotone multivalued elliptic problems*, Nonlinear Anal., 65 (2006), 1532–1546.
[10] S. Carl and D. Motreanu, *Constant-sign and sign-changing solutions of a nonlinear eigenvalue problem involving the p-Laplacian*, Differential Integral Equations, 20 (2007), 309–324.
[11] S. Carl and D. Motreanu, *Sign-changing and extremal constant-sign solutions of nonlinear elliptic problems with supercritical nonlinearities*, Comm. Appl. Nonlinear Anal., 14 (2007), 85–100.
[12] S. Carl and D. Motreanu, *Constant-sign and sign-changing solutions for nonlinear eigenvalue problems*, Nonlinear Anal., 68 (2008), 2668–2676.
[13] S. Carl and K. Perera, *Sign-changing and multiple solutions for the p-Laplacian*, Abstr. Appl. Anal., 7 (2002), 613–625.
[14] F. H. Clarke, *A new approach to Lagrange multipliers*, Math. Oper. Res., 1 (1976), 165–174.
[15] J. Crînganu and G. Dinca, *Multiple solutions for a class of nonlinear equations involving a duality mapping*, Differential Integral Equations, 21 (2008), 265–284.
[16] M. Cuesta, D. de Figueiredo, and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations, 159 (1999), 212–238.
[17] L. Del Pezzo, J. Fernández Bonder, and J. D. Rossi, *An optimization problem for the first Steklov eigenvalue of a nonlinear problem*, Differential Integral Equations, 19 (2006), 1035–1046.
[18] M. del Pino and C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Differential Equations, 26 (2001), 2189–2210.
[19] J. I. Díaz, "Nonlinear Partial Differential Equations and Free Boundaries," Pitman (Advanced Publishing Program), Boston, MA, 1985.
[20] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), 827–850.
[21] M. Dobrowolski, "Applied Functional Analysis. Functional Analysis, Sobolev Spaces and Elliptic Differential Equations," Springer, Berlin, 2006.
[22] M. Dobrowolski, private communication, 2008.
[23] P. Drábek, *The least eigenvalues of nonhomogeneous degenerated quasilinear eigenvalue problems*, Math. Bohem., 120 (1995), 169–195.
[24] P. Drábek, *Nonlinear eigenvalue problem for p-Laplacian in $\mathbf{R}^N$*, Math. Nachr., 173 (1995), 131–139.
[25] P. Drábek and J. Hernández, *Existence and uniqueness of positive solutions for some quasilinear elliptic problems*, Nonlinear Anal., 44 (2001), 189–204.
[26] P. Drábek, A. Kufner, and F. Nicolosi, "Quasilinear Elliptic Equations with Degenerations and Singularities," Walter de Gruyter & Co., Berlin, 1997.

[27] P. Drábek and S. B. Robinson, *Resonance problems for the p-Laplacian*, J. Funct. Anal., 169 (1999), 189–200.
[28] S. El Habib and N. Tsouli, *On the spectrum of the p-Laplacian operator for Neumann eigenvalue problems with weights*, Electron. J. Differ. Equ. Conf., 14 (2006), 181–190.
[29] J. F. Escobar, *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math., 43 (1990), 857–883.
[30] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, 235 (2007), 397–417.
[31] J. Fernández Bonder, *Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities*, Abstr. Appl. Anal., vol. 2004 (2004), 1047–1055.
[32] J. Fernández Bonder, *Multiple solutions for the p-Laplace equation with nonlinear boundary conditions*, Electron. J. Differential Equations, vol. 2006 (2006), 1–7.
[33] J. Fernández Bonder, E. Lami Dozo, and J. D. Rossi, *Symmetry properties for the extremals of the Sobolev trace embedding*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), 795–805.
[34] J. Fernández Bonder, S. R. Martínez, and J. D. Rossi, *The behavior of the best Sobolev trace constant and extremals in thin domains*, J. Differential Equations, 198 (2004), 129–148.
[35] J. Fernández Bonder and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions*, J. Math. Anal. Appl., 263 (2001), 195–223.
[36] J. Fernández Bonder and J. D. Rossi, *A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding*, Publ. Mat., 46 (2002), 221–235.
[37] J. P. García Azorero, I. Peral Alonso, and J. J. Manfredi, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math., 2 (2000), 385–404.
[38] L. Gasiński and N. S. Papageorgiou, "Nonsmooth Critical Point Theory and Nonlinear Boundary Value problems," Chapman & Hall/CRC, Boca Raton, FL, 2005.
[39] Z. Guo and Z. Zhang, *$W^{1,p}$ versus $C^1$ local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl., 286 (2003), 32–50.
[40] J. Heinonen, T. Kilpeläinen, and O. Martio, "Nonlinear Potential Theory of Degenerate Elliptic Equations," Dover Publications Inc., Mineola, NY, 2006.
[41] Y. X. Huang, *On eigenvalue problems of p-Laplacian with Neumann boundary conditions*, Proc. Amer. Math. Soc., 109 (1990), 177–184.
[42] A. Kufner, O. John, and S. Fučík, "Function Spaces," Noordhoff International Publishing, Leyden, 1977.
[43] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal., 64 (2006), 1057–1099.
[44] C. Li and S. Li, *Multiple solutions and sign-changing solutions of a class of nonlinear elliptic equations with Neumann boundary condition*, J. Math. Anal. Appl., 298 (2004), 14–32.
[45] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12 (1988), 1203–1219.
[46] D. G. Luenberger, "Optimization by Vector Space Methods," John Wiley & Sons Inc., New York, 1969.

[47] S. R. Martínez and J. D. Rossi, *Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition*, Abstr. Appl. Anal., 7 (2002), 287–293.
[48] S. R. Martínez and J. D. Rossi, *Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance*, Electron. J. Differential Equations, vol. 2003 (2003), 1–14.
[49] S. R. Martínez and J. D. Rossi, *On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition*, Nonlinear Anal., 59 (2004), 813–848.
[50] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, *Multiple nontrivial solutions for nonlinear eigenvalue problems*, Proc. Amer. Math. Soc., 135 (2007), 3649–3658.
[51] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, *Nonlinear neumann problems near resonance*, Indiana Univ. Math. J., 58 (2009), 1257–1279.
[52] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, *Positive solutions and multiple solutions at non-resonance, resonance and near resonance for hemivariational inequalities with p-Laplacian*, Trans. Amer. Math. Soc., 360 (2008), 2527–2545.
[53] D. Motreanu and N. S. Papageorgiou, *Multiple solutions for nonlinear elliptic equations at resonance with a nonsmooth potential*, Nonlinear Anal., 56 (2004), 1211–1234.
[54] F. Papalini, *Nonlinear eigenvalue Neumann problems with discontinuities*, J. Math. Anal. Appl., 273 (2002), 137–152.
[55] F. Papalini, *A quasilinear Neumann problem with discontinuous nonlinearity*, Math. Nachr., 250 (2003), 82–97.
[56] J. P. Pinasco, *Asymptotic behavior of the Steklov eigenvalues for the p-Laplace operator*, Adv. Nonlinear Stud., 7 (2007), 319–328.
[57] P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.
[58] T. Runst and W. Sickel, "Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations," Walter de Gruyter & Co., Berlin, 1996.
[59] W. Sickel, private communication, 2008.
[60] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, 51 (1984), 126–150.
[61] O. Torné, *Steklov problem with an indefinite weight for the p-Laplacian*, Electron. J. Differential Equations, vol. 2005 (2005), 1–8.
[62] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
[63] H. Triebel, "Theory of Function Spaces," Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.
[64] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (1984), 191–202.
[65] J. Wloka, "Partielle Differentialgleichungen," B. G. Teubner, Stuttgart, 1982.
[66] X. Wu and K.-K. Tan, *On existence and multiplicity of solutions of Neumann boundary value problems for quasi-linear elliptic equations*, Nonlinear Anal., 65 (2006), 1334–1347.
[67] E. Zeidler, "Nonlinear Functional Analysis and its Applications II/B," Springer-Verlag, New York, 1990.

[68] J.-H. Zhao and P.-H. Zhao, *Infinitely many weak solutions for a p-Laplacian equation with nonlinear boundary conditions*, Electron. J. Differential Equations, vol. 2007 (2007), 1–14.

[69] J.-H. Zhao and P.-H. Zhao, *Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions*, Nonlinear Anal., 69 (2008), 1343–1355.